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FLAG MANIFOLDS

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I. Basic facts about Lie groups and Lie algebras

I.1. Lie groups and Lie algebras. Functor Lie. Recall that a Lie group is a smooth manifold G together with structure of a group such that the group operations

$$\mu : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 \cdot g_2$$
$$i : G \to G, \quad g \mapsto g^{-1}$$

are smooth mappings. Here we will assume that either all objects are real and then G is a real Lie group, or all objects are complex and holomorphic, then G is a complex Lie group. An example of Lie group is the general linear group $GL(V) = \mathrm{Aut}(V)$, i.e. the group of all automorphisms of a finite dimensional vector space $V \approx \mathbb{R}^n$ or $\mathbb{C}^n$. If V is the arithmetic vector space k^n, $k = \mathbb{R}$ or $\mathbb{C}$, then the group $GL(k^n)$ is denoted also by $GL_n(k)$ and it is called the general matrix group. Any closed subgroup G of $GL(V)$ is a Lie group, called a linear group. Example of linear groups are the following classical Lie groups. The group

$$SL(V) = \{A \in GL(V), \ \det A = 1\}$$

of unimodular transformations and the group

$$\mathrm{Aut}(V, b) = \{A \in GL(V), \ A \cdot b := b(A\cdot, A\cdot) = b\}$$

of automorphisms of a nondegenerate bilinear form b. If $b = \omega$ is skew-symmetric, then the group $\mathrm{Aut}(V, \omega) = Sp_\omega(V) = Sp_l(\mathbb{C})$, $V = \mathbb{C}^{2l}$ is called symplectic group and it is connected. For $V = k^n$, $n = 2m$, the standard notation for symplectic group is $Sp_m(k)$. If $b = g$ is a symmetric form, then the group $\mathrm{Aut}_g(V, g) = O_g(V)$

AMS Subject Classification (1991): Primary 53C30, 53C55, 53C57 is called the orthogonal group. The connected component of the unity is denoted by $SO_g(V) = \{A \in O_g(V), \det A = 1\}$ or by $SO_n(\mathbb{C})$, if $V = \mathbb{C}^n$.

Lie algebra (real or complex) is a vector space (over $k = \mathbb{R}, \mathbb{C}$) with a bilinear operation

$$\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto [X, Y] \in \mathfrak{g}$$

which satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad X, Y, Z \in \mathfrak{g}.$$

Let G be a Lie group. Denote by L the left action of G on G, that is the homomorphism

$$L : G \to \mathrm{Diff}(G), \quad g \mapsto L_g, \quad L_g g_1 = g g_1, \quad g, g_1 \in G.$$

The space

$$\mathfrak{g} = \mathrm{Lie}\, G := \mathfrak{X}(G)^{L_G} = \{X \in \mathfrak{X}(G),\ L_g^* X = X,\ g \in G\}$$

of all L_G-invariant vector fields on G is a subalgebra of the Lie algebra $\mathfrak{X}(G)$ of vector fields (with respect to the Lie bracket $(X, Y) \mapsto [X, Y] = X \circ Y - Y \circ X$; here vector fields are considered as derivations of the algebra $C^\infty(G)$ of smooth functions on G and $\circ$ means the composition of derivations). Lie algebra of a Lie group G is called the tangent Lie algebra $\mathfrak{g}$ of G.

Any L_G-invariant vector field X is defined by its value X_e in the point $e \in G$:

$$X_g = X_e \cdot g := (R_g)_* X_e \quad g \in G,$$

where $R_g g_1 = g_1 g$ is the right multiplication. (In the case of a linear Lie group G, $X_g = X_e \cdot g$, where dot stands for matrix multiplication). The map $X \mapsto X_e$ defines isomorphism $\mathfrak{X}(G)^{L_G} \to T_e G$ which allows identify the tangent Lie algebra $\mathfrak{X}(G)^{L_G}$ with the tangent space $T_e G$. If $G \subset GL(V)$ is a linear Lie group, then the Lie bracket on $\mathfrak{g} = T_e G$ is the commutator:

$$[X, Y] = X \cdot Y - Y \cdot X, \quad X, Y \in gl(V) = \mathrm{End}\, V.$$

The map $G \mapsto \mathfrak{g} = \mathrm{Lie}\, G$ defines a functor from category of Lie groups to category of Lie algebras. It is called Lie functor and is very closed to be an isomorphism of categories. Indeed, any Lie algebra $\mathfrak{g}$ is the tangent Lie algebra of some simply connected Lie group G, defined up to an isomorphism. Any Lie group G_1 with tangent Lie algebra $\mathfrak{g}$ is isomorphic to the quotient G/Γ of G by a central discrete subgroup Γ of G. All Lie groups with the same tangent Lie algebra are locally isomorphic, i.e. they have the same group operations in some neighborhood of the unity. Many properties of a Lie group G can be described in terms of properties of the tangent Lie algebra $\mathfrak{g}$.

For example, Lie group G is *solvable, nilpotent, commutative* iff the tangent Lie algebra $\mathfrak{g}$ is *solvable, nilpotent, commutative.* Lie group G is *simple* (i.e. has no nondiscrete normal subgroup), *semisimple* (i.e. has no nondiscrete commutative normal subgroup), *reductive* (i.e. has a normal semisimple subgroup S with commutative quotient G/S) iff the tangent Lie algebra $\mathfrak{g}$ is *simple* (has no proper ideal), *semisimple* (has no proper commutative ideal) or *reductive* (is a direct sum of commutative and semisimple Lie algebras).

There is 1-1 correspondence between subalgebras (resp., ideals) of a Lie algebra $\mathfrak{g}$ and *virtual subgroups* (resp., *virtual normal subgroups*) of a Lie group G with $\operatorname{Lie} G = \mathfrak{g}$. Subgroup H of a Lie group G is called *virtual* if it is an immersed submanifold of G. Such subgroup H has a structure of Lie group, but H is not necessary a closed subgroup of G (even if it is simple).

Exercise. Prove that the classical Lie groups $SL(V)$, $SO_g(V)$, $Sp_\omega(V)$ have tangent Lie algebras

$$\begin{aligned}
sl(V) &= \{A \in gl(V),\ \operatorname{tr} A = 0\}\,,\\
so_g(V) &= \{A \in sl(V)\ \ g(Ax,y) = g(x,Ay) = 0,\ \forall x,y \in V\}\,,\\
sp_\omega(V) &= \{A \in sl(V),\ \omega(Ax,y) + \omega(x,Ay) = 0,\ \forall x,y \in V\}\,.
\end{aligned}$$

Prove that all of them are simple with the exception $SO_g(V)$, where $V = \mathbb{C}^4$ or $V = \mathbb{R}^4$ and g has signature (4,0), (2,2) or (0,4).

I.2. Basic results about structure of Lie groups and Lie algebras. We collect here some useful general results about Lie groups and Lie algebras.

(1) (Levi-Maltsev theorem). *Any Lie algebra $\mathfrak{g}$ can be decomposed into a sum*

$$\mathfrak{g} = \mathfrak{s} + \mathfrak{r}, \quad \mathfrak{s} \cap \mathfrak{r} = 0$$

of a maximal semisimple subalgebra $\mathfrak{s}$ and the radical $\mathfrak{r}$ (maximal solvable ideal).

Any two maximal semisimple subalgebras are conjugated by an automorphism of $\mathfrak{g}$.

Any connected Lie group G can be decomposed into a product $G = S \cdot R$ of a maximal semisimple subgroup S and the radical R (i.e. the maximal solvable normal subgroup), such that $S \cap R$ is a discrete subgroup of G. Any two maximal semisimple subgroups of G are conjugated by an automorphism of G.

(2) *Any semisimple (respectively, reductive) Lie algebra $\mathfrak{g}$ is a direct sum of noncommutative simple ideals:*

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_k.$$

(respectively, simple ideals and the center).

Any simply connected semisimple Lie group G is a direct product of simple connected normal subgroups:

$$G = G_1 \times \ldots \times G_k.$$

Any simply connected connected reductive Lie group G is a direct product of simple Lie groups G_i, $i > 0$ and the connected component of the center of connected Lie group $Z_0(G) \simeq \mathbb{R}^p \times T^q$, where

$$T^q = \underbrace{S^1 \times ... \times S^1}_{q}$$

is the q-torus $G = Z_0(G) \times G_1 \times ... \times G_k$.

(3) (Ado theorem). *Any Lie algebra* $\mathfrak{g}$ *(over* $\mathbb{R}, \mathbb{C}$*) admits exact linear representation, that is a homomorphism* $\rho : \mathfrak{g} \to gl(V)$ *into general linear Lie algebra* $gl(V) = \operatorname{End} V$ *with trivial kernel.*

For any simply connected connected Lie group G, there exists a discrete central subgroup Γ such that G/Γ admits an exact linear representation $\rho : G/\Gamma \to GL(V)$ (monomorphism into $GL(V)$).

Let G be a Lie group and denote by Ad the natural homomorphism of G into the group $\operatorname{Aut}(G)$ of automorphisms of G given by $\operatorname{Ad}_g g_1 = g g_1 g^{-1}$, $g, g_1 \in G$. Since any automorphism of G preserves the unity e, the group $\operatorname{Ad}_G \subset \operatorname{Aut}(G)$ acts on the tangent space $\mathfrak{g} = T_e G$ by linear transformations and this action preserves the Lie bracket on $\mathfrak{g}$:

$$[\operatorname{Ad}_g X, \operatorname{Ad}_g Y] = \operatorname{Ad}_g [X, Y], \quad g \in G, \ X, Y \in T_e M.$$

If $G \subset GL(V)$ is a linear Lie group,

$$\operatorname{Ad}_g X = g \cdot X \cdot g^{-1},$$

where $\cdot$ denotes the matrix multiplication. We get a representation

$$\operatorname{Ad} : g \mapsto \operatorname{Ad}_g \in \operatorname{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g}),$$

which is called the *adjoint representation* of a Lie group G. It induces the *adjoint representation* ad of the tangent Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ by derivations of $\mathfrak{g}$:

$$\operatorname{ad} : \mathfrak{g} \to \operatorname{Der}(\mathfrak{g}), \quad X \mapsto \operatorname{ad}_X, \quad \operatorname{ad}_X Y = [X, Y] \quad X, Y \in \mathfrak{g}.$$

Remark that the kernel of Ad coincides with $Z(G)$ (the center of G) and $\ker \operatorname{ad} = Z(\mathfrak{g})$ (the center of $\mathfrak{g}$).

Define a symmetric bilinear form B on a Lie algebra $\mathfrak{g}$ by

$$\mathrm{B}(X, Y) = \operatorname{tr} \operatorname{ad}_X \operatorname{ad}_Y, \quad X, Y \in \mathfrak{g}.$$

It is called the *Killing form*. The adjoint action of a Lie group G with $\operatorname{Lie} G = \mathfrak{g}$ preserves B:

$$\mathrm{B}(\operatorname{Ad}_g X, \operatorname{Ad}_g Y) = \mathrm{B}(X, Y), \quad g \in G, \ X, Y \in \mathfrak{g}. \tag{1}$$

In terms of the adjoint action of the Lie algebra this condition can be written as

$$\mathrm{B}(\mathrm{ad}_Z X, Y) + \mathrm{B}(X, \mathrm{ad}_Z Y) = 0 \quad X, Y, Z \in \mathfrak{g}. \tag{2}$$

This condition follows from previous relation if we remark that any vector $Z \in \mathfrak{g}$ generates an 1-parametric subgroup $g_t = \exp tZ$ of G with tangent vector Z at $e : dg_t/dt|_{t=0} = Z$. Then to get (2), it is sufficient to differentiate (1) with $g = g_t$ with respect to t and put $t = 0$.

The following useful criterion of solvability and semi-simplicity of a Lie algebra $\mathfrak{g}$ is due to Cartan.

(4) *A Lie algebra* $\mathfrak{g}$ *is solvable if the commutant*

$$[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\,\{[X, Y],\ X, Y \in \mathfrak{g}\}$$

is in the kernel of B, *i.e.*

$$\mathrm{B}([X, Y], Z) = 0 \quad \forall X, Y, Z \in \mathfrak{g}.$$

A Lie algebra $\mathfrak{g}$ *is semisimple if the Killing form* B *is nondegenerate.*

I.3. Structure of complex semisimple Lie algebras.

I.3.1. Cartan decomposition of a semisimple Lie algebra. Let now $\mathfrak{g}$ be a semisimple Lie algebra. Since $Z(\mathfrak{g}) = 0$, the adjoint representation ad : $X \mapsto \mathrm{ad}_X$, $\mathrm{ad}_X Y = [X, Y]$ is exact, and we can identify $\mathfrak{g}$ with linear Lie algebra $\mathrm{ad}_{\mathfrak{g}} \subset \mathrm{der}(\mathfrak{g}) \subset \mathrm{gl}(\mathfrak{g})$. Remark that the connected group of automorphisms $\mathrm{Ad}\, G = \mathrm{Aut}(\mathfrak{g})_0$ has $\mathfrak{g}$ as Lie algebra. It is called the *adjoint group.*

Define a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ as a maximal subalgebra such that the linear subalgebra $\mathrm{ad}_{\mathfrak{h}} \subset \mathrm{ad}\,\mathfrak{g}$ is diagonalizable, i.e. with respect to some basis of $\mathfrak{g}$ all the endomorphisms from $ad_{\mathfrak{h}}$ are represented by diagonal matrices. Cartan subalgebra $\mathfrak{h}$ always exists and coincides with its normalizer

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g},\ [x, \mathfrak{h}] \subset \mathfrak{h}\},$$

and any two Cartan subalgebras are conjugated by an automorphism of $\mathfrak{g}$.

A linear form α on Cartan subalgebra $\mathfrak{h}$ is called a *root* if the corresponding *root space*

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g},\ \mathrm{ad}_h\, x = \alpha(h)x,\ \forall h \in \mathfrak{h}\}$$

is not zero. A nonzero vector from $\mathfrak{g}_\alpha$ is called a *root vector.* Denote by R the (finite) set of all roots. Then we have the following Cartan decomposition of $\mathfrak{g}$ into direct sum of subspaces:

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha.$$

The main properties of such a decomposition are:

1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, where $\mathfrak{g}_{\alpha+\beta} = 0$ if $\alpha + \beta \notin R$.
2) $\mathrm{B}(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$, $\mathrm{B}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$.
3) $R = -R$
4) $B|_{\mathfrak{h}}$ is nondegenerate and define isomorphism

$$\mathrm{B} : \mathfrak{h}^* \mapsto \mathfrak{h}, \quad \alpha \mapsto u_\alpha = \mathrm{B}^{-1} \circ \alpha$$

5) Put $E = \langle R \rangle = \mathrm{span}_{\mathbb{R}} R$, $\mathfrak{h}(\mathbb{R}) = B^{-1}E$. Then $\mathfrak{h} = \mathfrak{h}(\mathbb{R}) + i\mathfrak{h}(\mathbb{R})$ and $B|_{\mathfrak{h}(\mathbb{R})}$ is positively defined. We will denote the corresponding scalar product on E and $\mathfrak{h}(\mathbb{R})$ by $(\cdot,\cdot)$.
6) $\dim \mathfrak{g}_\alpha = 1$.

Choose a vector $E_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in R$ such that $\mathrm{B}(E_\alpha, E_{-\alpha}) = 2/(\alpha,\alpha)$ and put $H_\alpha = [E_\alpha, E_{-\alpha}]$. Then $[H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$ and $\mathfrak{g}(\alpha) = \mathbb{C}H_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ is a subalgebra isomorphic to $sl_2(C)$. The isomorphism is given by

$$H_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

There is some freedom in choice of root vector E_α. Using this freedom, one can choose E_α, $\alpha \in R$, such that the following commutation relations hold:

$$7) \qquad [E_\alpha, E_\beta] = \begin{cases} 0, & \text{if} \, \alpha + \beta \notin R \\ \pm(p+1)E_{\alpha+\beta}, & \text{if} \, \alpha + \beta \in R, \end{cases}$$

where $p \geq 0$ is the maximal integer such that $\beta - p\alpha \in R$. Moreover, there exist an algorithm for determination the sign $\pm$ in this formula [Tits].

As a corollary we have

PROPOSITION. *The system of roots R of a semisimple Lie algebra $\mathfrak{g}$ determines $\mathfrak{g}$ up to an isomorphism.*

I.3.2. Cartan decomposition and root system of classical complex Lie algebras. Now we describe the Cartan decomposition of the classical Lie algebras

$$A_l = sl_{l+1}(\mathbb{C}), \quad B_l = so_{2l+1}(\mathbb{C}), \quad C_l = sp_l(\mathbb{C}), \quad D_l = so_{2l}(\mathbb{C})$$

which are tangent Lie algebras of the classical groups

$$SL_{l+1}(\mathbb{C}), \quad SO_{2l+1}(\mathbb{C}), \quad Sp_l(\mathbb{C}), \quad SO_{2l}(\mathbb{C}).$$

A_l) Denote by e_i, $i = 1, ..., l+1$ the standard basis of the vector space $V = \mathbb{C}^{l+1}$ and identify the Lie algebra $gl(V)$ of endomorphisms with $V \otimes V^*$. The Lie algebra $A_l = sl_{l+1}(\mathbb{C})$ consists of all traceless endomorphisms. The subalgebra

$$\mathfrak{h} = \left\{ h = \sum_{i=1}^{l+1} x_i e_i \otimes e_i^*, \ \sum x_i = 0 \right\}$$

of all diagonal (with respect to the basis $\{e_i\}$) elements of A_l is a Cartan subalgebra of A_l. Set $E_{ij} = e_i \otimes e_j^*$ and denote by $\varepsilon_i \in \mathfrak{h}^*$ the linear form on $\mathfrak{h}$ defined by $\varepsilon_i(h) = x_i$. Then $R = \{\varepsilon_i - \varepsilon_j,\ i \neq j\}$ is the system of roots and E_{ij} is the root vector with root $\varepsilon_i - \varepsilon_j$. The Cartan decomposition of A_l is given by

$$A_l = \mathfrak{h} + \sum_{i \neq j} \mathbb{C} E_{ij}.$$

B_l, C_l, D_l). To describe the Lie algebras B_l, C_l, D_l in a unified way, we denote by b a nondegenerate bilinear form in the space $V = \mathbb{C}^n$ which is either symmetric ($b = g$), or skew-symmetric ($b = \omega$) and by

$$\operatorname{aut}(V, b) = \{A \in gl(V),\ b\,(A\cdot, \cdot) + b\,(\cdot, A\cdot) = 0\}$$

the Lie algebra of endomorphisms which preserve b. Then

$$B_l = \operatorname{aut}\left(\mathbb{C}^{2l+1}, g\right), \quad C_l = \operatorname{aut}\left(\mathbb{C}^{2l}, \omega\right), \quad D_l = \operatorname{aut}\left(\mathbb{C}^{2l}, g\right).$$

We choose a basis e_0, e_i, e_{-i}, $i = 1, \dots, l$ of $\mathbb{C}^n$ for $n = 2l + 1$ and a basis e_i, e_{-i}, $i = 1, \dots, l$ of $\mathbb{C}^n$ for $n = 2l$ such that the form b is given by

$$B_l : b = g = e_0^* \otimes e_0^* + \sum_{i=1}^{l} e_i^* \vee e_{-i}^*$$

$$C_l : b = \omega = \sum_{i=1}^{l} e_i^* \wedge e_{-i}^*$$

$$D_l : b = g = \sum_{i=1}^{l} e_i^* \vee e_{-i}^*$$

where e_i^* is the dual basis of V^* and

$$x \wedge y = x \otimes y - y \otimes x, \quad x \vee y = x \otimes y + y \otimes x$$

are wedge and symmetric product. Such basis is called the standard basis of V. We identify the dual space V^* with V by means of the bilinear form b:

$$V \stackrel{\cong}{\to} V^*, \quad x \mapsto bx = b\,(x, \cdot) \quad x \in V.$$

Then the Lie algebras B_l, D_l are identified with the space $\wedge^2 V$ of bivectors and C_l with the space $\vee^2 V$ of symmetric (2,0)-tensors. For example, a decomposable bivector $x \wedge y$ defines the endomorphism $z \mapsto (x \wedge y)\, z = g(y, z)x - g(x, z)y$ which belongs to $\operatorname{aut}(V, g)$.

The set of all diagonals with respect to the standard basis elements forms a Cartan subalgebra $\mathfrak{h}$ of $\mathrm{aut}(V, b)$. More precisely, we have

$$B_l : \mathfrak{h} = \left\{ h = \sum_{i=1}^{l} x_i \cdot e_i \wedge e_{-i} \right\},$$

$$C_l : \mathfrak{h} = \left\{ h = \sum_{i=1}^{l} x_i \cdot e_i \vee e_{-i} \right\},$$

$$D_l : \mathfrak{h} = \left\{ h = \sum_{i=1}^{l} x_i \cdot e_i \wedge e_{-i} \right\}.$$

The corresponding Cartan decomposition is given by

$$B_l = \mathfrak{h} + \sum_{i,j=0}^{l} \mathbb{C} e_i \wedge e_{-j} + \sum_{0 \le i < j} \mathbb{C} e_i \wedge e_j + \sum_{0 \le i < j} \mathbb{C} e_{-i} \wedge e_{-j},$$

$$C_l = \mathfrak{h} + \sum_{i,j=1}^{l} \mathbb{C} e_i \vee e_{-j} + \sum_{i,j=1}^{l} \mathbb{C} e_i \vee e_j + \sum_{i,j=1}^{l} \mathbb{C} e_{-i} \vee e_{-j},$$

$$D_l = \mathfrak{h} + \sum_{i,j=1}^{l} \mathbb{C} e_i \wedge e_{-j} + \sum_{0 < i < j} \mathbb{C} e_i \wedge e_j + \sum_{0 < i < j} \mathbb{C} e_{-i} \wedge e_{-j}.$$

Denote by ε_i, $i = 1, \dots, l$ the standard basis of $\mathfrak{h}^*$, defined by $\varepsilon_i(h) = x_i$. Then the roots and corresponding root vectors are given in the following table.

$\mathfrak{g}$	roots	roots vectors
B_l	$\varepsilon_i - \varepsilon_j$, $i, j > 0$, $i \neq j$	$e_i \wedge e_{-j}$
	$\pm\varepsilon_i$, $i > 0$	$e_0 \wedge e_{\pm i}$
	$\varepsilon_i + \varepsilon_j$, $i, j > 0$, $i < j$	$e_i \wedge e_j$
	$-\varepsilon_i - \varepsilon_j$, $i, j > 0$, $i < j$	$e_{-i} \wedge e_{-j}$
C_l	$\varepsilon_i - \varepsilon_j$, $i \neq j$	$e_i \vee e_{-j}$
	$\varepsilon_i + \varepsilon_j$, $i \neq j$	$e_i \vee e_j$
	$\pm 2\varepsilon_i$	$e_{\pm i} \vee e_{\pm i}$
	$-\varepsilon_i - \varepsilon_j$, $i \neq j$	$e_{-i} \vee e_{-j}$
D_l	$\varepsilon_i - \varepsilon_j$, $i \neq j$	$e_i \wedge e_{-j}$
	$\varepsilon_i + \varepsilon_j$, $i \neq j$	$e_i \wedge e_j$
	$-\varepsilon_i - \varepsilon_j$, $i \neq j$	$e_{-i} \wedge e_{-j}$

I.3.3. Root systems. Let $R \subset E = \mathrm{span}_{\mathbb{R}} R \subset \mathfrak{h}^*$ be the system of roots of a semi-simple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$. Denote by $G(\alpha)$ the 3-dimensional subgroup $\mathfrak{g}$ of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ associated with the Lie subalgebra $\mathfrak{g}(\alpha) = \mathbb{C}H_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$, $\alpha \in R$. Studying the adjoint action of this group on root vectors, one can establish the following fundamental properties of the set R:

i) $\forall \alpha, \beta \in R$, $\langle \beta | \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$;

ii) $\forall \alpha \in R$, $S_\alpha R = R$, where $S_\alpha : \beta \mapsto \beta - \langle \beta | \alpha \rangle \alpha$ is the reflection with respect to hyperplane $\alpha^\perp$ in E;

iii) α, $\lambda\alpha \in R$ for $\lambda \in \mathbb{R}$ implies $\lambda = \pm 1$.

Definition. A finite set R of vectors in Euclidian space E is called an (abstract reduced) root system, if R generates E and satisfies i)–iii).

From the definition one can derive the following two additional properties of root system R.

1) Let $\alpha, \beta \in R$ be roots such that $|\beta| \geq |\alpha|$ and $(\alpha, \beta) \leq 0$. Denote by θ the angle between α, β. Then all possibilities for θ, $\langle \alpha | \beta \rangle$, $\langle \beta | \alpha \rangle$, $|\beta|^2/|\alpha|^2$ are given in the following table.

θ	$\langle \alpha \vert \beta \rangle$	$\langle \beta \vert \alpha \rangle$	$\vert\beta\vert^2/\vert\alpha\vert^2$
$\pi/2$	0	0	
$2\pi/3$	-1	-1	1
$3\pi/4$	-1	-2	2
$5\pi/6$	-1	-3	3

Let α, $\beta \neq \pm\alpha \in R$. The α-series of roots, containing β is defined as the set of all roots of the form $\beta + k\alpha$, $k \in \mathbb{Z}$.

2) α-series of roots containing β has the form $\{\beta + k\alpha,\ -p \leq k \leq q\}$ where $p, q \geq 0$ and $p - q = \langle \beta | \alpha \rangle$. In particular, if $(\beta, \alpha) < 0$, then $\beta + \alpha \in R$, if $\beta - \alpha \notin R$ and $\beta + \alpha \in R$, then $(\beta, \alpha) < 0$.

We associate with a semisimple Lie algebra $\mathfrak{g}$ a root system R. Two natural questions arise:

1) Are two semisimple Lie algebras with isomorphic root system isomorphic?

2) Is it true that any abstract root system R is the root system of some semisimple Lie group?

The answer for both questions is positive.

THEOREM. 1) *Let $\mathfrak{g}$ resp. $\mathfrak{g}'$ are semisimple Lie algebras and R, resp. R' is root system of $\mathfrak{g}$, resp. $\mathfrak{g}'$ with respect to a Cartan subalgebra $\mathfrak{h}$, $\mathfrak{h}'$. Then any isomorphism of Euclidean vector spaces $\langle R \rangle$, $\langle R' \rangle$ which maps R onto R' can be extended to an isomorphism $\mathfrak{g} \to \mathfrak{g}'$ of Lie algebras.*

2) Any abstract root system R is isomorphic to the root system of some semisimple Lie algebra.

The semisimple Lie algebra $\mathfrak{g}$ with given root system $R \subset E$ may be constructed as follows. Let $E^{\mathbb{C}}$ be the complexification of the vector space E and

$\mathfrak{h} = (E^{\mathbb{C}})^*$ the dual vector space. Consider direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathbb{C} E_\alpha,$$

where E_α is a basis of 1-dimensional vector space $\mathbb{C}E_\alpha$ associated with a root $\alpha \in R$. $\mathfrak{g}$ become a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system R, if the Lie bracket is defined by:

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &= 0, \\ [h, E_\alpha] &= \alpha(h) E_\alpha, \quad h \in \mathfrak{h}, \ \alpha \in R \\ [E_\alpha, E_\beta] &= \begin{cases} 0 & \text{if } \alpha + \beta \notin R \\ \pm(p+1) E_{\alpha+\beta} & \text{if } \alpha + \beta \in R \end{cases} \end{aligned}$$

where α-series of roots, containing β is given by $\{\beta - p\alpha, ..., \beta + q\alpha\}$. An algorithm for determination of the sign $\pm$ is given in [Tits].

I.3.4. System of simple roots. The classification of root systems can be reduced to the classification of some special bases of the Euclidean vector space E.

Definition. Let R be a root system in Euclidean vector space E. A set $\Pi = \{\gamma_1, ..., \gamma_l\}$ of roots is called a basis of R or a system of simple roots if Π is a basis of E and any root $\alpha \in R$ has integer coordinates with respect to Π of the same sign:

$$\alpha = \sum_{i=1}^{l} k_i \alpha_i, \quad k_i \in \mathbb{Z}$$

where either $k_i \geq 0$, or $k_i \leq 0$ for $i = 1, ..., l$. If $k_i \geq 0$ (resp., $k_i \leq 0$), then the root α is called positive (resp. negative).

Hence, a basis Π defines a decomposition $R = R^+ \cup R^-$ of the root system into disjoint sum of positive roots R^+ and negative roots $R^- = -R^+$.

To construct a system of simple roots, we define the set $E^*_{\mathrm{reg}} = E^* \setminus \{\bigcup_{\alpha \in R} (\alpha = 0)\}$ of regular elements in E^* as the set of vectors from E^* on which all roots take nonzero values. A connected component C of E^*_{reg} is called a Weyl chamber. Any Weyl chamber C is defined by inequalities $C = \{\alpha_1 > 0, ..., \alpha_l > 0\}$, where α_i are some roots. These roots form a basis $\Pi = \{\alpha_1, ..., \alpha_l\}$ of R and any basis of R can be obtained in such a way. The finite group $W = \langle S_\alpha, \alpha \in R \rangle$ generated by reflections S_α (the *Weyl group* of a root system R) acts simply transitively on the set of Weyl chambers and, hence, bases of R. A practical way for constructing a basis of R is the following:

Choose a regular element $h \in E^*_{\mathrm{reg}}$ and define the set of positive roots R^+ as the set of roots which have positive value on h:

$$R^+ = \{\alpha \in R, \ \alpha(h) > 0\}.$$

A positive root $\alpha \in R^+$ is called *simple* if it is not a sum of two positive roots. The set $\Pi = \{\alpha_1, ..., \alpha_l\}$ of simple roots is a basis of R.

Due to the formula $p - q = \langle\beta|\alpha\rangle$ for α-series of roots $\{\beta + k\alpha, -p \leq k \leq q\}$, a root system R can be reconstructed from simple root system Π inductively, starting from simple roots. At first we determine all the roots which are sums of two simple roots, then the roots which are sums of three simple roots etc.

Similar to root systems, one can give an intrinsic characterization of a simple root system as follows. A basis $\Pi = \{\alpha_1, ..., \alpha_l\}$ of the Euclidean vector space E^l is called a simple root system if

$$a_{ij} = \langle\alpha_i|\alpha_j\rangle := \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

is a nonpositive integer for any $i \neq j$. The matrix $A = ||a_{ij}||$, $a_{ij} = \langle\alpha_i|\alpha_j\rangle$, is called the *Cartan matrix* of the simple root system Π. It determines Π up to an isometry and is characterized by the following properties:

i) $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$;

ii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$;

iii) $m_{ij} = a_{ij}a_{ji} = 0, 1, 2, 3$ for $i \neq j$;

iv) The matrix $G(A) = ||g_{ij}||$, $g_{ii} = 1$, $g_{ij} = -\frac{1}{2}\sqrt{m_{ij}}$, $i \neq j$ is positively defined.

The last property follows from the fact that $G(\Gamma)$ is the Gram matrix of the basis $\{\alpha_i/||\alpha_i||\}$.

A nice way for visualization of a simple root system Π and its Cartan matrix A was proposed by E.B. Dynkin. He associates with Π a graph $\Gamma = \Gamma(\Pi)$, which is called Dynkin graph, by the following rules:

Any simple root $\alpha_i \in \Pi$ is represented by a vertex of Γ;

Two verteces α_i, α_j are jointed by $m_{ij} = a_{ij}a_{ji} = \langle\alpha_i|\alpha_j\rangle\langle\alpha_j|\alpha_i\rangle$ lines;

If $|\alpha_i| > |\alpha_j|$, and $(\alpha_i, \alpha_j) \neq 0$ (and hence, $a_{ij} = \langle\alpha_i|\alpha_j\rangle > a_{ji} = \langle\alpha_j|\alpha_i\rangle$), then we draw arrow which indicate the direction from long root α_i to short root α_j.

The Dynkin diagram determines simple root system up to isometry.

Remark that if $R' \subset E'$, $R'' \subset E'$ are root systems in spaces E', E'', respectively, then $R = (R' + 0) \cup (0 + R'')$ is a root system in the space $E = E' \oplus E''$. Such system R is called decomposable and it corresponds to a semisimple Lie algebra which is a direct sum of two semisimple ideals. Root system is indecomposable iff the associated Dynkin graph is connected. It is equivalent to the condition that the associated Lie algebra is simple, or, in terms of Cartan matrix A, that the Cartan matrix A is indecomposable, i.e., it can not be transformed into block diagonal form by means of a permutation of rows and the same permutation of columns.

The classification of indecomposable Cartan matrices leads to the following result.

CLASSIFICATION THEOREM. *Besides the root systems A_l, B_l, C_l, D_l of classical complex Lie algebras $sl_{l+1}(\mathbb{C})$, $so_{2l+1}(\mathbb{C})$, $sp_{2l}(\mathbb{C})$, $so_{2l}(\mathbb{C})$ there exist only 5 indecomposable root systems G_2, F_4, E_6, E_7, E_8.*

The low index indicate the rank of the system, i.e. the dimension l of the corresponding Euclidean space E^l. These root systems and the corresponding Lie algebra are called exceptional. The exceptional root systems are described in the following table taken from [G-O-V].

Type	$\dim G$	R	Π, δ	W
E_8	248	$\varepsilon_i - \varepsilon_j$ $\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i < 8$ $\alpha_8 = \varepsilon_6 + \varepsilon_7 + \varepsilon_8$ $\delta = \varepsilon_1 - \varepsilon_9$	$\operatorname{Aut} R$
E_7	133	$\varepsilon_i - \varepsilon_j$ $\varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i < 7$ $\alpha_7 = \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8$ $\delta = -\varepsilon_7 + \varepsilon_8$	$\operatorname{Aut} R$
E_6	78	$\varepsilon_i - \varepsilon_j,\ \pm 2\varepsilon$ $\varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i < 6$ $\alpha_6 = \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon$ $\delta = 2\varepsilon$	$\operatorname{Aut} R =$ $W \times \{\pm 1\}$
F_4	52	$\pm\varepsilon_i \pm \varepsilon_j; \pm\varepsilon_i$ $(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2$	$\alpha_1 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$ $\alpha_2 = \varepsilon_4$ $\alpha_3 = \varepsilon_3 - \varepsilon_4$ $\alpha_4 = \varepsilon_2 - \varepsilon_3$ $\delta = \varepsilon_1 + \varepsilon_2$	$\operatorname{Aut} R$
G_2	14	$\varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i$	$\alpha_1 = -\varepsilon_2$ $\alpha_2 = \varepsilon_2 - \varepsilon_3$ $\delta = \varepsilon_1 - \varepsilon_3$	$\operatorname{Aut} R$

The following notations are used in this table. For F_4, ε_i, $(i = 1,2,3,4)$ is an orthonormal basis of the 4-dimensional Euclidean space E^4. For all other exceptional root systems of rank l, $\varepsilon_1, ..., \varepsilon_{l+1}$ is the standard basis of the Euclidean vector space $\mathbb{R}^{l+1}$ restricted to the hypersurface

$$E^l = \left\{\alpha = \sum x_i\varepsilon_i, \quad \sum_{i=1}^{l+1} x_i = 0\right\}.$$

In particular,

$$(\varepsilon_i, \varepsilon_i) = \frac{l}{l+1}, \qquad (\varepsilon_i, \varepsilon_j) = -\frac{1}{l+1}, \quad i \neq j.$$

For E_6, ε is the vector with $(\varepsilon, \varepsilon) = 1/2$, orthogonal to all vectors ε_i.

We indicate the dimension of the corresponding exceptional Lie algebra $\mathfrak{g}$, system of simple roots $\Pi = \{\alpha_1, ..., \alpha_l\}$ and the maximal root δ and give a description of the Weyl group W. For completeness, we also give a similar description for classical root system. Here we indicate also the root vectors.

Type	$\mathfrak{g}$	$\dim \mathfrak{g}$	$\mathfrak{h}$	Root vectors	Root system R	Π	W
A_l	$sl_{l+1}(\mathbb{C})$	$l(l+2)$	$\sum_1^{l+1} x_i e_i \otimes e_i^*$ $\sum x_i = 0$	$e_i \otimes e_j^*$ $i \neq j$	$\varepsilon_i - \varepsilon_j$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$	S_{l+1}
B_l	$so_{2l+1}(\mathbb{C})$	$l(2l+1)$	$\sum_{i=1}^{l} x_i e_i \wedge e_{-i}$	$e_i \wedge e_{-j}$ $e_i \wedge e_j$ $e_{-i} \wedge e_{-j}$ $e_{-i} \wedge e_j$	$\varepsilon_i - \varepsilon_j$ $\varepsilon_i + \varepsilon_j$ $-\varepsilon_i - \varepsilon_j$ $\pm \varepsilon_i$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $i = 1, ..., l-1$ $\alpha_l = \varepsilon_l$	$S_l \cdot \mathbb{Z}_2^l$
C_l	$sp_{2l+1}(\mathbb{C})$	$l(2l+1)$	$\sum_{i=1}^{l} x_i e_i \vee e_{-i}$	$e_i \vee e_{-j}$ $e_i \vee e_j$ $e_{-i} \vee e_j$	$\varepsilon_i - \varepsilon_j$ $\varepsilon_i + \varepsilon_j$ $-\varepsilon_i - \varepsilon_j$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $i = 1, ..., l-1$ $\alpha_l = 2\varepsilon_l$	$S_l \cdot \mathbb{Z}_2^l$
D_l	$so_{2l}(\mathbb{C})$	$l(2l-1)$	$\sum_{i=1}^{l} x_i e_i \wedge e_{-i}$	$e_i \wedge e_{-j}$ $e_i \wedge e_j$ $e_{-i} \wedge e_{-j}$	$\varepsilon_i - \varepsilon_j$ $\varepsilon_i + \varepsilon_j$ $-\varepsilon_i - \varepsilon_j$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $i = 1, ..., l-1$ $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$	$S_l \cdot \mathbb{Z}_2^{l-1}$

I.4. Real forms of a complex semisimple Lie algebra

Classification of real semisimple Lie algebras reduces to description of real forms of complex semisimple Lie algebras. Recall that any real Lie algebra $\mathfrak{k}$ can be naturally extended to a complex Lie algebra $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ which is called the complexification of $\mathfrak{k}$. The real subalgebra $=\mathfrak{k}$ of the Lie algebra $\mathfrak{g}$ is called a *real form* of the complex Lie algebra $\mathfrak{g}$. Remark that the complex conjugation of the space $\mathfrak{g}$ with respect to the real subspace $\mathfrak{k}$ is an antiholomorphic involutive automorphism σ of $\mathfrak{g}$ and the real form $\mathfrak{k}$ can be reconstructed as the fixed point set $\mathfrak{g}^\sigma$ of σ.

Any complex semisimple Lie algebra $\mathfrak{g}$ admits unique (up to a conjugation by an automorphism of $\mathfrak{g}$) compact real form. It is the fixed point set $\mathfrak{g}^\tau$ of the canonical antiholomorphic involutive automorphism τ of $\mathfrak{g}$, defined as follows.

Let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathbb{C} E_\alpha$$

be a Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{h}(\mathbb{R}) = B^{-1} < R >$ the real form of $\mathfrak{h}$. Then

$$\tau|\mathfrak{h}(\mathbb{R}) = -1, \quad \tau E_\alpha = E_{-\alpha}.$$

More precisely,

$$\mathfrak{g}^\tau = i\mathfrak{h}(\mathbb{R}) + \sum_{\alpha \in R^+} \mathrm{span}_{\mathbb{R}}(E_\alpha + E_{-\alpha}, i(E_\alpha - E_{-\alpha})).$$

I.5. Parabolic subalgebras of complex semisimple Lie algebra.

I.5.1. Regular subalgebra. Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathbb{C}E_\alpha$ be a Cartan decomposition of a complex semisimple Lie algebra $\mathfrak{g}$

Definition. 1) Subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is called to be regular if $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$.

2) Subset $Q \subset R$ is called to be closed (or a root subsystem) if $(Q + Q) \cap R \subset Q$, i.e., $\alpha, \beta \in Q,\ \alpha + \beta \in R \Rightarrow \alpha + \beta \in Q$.

3) Subset $Q \subset R$ is called to be symmetric (resp. asymmetric) iff $Q = -Q$ (resp., $Q \cap (-Q) = \emptyset$).

Any $Q \subset R$ can be decomposed into a disjoint sum $Q = Q^s \cup Q^a$, where $Q^s = Q \cap (-Q)$ (resp., $Q^a = Q \setminus Q^s$) are symmetric and asymmetric parts of Q.

LEMMA. *If Q is closed, then Q^s, Q^a are closed.*

PROPOSITION. *A closed set $Q \subset R$ defines a regular subalgebra*

$$\mathfrak{g}(Q) := \langle [E_\alpha, E_{-\alpha}] = H_\alpha, \alpha \in Q^s \rangle + \sum_{\alpha \in Q} \mathbb{C}E_\alpha = \mathfrak{h}_Q + \sum_{\alpha \in Q} \mathbb{C}E_\alpha.$$

Its Levi-Maltcev decomposition is $\mathfrak{g}(Q) = \mathfrak{g}(Q^s) \oplus \mathfrak{g}(Q^a)$.

Conversely, any regular Lie algebra $\mathfrak{k}$ of $\mathfrak{g}$ has the form $\mathfrak{k} = \mathfrak{h}' + \mathfrak{g}(Q)$, where $\mathfrak{h}'$ is a subalgebra of $\mathfrak{h}$ and Q is a closed subset of R.

Example. A system of positive (respectively, negative) roots R^+ (resp. R^-) is closed and asymmetric. The corresponding Lie algebras $\mathfrak{b}^\pm = \mathfrak{h} + \mathfrak{g}(R^\pm)$ are Borel subalgebras, that is, maximal solvable subalgebra of $\mathfrak{g}$. Any two Borel subalgebras are conjugated by an automorphism of $\mathfrak{g}$.

LEMMA. *A maximal asymmetric closed subset Q of R is the set of all positive roots with respect to some Weyl chamber.*

I.5.2. Parabolic subalgebras. A *parabolic subalgebra* $\mathfrak{p}$ is a subalgebra which contains a Borel subalgebra $\mathfrak{b}$.

Construction of parabolic subalgebras. Let Π is a basis of R, $R = R^+ \cup R^-$ is the corresponding decomposition of R and $\Pi_0 \subset \Pi$ is a subset. Define

$$[\Pi_0] = \langle \Pi_0 \rangle \cap R, \quad [\Pi_0]_\pm = \langle \Pi_0 \rangle \cap R_\pm.$$

Then $R' = R_{\Pi_0} = [\Pi_0]_- \cup R^+$ is a closed set. Moreover $R_0 = R^s_{\Pi_0} = [\Pi_0]$ and $R^a_{\Pi_0} = R^+ \setminus [\Pi_0]_+$. Hence,

$$\mathfrak{p}_{\Pi_0} = \mathfrak{h} + \mathfrak{g}\left(R_{\Pi_0}\right) = \mathfrak{z} + \mathfrak{g}\left([\Pi_0]\right) + \mathfrak{g}\left(R^a_{\Pi_0}\right)$$

is a parabolic subalgebra ($\mathfrak{p}_{\Pi_0} \supset \mathfrak{b}^+ = \mathfrak{h} + \mathfrak{g}(R^+)$) with the radical $\mathfrak{z} + \mathfrak{g}(R^a_{\Pi_a})$ and semisimple part $\mathfrak{g}\left([\Pi_0]\right)$. Here $\mathfrak{z}$ is the B-orthogonal complement to the Cartan subalgebra $\mathfrak{h}_0 = \langle H_\alpha, \alpha \in R_0 \rangle$ of $\mathfrak{g}\left([\Pi_0]\right)$ in $\mathfrak{h}$. We have

$$[\mathfrak{z}, \mathfrak{g}\left([\Pi_0]\right)] = 0, \quad \left[\mathfrak{z} + \mathfrak{g}([\Pi_0]), \mathfrak{g}\left(R^a_{\Pi_0}\right)\right] \subset \mathfrak{g}\left(R^a_{\Pi_0}\right).$$

Moreover, $\mathfrak{z}+\mathfrak{g}([\Pi_0])$ is a maximal reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{g}(R^a_{\Pi_0})$ is a regular nilpotent subalgebra of $\mathfrak{g}$.

PROPOSITION. *Any parabolic subalgebra is conjugated to a subalgebra of the form* $\mathfrak{p}_{\Pi_0}$.

Proof. The proof is immediate. Let $\mathfrak{k}$ be a parabolic subalgebra of $\mathfrak{g}$. Since any two Borel subalgebras are conjugated, we may assume that $\mathfrak{k}$ contains the Borel subalgebra $\mathfrak{b}^+ = \mathfrak{h} + \mathfrak{g}(R^+)$. Then $\mathfrak{k} = \mathfrak{h} + \mathfrak{g}(R')$, where $R' \supset R^+$ is a closed subset of R. This implies that $R' = R_{\Pi_0}$ for some subset Π_0 of Π.

Remark that any subset Π_0 of Π defines a decomposition of the Lie algebra $\mathfrak{g}$ into sum of three subalgebras

$$\mathfrak{g} = \mathfrak{g}(-R^a_{\Pi_0}) + \mathfrak{k} + \mathfrak{g}(R^a_{\Pi_0}),$$

where $\mathfrak{k} = \mathfrak{h} + \mathfrak{g}([\Pi_0]) = \mathfrak{z} + \mathfrak{g}([\Pi_0])$ is a reductive subalgebra and $\mathfrak{g}(\pm R^a_{\Pi_0})$ are nilpotent subalgebras. This decomposition is called generalized Gauss decomposition.

Remark also that the center $\mathfrak{z}$ of $\mathfrak{k}$ is always different from zero and $\mathfrak{k}$ coincides with the centralizer of $\mathfrak{z}$:

$$\mathfrak{k} = Z_{\mathfrak{g}}(\mathfrak{z}).$$

II. Homogeneous manifolds

A homogeneous manifold is a manifold M together with a transitive action of a Lie group G on M. Transitivity means that for any $x, y \in M$ there exist $g \in G$ such that $gx = y$. In other words, G has only one orbit on M. A homogeneous manifold M may be identified with the coset space G/K, where $K = \{g \in G, go = o\}$ is the stabilizer (or, stability subgroup) of a point $o \in M$. It is clear that K is closed, but not necessary connected subgroup of G. Conversely, any closed subgroup K of a Lie group G defines a homogeneous manifold G/K. There exist unique smooth structure on G/K such that the natural action of G on G/K is smooth.

Assume, for simplicity, that the homogeneous space $M = G/K$ is reductive, that is Lie subalgebra $\mathfrak{k} = \operatorname{Lie} K$ of $\mathfrak{g} = \operatorname{Lie} G$ admits Ad_K-invariant complement $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is an Ad_K-invariant decomposition. It is always the case if K is compact. Then we can identify $\mathfrak{m}$ with the tangent space T_oM by

$$\mathfrak{m} \ni X \leftrightarrow \frac{d}{dt}(\exp tX)\, o\Big|_{t=0},$$

where $\exp tX$ is the 1-parametric subgroup of G, generated by X.

Under this identification, the isotropy representation j of K is identified with the restriction of the adjoint representation $\operatorname{Ad}_K|_{\mathfrak{g}}$ to $\mathfrak{m}$:

$$j(K) = \operatorname{Ad} K|_{\mathfrak{m}}.$$

Many geometrical questions about homogeneous manifold M may be reformulated in terms of the pair (G, K) of Lie group and then in terms of the corresponding pair ($\mathfrak{g} = \operatorname{Lie} G, \mathfrak{k} = \operatorname{Lie} K$) of Lie algebras. For example, the classification of G-invariant tensor fields of given type (p, q) reduces to the description of (p, q) tensors of the tangent space T_oM, which are invariant under isotropy representation

$$j : K \to GL(T_oM)$$

of the stabilizer. Recall that j is defined by

$$j(k)\left(\frac{d}{dt}\gamma_t\bigg|_{t=0}\right) = \frac{d}{dt}(k\gamma_t)\bigg|_{t=0} \qquad k \in K,$$

where γ_t is a curve with $\gamma_0 = 0$.

Hence, the classification of G-invariant tensor fields of type (p, q) reduces to a description of the space $T^p_q(m)^{\operatorname{Ad} K}$ of $\operatorname{Ad} K$-invariant tensors of type (p, q) on $\mathfrak{m}$.

If the group K is connected (this is the case if G is connected and M is simply connected), then

$$T^p_q(\mathfrak{m})^{\operatorname{Ad}_K} = T^p_q(\mathfrak{m})^{\operatorname{ad}_\mathfrak{k}}, \quad \mathfrak{k} = \operatorname{Lie} K,$$

where $T^p_q(\mathfrak{m})^{\operatorname{ad}_\mathfrak{k}} = \{A \in T^p_q\mathfrak{m},\ \operatorname{ad}_X A = 0,\ \forall X \in \mathfrak{k}\}$, and the problem reduces to a description of tensors on $\mathfrak{m}$ invariant under the adjoint action of Lie algebra $\mathfrak{k}$.

For example, a homogeneous space $M = G/K$ admits G-invariant Riemannian metric g, almost complex structure J or almost symplectic structure ω iff the representation $\operatorname{Ad} K|\mathfrak{m}$ is orthogonal (i.e. preserves an Euclidean metric on $\mathfrak{m}$), complex (commute with a complex structure J) or, respectively, symplectic (preserves some nondegenerate skew-symmetric 2-form).

Remark that any linear representation of a compact Lie group K is orthogonal. Hence, any homogeneous manifold $M = G/K$ with compact subgroup K admits an invariant Riemannian metric.

It is not difficult to express the condition of integrability of an invariant almost complex structure J or an invariant almost symplectic structure ω on a homogeneous manifold $M = G/K$ in terms of Lie algebras $\mathfrak{g}, \mathfrak{k}$. Recall that an almost complex structure J (i.e. field of endomorphisms J with $J = -id$) is integrable iff the corresponding complex eigendistribution

$$T^{10}M = \{v \in T^{\mathbb{C}}M = TM \otimes \mathbb{C},\ Jv = iv\}$$

of J with eigenvalue $+i$ is involutive, i.e. the space of sections of $T^{10}M$ is closed under Lie bracket.

Let $M = G/K$ be a reductive homogeneous manifold with reductive decomposition $\mathfrak{g} = k + \mathfrak{m}$. Then we have natural identification $T^{\mathbb{C}}_0 M = \mathfrak{m}^{\mathbb{C}}$. Let J_M be an invariant almost complex structure on $M = G/K$ and $J = J_M|_0$ the corresponding complex structure on $T_0M = \mathfrak{m}$. Denote by $\mathfrak{m}^{10}$ and $\mathfrak{m}^{01}$ the eigenspaces of J on

$m^{\mathbb{C}}$ with eigenvalues $+i$ and $-i$, respectively. They are $\operatorname{Ad} K$-invariant and define invariant eigendistributions $T^{10}M$ and $T^{01}M$ of J_M. One can easity check that almost complex structure J_M is integrable iff $\mathfrak{g}_+ = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{10}$ is a complex subalgebra of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$.

Now we assume for simplicity, that G is connected compact semisimple Lie group and describe all homogeneous spaces G/K which admits an invariant symplectic structure ω (i.e. nondegenerate closed 2-form). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be a reductive decomposition. The value ω_o of ω at $o = eK$ is Ad_K-invariant nondegenerate 2-form on $m, \omega_o \in \left(\wedge^2 m^*\right)^{\operatorname{Ad}_K}$. It may be considered as 2-form ω_0 on $\mathfrak{g}$ with $\operatorname{Ker}\omega_0 = \mathfrak{k}$. The closedness of ω means that ω_o is closed in the complex $\wedge\mathfrak{g}^*$ of exterior forms on the Lie algebra $\mathfrak{g}$:

$$d\omega_0(X,Y,Z) = \operatorname{cycl}\omega_o([X,Y],Z) = 0, \quad X,Y,Z \in \mathfrak{g},$$

where cycl means the sum of cyclic permutations . It is known that $H^1(\mathfrak{g}, \mathbb{R}) = 0$ for semisimple Lie algebra $\mathfrak{g}$. In other words, any closed 2-form ω_o is exact: $\omega_0 = d\xi$, where $\xi \in \mathfrak{g}^*$ is 1-form and $d\xi(X,Y) = \xi([X,Y])$, $X, Y \in \mathfrak{g}$. This 1-form ξ is canonically defined. Using nondegeneracy of Killing form B we can write $\xi(X)$ as

$$\xi(X) = \mathrm{B}(h_o, X), \quad X \in \mathfrak{g},$$

where h_o is some element of Lie algebra $\mathfrak{g}$. Hence,

$$\omega_0(X,Y) = B(h_0, [X,Y]) = B([h_0, X], Y), \quad X, Y \in \mathfrak{g}$$

and

$$\mathfrak{k} = \ker\omega_o = Z_{\mathfrak{g}}(h_o) := \{X \in \mathfrak{g},\ [h_o, X] = 0\}.$$

This implies that K contains the centralizer $Z_G(h_o) = \{g \in G,\ \operatorname{Ad}_g h_o = h_o\}$ because this centralizer is connected. Since element $h_o \in \mathfrak{g}$ is canonically defined by ω_o, it is invariant under Ad_K. Thus $K \subset Z_G(h_o)$. Both inclusions imply equality $K = Z_G(h_o)$. We have

THEOREM (Kirillov-Kostant-Souriau). *Let $M = G/K$ be a homogeneous manifold of compact semisimple Lie group G with invariant symplectic structure ω. Then $K = Z_G(h_o)$ is the centralizer of some element $h_o \in \mathfrak{g} = \operatorname{Lie} G$ and hence, M can be identified with the adjoint orbit $M = (\operatorname{Ad} G)h_o$ of this element. Moreover, the value ω_h of the symplectic form ω at a point $h \in M = (\operatorname{Ad} G)h_o$ is given by $\omega_h(X,Y) = \mathrm{B}(h, [X,Y])$, $X, Y \in T_h M \subset \mathfrak{g}$.*

COROLLARY. *There exists 1-1 correspondence between invariant symplectic structures on a homogeneous manifold $M = G/K$ of a compact semisimple Lie group G and elements $h \in \mathfrak{g}$ with the stabilizer K in G.*

Such elements have the orbit isomorphic to G/K.

Homogeneous manifolds G/K of compact semisimple Lie group G which are G-diffeomorphic to adjoint orbits of G are called *flag manifolds.*

III. Flag manifolds as homogeneous spaces of a complex semisimple Lie group

III.1. Flag manifolds of classical groups $SL_n(\mathbb{C})$, $SO_n(\mathbb{C}), Sp_n(\mathbb{C})$. Now we describe the manifolds of complex flags and complex isotropic flags and identify them with coset spaces of classical Lie groups modulo parabolic subgroups.

Let $V = \mathbb{C}^n$ be a complex vector space and $\bar{p} = (p_1, ..., p_r)$ an r-tuple of natural numbers such that $0 < p_1 < ... < p_r < n$.

Definition. A flag of type $\bar{p}$, or $\bar{p}$-flag, is a system $f = (V_1, V_2, ..., V_r)$ of subspaces of V such that $V_i \subset V_{i+1}$ and $\dim V_i = p_i$.

The unimodular group $SL_n(\mathbb{C}) = \{A \in GL(\mathbb{C}), \det A = 1\}$ acts transitively on the set $\mathcal{F}_{\bar{p}}(V)$ of all $\bar{p}$-flags. Let $e_1, ..., e_i, ..., e_n$ be the standard basis of $V = \mathbb{C}^n$. We will denote by $\langle e_1, ..., e_k \rangle$ the complex subspace of V spanned by $e_1, ..., e_k$. The $\bar{p}$-flag

$$f_0 = (\langle e_1, ..., e_{p_1} \rangle, \langle e_1, ..., e_{p_2} \rangle, ..., \langle e_1, ..., e_{p_r} \rangle)$$

is called the standard $\bar{p}$-flag (or $\bar{p}$-flag associated with the standard basis). Its stabilizer $P_{\bar{p}}$ in $SL_n(\mathbb{C})$ consists of all upper block-triangular matrices with the blocks of size $p_1, p_2 - p_1, p_3 - p_2, \ldots$. Hence, we may identify $\mathcal{F}_{\bar{p}}(V)$ with the coset space $SL_n(\mathbb{C})/P_{\bar{p}}$.

A flag of type $(1, 2, ..., n-1)$ is called a full flag. Remark that the stabilizer $P_{12\ldots n-1}$ of the standard full flag f_0 associated with the standard basis is the subgroup B of all upper triangular matrices from $SL_n(\mathbb{C})$, that is a Borel subgroup of $SL_n(\mathbb{C})$. The stabilizer $P_{\bar{p}}$ of the standard flag of any type $\bar{p}$ contains B.

Now we describe the manifolds of isotropic flags which are homogeneous spaces of other classical Lie groups $SO_n(\mathbb{C})$, $Sp_n(\mathbb{C})$.

Assume that a nondegenerate bilinear form b in $V = \mathbb{C}^n$ is given, symmetric ($b = g$) or skew-symmetric ($b = \omega$). Last case is possible only if $n = 2l$. The classical groups $SO_n(\mathbb{C})$, $Sp_{n/2}(\mathbb{C})$ are defined as the connected component $\mathrm{Aut}(V, b)_0$ of the group $\mathrm{Aut}(V, b)$ of automorphisms of V, which preserve b. As in I.3.2, we identify the Lie algebras $so_g(V)$ and $sp_\omega(V)$ with the space $\wedge^2 V$ of bivectors and with the space $\vee^2(V)$ of symmetric 2-tensors.

We will fix a basis e_i, e_{-i} $i = 1, ..., l$ of V for $n = 2l$ and e_0, e_i, e_{-i}, $i = 1, ..., l$ for $n = 2l + 1$ such that $b(e_i, e_{-j}) = \delta_{ij}, b(e_0, e_0) = 0$ and all other products are zero. In such standard basis, the form b is given by

$$\begin{aligned} n = 2l: \quad & b = g = \sum e_{-i} \vee e_i, \quad b = \omega = \sum e_{-i} \wedge e_i \\ n = 2l+1: \quad & b = g = e_0 \otimes e_0 + \sum e_{-i} \vee e_i. \end{aligned}$$

Definition. A flag $f = (V_1, ..., V_r)$ of type $\bar{p} = (p_1, ..., p_r)$ in the space (V, b) is called isotropic if $b|_{V_r} = 0$.

In this case $r \leq l = \left[\frac{n}{2}\right]$. The group $\mathrm{Aut}(V, b)$ acts transitively on the manifold $I\mathcal{F}_{\bar{p}}(V)$ of isotropic $\bar{p}$-flags, $\bar{p} = (p_1, ..., p_r)$. Moreover, its connected component $\mathrm{Aut}(V, b)_0$ acts on $I\mathcal{F}_{\bar{p}}(V)$ transitively if $b = \omega$ or $b = g$ and $n = 2l + 1$, or $b = g$, $n = 2l$ and $p_r < l$. In the case $b = g$, $n = 2l$, $p_r = l$, the group $\mathrm{Aut}(V, g)_0 = SO_{2l}(\mathbb{C}) = D_l$ has two open orbits $SO_{2l}(\mathbb{C})f_0$, $SO_{2l}(\mathbb{C})f_1$, where f_0, f_1, are flags associated with the standard basis $e_1, ..., e_l, e_{-1}, ..., e_{-l}$ and the basis $e_{-1}, e_2, ..., e_l, e_1, e_{-2}, ..., e_{-l}$ on the manifold $I\mathcal{F}_{\bar{p}}(V)$ of flags of type $\bar{p} = (p_1, ..., p_r = l)$, see [G-O-V].

Denote by $f_0 = (\langle e_1, ..., e_{p_1}\rangle, \langle e_1, ..., e_{p_2}\rangle, \dots, \langle e_1, ..., e_{p_r}\rangle)$ the standard isotropic $\bar{p}$-flag in (V, b), associated with the standard basis $e_i, e_{-i}, (e_0)$.

Exercise. Describe the stability subalgebra $\mathfrak{p}_{f_0}$ of the point f_0. For example, for D_l-case ($b = g$, $n = 2l$) it is given by

$$\mathfrak{p}_{\bar{p}} = \langle e_i \wedge e_j,\ \forall i, j;\quad e_i \wedge e_{-j},\ \text{s.t. if } p_s < j < p_{s+1},\ \text{then } i < p_{s+j}\rangle.$$

III.2. Flag manifolds of a complex semisimple Lie group. Now we define flag manifolds associated with any connected complex semisimple Lie group G. Recall that a subgroup P of G is called parabolic if it contains a Borel subgroup (i.e. a maximal solvable subgroup) B of G.

Definition. A flag manifold of a complex semisimple Lie group G is the quotient $M = G/P$ of G by a parabolic subgroup P.

It is known that a parabolic subgroup P of G is always connected and coincides with its normalizer $N_G(P) = P$. This implies that a flag manifold $M = G/P$ is simply connected complex manifold. Moreover, $M = G/P$ can be realized as a closed (hence, compact) orbit of G in the projective space PV, associated with some linear representation $G \to GL(V)$ of G (more precisely, the orbit of the highest weight vector).

Flag manifolds of the classical Lie groups are exactly the manifolds of $\bar{p}$-flags or isotropic $\bar{p}$-flags, described above.

Lie algebra $\mathfrak{p}$ of a parabolic subgroup P is a parabolic subalgebra. Hence, we can write

$$\mathfrak{p} = \mathfrak{p}_{\Pi_0} = \mathfrak{k} + \mathfrak{g}\left(R^a_{\Pi_0}\right), \quad \mathfrak{k} = \mathfrak{z} + \mathfrak{g}(\langle \Pi_0\rangle), \quad R^a_{\Pi_0} = R^+ \setminus [\Pi_0]_+ .$$

The generalized Gauss decomposition

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{k} + \mathfrak{n}_+ = \mathfrak{g}\left(-R^a_{\Pi_0}\right) + \mathfrak{k} + \mathfrak{g}\left(R^a_{\Pi_0}\right)$$

induces a decomposition of some open subset G_{reg} of G into product of the corresponding subgroups:

$$G_{\mathrm{reg}} = N_- \cdot K \cdot N_+, \quad K \cap N_\pm = N_+ \cap N_- = \{e\},$$

where $\mathrm{Lie}\, K = \mathfrak{k}$, $\mathrm{Lie}\, N_{\pm} = \mathfrak{g}\left(\pm R^a_{\Pi_0}\right)$. Moreover, $P = K \cdot N_+$. This decomposition shows that the nilpotent subgroup N_- acts simply transitively on the open dense subset $M_{\mathrm{reg}} = G_{\mathrm{reg}}/P$ of the flag manifold $M = G/P$. Hence, any complex coordinates on N_- (for example, independent matrix elements of a matrix representation of N_-) define local complex coordinates on M. In the case of Grassmannian, such coordinates are standard local complex coordinates of the Grassmannian.

Remark that any chain of subsets

$$\Pi_0 = \emptyset \subset \Pi_1 \subset \Pi_2 \subset \ldots \subset \Pi_k = \Pi$$

of a system Π of simple roots of the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ defines a tower of holomorphic G-equivariant fiberings

$$G/B \to G/P_{\Pi_1} \to G/P_{\Pi_2} \to \ldots \to G/P_{\Pi} = \{\text{point}\},$$

where P_{Π_i} is the parabolic subgroup with tangent Lie algebra

$$\mathfrak{p}_{\Pi_i} = \mathfrak{h} + \mathfrak{g}\left(R_{\Pi_i}\right), \quad R_{\Pi_i} = [\Pi_i]_- \cup R^+ = [\Pi_i] \cup \left(R^+ \setminus [\Pi_i]_+\right).$$

In particular, if $\Pi_i = \Pi_{i-1} \cup \{\alpha_i\} = \{\alpha_1, \ldots, \alpha_i\}$ then we have the tower

$$G/B \to G/P_{\{\alpha_1\}} \to G/P_{\{\alpha_1,\alpha_2\}} \to \ldots \to G/P_{\{\alpha_1,\ldots,\alpha_i\}} \to \ldots \to G/G = \{\text{point}\}.$$

All fibers $P_{\{\alpha_1,\ldots,\alpha_i\}}/P_{\{\alpha_1,\ldots,\alpha_{i-1}\}}$ can be identified with a primitive flag manifold, i.e. the quotient of a simple Lie group G_i modulo a maximal parabolic subgroup P_i which corresponds to the Dynkin graph of G_i with one deleted root α_i.

III.3. The action of a maximal compact subgroup of G on a flag manifold $M = G/P$. Recall that a flag manifold $M = G/P$ of a connected complex semisimple Lie group G is simply connected and compact. Then, by Mostow theorem, a maximal compact subgroup G^τ of G acts on M transitively. From the exact sequence of homotopy groups of the fibration $G^\tau \to M = G^\tau/G^\tau_x$, one can derive easily that the stabilizer G^τ_x of a point $x \in M$ is a connected subgroup of G^τ. Hence, it is determined by the Lie algebra

$$\mathfrak{g}^\tau_x = \mathfrak{g}^\tau \cap \mathfrak{p} = \mathfrak{p}^\tau = \{x \in \mathfrak{p},\ \tau X = X\},$$

where $\mathfrak{g}^\tau = \mathrm{Lie}\, G^\tau$ is the Lie algebra of the maximal compact subgroup G^τ of G, that is the fixed point set of the standard antiholomorphic involution τ. We can write

$$\mathfrak{p} = \mathrm{Lie}\, P = \mathfrak{z} + \mathfrak{g}\left([\Pi_0]\right) + \mathfrak{g}\left(R^a_{\Pi_0}\right) = \mathfrak{k} + \mathfrak{g}\left(R^a_{\Pi_0}\right).$$

Since the subalgebra $\mathfrak{k} = \mathfrak{z} + \mathfrak{g}\left(R^a_{\Pi_0}\right)$ is invariant under the antiholomorphic involution τ and

$$\tau \mathfrak{g}\left(R^a_{\Pi_0}\right) = \mathfrak{g}\left(-R^a_{\Pi_0}\right),$$

we get

$$\mathfrak{g}_x^\tau = \mathfrak{p}^\tau = \mathfrak{k}^\tau = \mathfrak{z}^\tau + \mathfrak{g}\left(R_{\Pi_0}^a\right)^\tau .$$

Hence, the stability subalgebra $\mathfrak{g}_x^\tau$ is a compact form of the subalgebra $\mathfrak{k} = Z_\mathfrak{g}(\mathfrak{z})$. In particular, $\mathfrak{g}_x^\tau = Z_{\mathfrak{g}^\tau}(\mathfrak{z}^\tau)$ is the centralizer of a commutative subalgebra $\mathfrak{z}^\tau$ of $\mathfrak{g}^\tau$. This subalgebra $\mathfrak{z}$ generates a torus T in a compact connected semisimple Lie group G^τ. At is known that the centralizer of a torus in a connected compact semisimple Lie group is connected. This implies

$$G_x^\tau = Z_{G^\tau}(\mathfrak{z}^\tau) = Z_G(T) = Z_G(t_0),$$

where $t_0 \in \mathfrak{z}^\tau$ is an element generating a dense 1-parametric subgroup of the torus T. We get

PROPOSITION. *Any flag manifold $M = G/P$ can be identified with the quotient G^τ/K^τ of a semisimple compact Lie group $G^\tau \subset G$ modulo the centralizer $K^\tau = Z_{G^\tau}(t_0)$ of an element $t_0 \in \mathfrak{g}^\tau = \mathrm{Lie}\, G^\tau$ and, hence, with an adjoint orbit* $(\mathrm{Ad}_{G^\tau})t_0$.

IV. Flag manifolds as a quotient of a compact Lie group

IV.1. T-roots, T-chambers and T-Weyl group of a flag manifold. Let $M = G/P$ be a flag manifold of a complex semisimple Lie group G. We know that the Lie algebra $\mathfrak{p}$ of P has a semidirect decomposition

$$\mathfrak{p} = \mathfrak{p}_{\Pi_0} = \mathfrak{k} + \mathfrak{n} = (\mathfrak{z} + \mathfrak{g}([\Pi_0])) + \mathfrak{g}\left(R^+ \setminus [\Pi_0]_+\right).$$

into a sum of a reductive subalgebra $\mathfrak{k}$ and the maximal nilpotent ideal $\mathfrak{n}$. It defines a decomposition $P = K \cdot N$ of the Lie group P.

Denote by τ the standard antiholomorphic involution of G which determines a compact form G^τ of G. (G^τ is the fixed point set of τ). Then $P^\tau = K^\tau$ is a compact form of Lie group K and we can identify $M = G/P$ with G^τ/K^τ. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k} = \mathrm{Lie}\, K$. It is also a Cartan subalgebra of $\mathfrak{g} = LieG$. We will denote by R_0 (resp., R) the root system of $(\mathfrak{h}, \mathfrak{k})$ (resp., $(\mathfrak{h}, \mathfrak{g})$) and set $R' = R \setminus R_0$.

We can write

$$\mathfrak{k} = \mathfrak{h} + \sum_{\phi \in R_0} \mathbb{C}E_\phi = \mathfrak{z} + \mathfrak{g}(R_0),$$

where $\mathfrak{z} = Z(\mathfrak{k})$ is the center of $\mathfrak{k}$,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \qquad \mathfrak{m} = \sum_{\alpha \in R'} \mathbb{C}E_\alpha.$$

We will identify $\mathfrak{m}$ with the complexification $T_0^{\mathbb{C}}M$ of the tangent space T_0M of $M = G^\tau/K^\tau$ at the point $0 = eK^\tau$. Then $\mathfrak{m}^\tau = \{x \in M, \tau x = x\}$ is identified with T_0M and the isotropy representation of K^τ on $T_0^{\mathbb{C}}M$ is identified with the restriction to $\mathfrak{m}$ of the adjoint representation $\mathrm{Ad}_{K^\tau}|\mathfrak{g}$.

As before, we denote by $\mathfrak{h}(\mathbb{R})$ the real form of the Cartan subalgebra $\mathfrak{h}$, spanned by roots,

$$\mathfrak{h}(\mathbb{R}) = B^{-1}\langle R\rangle,$$

and we put $\mathfrak{t} = \mathfrak{z} \cap \mathfrak{h}(\mathbb{R})$. Then $\mathfrak{z}^\tau = i\mathfrak{t}$. Remark that roots from R are real linear forms on $\mathfrak{h}(\mathbb{R})$.

Definition. The restriction $\tilde{\alpha} = \alpha|_{\mathfrak{t}}$ of a root $\alpha \in R' = R \setminus R_0$ to $\mathfrak{t}$ is called a T-root.

We will denote the set of all T-roots by R_T. It is a finite set of nonzero vectors in the Euclidean vector space $\mathfrak{t}$. In general, R_T is not a root system.

Definition. 1) An element $t \in \mathfrak{t}$ is called to be regular, if any T-root $\tilde{\alpha}$ has nonzero value on $\mathfrak{t}$.

2) A connected component C of the set $\mathfrak{t}_{\mathrm{reg}} = \mathfrak{t} \setminus (\cup_{\tilde{\alpha}\in R_T}\{\tilde{\alpha} = 0\})$ of regular elements is called a T-chamber.

3) The subgroup $W^{R_0} = \{w \in W, wR_0 = R_0\}$ of the Weyl group W of R, which preserves R_0 is called T-Weyl group.

T-Weyl group W^{R_0} acts on the set of T-chambers, but the action is not transitive, in general.

IV.2. Realization of a flag manifold by an adjoint orbit. We say that an element $t_0 \in \mathfrak{g}^\tau$ defines a realization of the flag manifold G^τ/K^τ by an adjoint orbit if it has stabilizer K^τ in Ad_{G^τ}. Then we have canonical identification of the flag manifold G^τ/K^τ with the orbit $(\mathrm{Ad}_{G^\tau})\,t_0$ such that the point $o = eK^\tau$ is identified with t_0.

PROPOSITION. *1) An element $t_0 \in \mathfrak{g}^\tau$ defines a realization of the flag manifold $M = G^\tau/K^\tau$ iff $it_0 \in \mathfrak{t}_{\mathrm{reg}}$.*

2) Two elements t_0, t_1 with stabilizer K^τ belong to one G^τ-orbit if they belong to one W^{R_0} orbit:

$$t_1 = wt_0,\ w \in W^{R_0}$$

COROLLARY. *The set of adjoint orbits of G^τ isomorphic to $M = G^\tau/K^\tau$ is parametrized by elements of $\mathfrak{t}_{\mathrm{reg}}/W^{R_0}$.*

Proof of Proposition. 1) Let $t_0 \in \mathfrak{g}^\tau$ has stabilizer K^τ. Then $\mathfrak{k}^\tau = Z_{\mathfrak{g}^\tau}(t_0)$. Hence, t_o belongs to the center $\mathfrak{z}^\tau = Z(\mathfrak{k}^\tau) \subset \mathfrak{h}^\tau \subset \mathfrak{h}$ of $\mathfrak{k}^\tau$ and

$$R_0 = \{\alpha \in R,\ \alpha(t_0) = 0\} = R \cap \mathfrak{t}_0^\perp.$$

This shows that $it_0 \in t_{\mathrm{reg}}$.

2) Let t_0, t_1 be elements with the common stabilizer K^τ which belong to the same orbit. Then $t_1 = \mathrm{Ad}_g\, t_0$, for some $g \in N_{G^\tau}(K^\tau)$. The element Ad_g

preserves the center $\mathfrak{z}^\tau$ of $\mathfrak{k}^\tau$ and transforms a Cartan subalgebra $\mathfrak{h}^\tau = \mathfrak{z}^\tau + \mathfrak{h}_0$ of k^τ into a Cartan subalgebra $\mathfrak{h}' = \mathfrak{z}^\tau + \mathfrak{h}_1$, where $\mathfrak{h}_0, \mathfrak{h}_1$ are Cartan subalgebras of the semisimple part $[\mathfrak{k}^\tau, \mathfrak{k}^\tau]$ of $\mathfrak{k}^\tau$. Using the fact that any two Cartan subalgebras of $\mathfrak{k}^\tau$ are conjugated, we can change g to $g' = ga$, for some $a \in K^\tau$, such that $\mathrm{Ad}_{g'}$ preserves the Cartan subalgebra $\mathfrak{h}^\tau$ and its decomposition $\mathfrak{h}^\tau = \mathfrak{z}^\tau + \mathfrak{h}_0$. Then $\mathrm{Ad}_{g'} | \mathfrak{h}^\tau = w \in W^{R_0}$. On the other hand,

$$t_1 = \mathrm{Ad}_g\, t_0 = \mathrm{Ad}_{g'}\, t_0 = wt_0,$$

because $t_0, t_1 \in \mathfrak{z}^\tau$ and $\mathrm{Ad}_a | \mathfrak{z}^a = id$. □

IV.3. Closed invariant 2-forms on a flag manifold. We now state the folowing proposition.

PROPOSITION. *Let $M = G^\tau/K^\tau$ be a flag manifold and $\mathfrak{k}^\tau = i\mathfrak{t} + \mathfrak{g}(R_0)$ the decomposition of the Lie algebra of K^τ into sum of the center $i\mathfrak{t}$ and the semisimple ideal $\mathfrak{g}(R_0)$. There exist natural one-to-one correspondence between elements from $\mathfrak{t}$ and closed invariant 2-forms on M:*

$$\mathfrak{t} \ni t \Leftrightarrow \omega_t \in \Omega^2(M)^{G^\tau},$$

where ω_t is G^τ-invariant 2-form on M, whose value at $o = eK^\tau$ is given by

$$\omega_t|_0 = \frac{i}{2\pi} d(B \circ t) \in \wedge^2(\mathfrak{m}^\tau)^* = \wedge^2 T_0^* M.$$

Here $\mathfrak{m}^\tau$ is the $\mathrm{Ad}\, K^\tau$-invariant complement to $\mathfrak{k}^\tau$ in $\mathfrak{g}^\tau$, identified with the tangent space T_0M, B is the Killing form of $\mathfrak{h}^*$, and $d : \mathfrak{g}^* \to \wedge^2\mathfrak{h}^*$ is the exterior differential

$$d\xi(X,Y) = -\frac{1}{2}\xi([X,Y]) \quad X, Y \in \mathfrak{g},\ \xi \in \mathfrak{g}^*.$$

Moreover, symplectic (i.e., nondegenerated) 2-forms ω_t correspond to regular elements $t \in \mathfrak{t}_{\mathrm{reg}}$.

The proof is straightforward. Remark that a regular element $t_0 \in \mathfrak{t}_{\mathrm{reg}}$ defines an identification $M = G^\tau/K^\tau \leftrightarrow \mathrm{Ad}_{G^\tau}\, it_0$. The corresponding symplectic form ω_{t_0} on $M = \mathrm{Ad}_{G^\tau}\, it_0$ is exactly the Kirillov-Kostant-Souriau form of the orbit $\mathrm{Ad}_{G^\tau}\, t_0$:

$$\omega_{t_0}(X,Y) = B(t, [X,Y]), \quad X, Y \in T_tM \subset \mathfrak{g}^\tau.$$

IV.4. Decomposition of the isotropy representation into irreducible submodules. Now we fix a system of simple roots Π_0 of the root system R_0 and denote by R_0^+, R_0^- the corresponding systems of positive and negative roots.

Definition. A root $\alpha \in R' = R \setminus R_0$ is called K-simple, if $\alpha - \phi \notin R$ for any $\phi \in R_0^+$.

PROPOSITION. *There exist natural one-to-one correspondence between T-roots, irreducible Ad_{K^τ}-submodules of the complexified tangent space $\mathfrak{m} = \sum_{\alpha\in R'} \mathbb{C}E_\alpha = (\mathfrak{m}^\tau)^{\mathbb{C}}$ of the flag manifold G^τ/K^τ and the set of K-simple roots from $R' = R\setminus R_0$, given by*

$$R_T \ni \tilde{\alpha} \Longleftrightarrow \mathfrak{m}_{\tilde{\alpha}} = \sum_{\alpha\in R',\, \alpha|_{\mathfrak{t}}=\tilde{\alpha}} \mathbb{C}E_\alpha \Longleftrightarrow (\tilde{\alpha})_- \,,$$

where $(\tilde{\alpha})_-$ is the lowest weight of irreducible Ad_{K^τ}-module $\mathfrak{m}_{\tilde{\alpha}}$.

For proof see [Sieb], [Al-Per], [G-O-V].

IV.5. Invariant complex structures and Kahler metrics. A regular element $t_0 \in \mathfrak{t}$ defines a partial order in $\mathfrak{t}^*$. We say that a T-root $\tilde{\alpha}$ is *positive* (resp., *negative*) if $\tilde{\alpha}(t_0) > 0$ (resp., $\tilde{\alpha}(t_0) < 0$. A positive T-root $\tilde{\alpha}$ is called *simple* if it is not a sum of two positive T-roots. The set $\Pi_T = \{\tilde{\alpha}_1, ..., \tilde{\alpha}_m\}$of all simple T-roots is called a T-basis. It is a basis of $\mathfrak{t}^*$ such that any T-root $\tilde{\alpha}$ can be written as linear combination $\tilde{\alpha} = \sum k_i\tilde{\alpha}_i$ with integer coefficients k_i of the same sign (all $k_i \geq 0$ or all $k_i \leq 0$). The T-chamber $C(t_0)$ which contains t_0 is defined by inequalities

$$C(t_0) = \{\tilde{\alpha}_1 > 0, ..., \tilde{\alpha}_m > 0\}.$$

Denote by

$$\Pi' = \{\alpha_1 = (\tilde{\alpha}_1)_-, ..., \alpha_m = (\tilde{\alpha}_m)_-\}$$

the set of the corresponding K-simple roots. Then $\Pi = \Pi_0 \cup \Pi'$ is a system of simple roots of R. It contains Π_0 and depends only on T-chamber $C(t_0)$.

PROPOSITION. [Al-Per] *There exist natural one-to-one correspondence between*

1) *T-bases $\Pi_T = \{\tilde{\alpha}_1, ..., \tilde{\alpha}_m\}$;*
2) *T-chambers $C = \{\tilde{\alpha}_1 > 0, ..., \tilde{\alpha}_m > 0\}$;*
3) *systems $\Pi = \{(\tilde{\alpha}_1)_-, ..., (\tilde{\alpha}_m)_-\} \cup \Pi_0$ of simple roots of R which contain fixed system Π_0 of simple roots of R_0;*
4) *decomposition $R' = R'_+ \cup R'_-$ of $R' = R\setminus R_0$ into disjoint union of asymmetric closed subsets R'_+ and $R'_- = -R'_+$;*
5) *invariant complex structures on flag manifold G^τ/K^τ (defined up to a sign).*

Proof. Let $\Pi_T = \{\tilde{\alpha}_1, ..., \tilde{\alpha}_m\}$ be a T-basis. Then $C = \{\tilde{\alpha}_1 > 0, ..., \tilde{\alpha}_m > 0\}$ is the associated T-chamber. (The proof is the same as for ordinary Weyl chamber). Now we put $\Pi' = \{(\tilde{\alpha}_1)_-, ..., (\tilde{\alpha}_m)_-\}$. It follows from the properties of weights of an irreducible representation of semisimple Lie algebra, that $\Pi = \Pi_0 \cup \Pi'$ contains a basis of $\mathfrak{t}^*$. Using the formula for α-series of roots, one can prove that scalar product between different elements from Π is nonpositive. This implies that Π is a basis of R.

Denote by $R^+ = [\Pi]_+$, $R^- = [\Pi]_-$ the corresponding systems of positive and, respectively, negative roots and put $R'_\pm = R' \cap R^\pm = R^\pm \setminus [\Pi_0]_\pm$. Then it is clear that R'_+, $R'_- = -R'_+$ are closed sets of roots and $R' = R'_+ \cup R'_-$ is the disjoint sum. Now we define a decomposition of the complexified tangent space $T_0^{\mathbb{C}}M = \mathfrak{m} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$, where $\mathfrak{m}^{10} = \mathfrak{g}(R'_+)$, $\mathfrak{m}^{01} = \mathfrak{g}(R'_-)$.

Since the subspaces $\mathfrak{m}^{10}$, $\mathfrak{m}^{01}$ are Ad_{K^τ}-invariant, they can be extended to two complex invariant distributions $T^{10}M$ and $T^{01}M$. We define an invariant almost complex structure J on $M = G^\tau/K^\tau$ such that $T^{10}M$ and $T^{01}M$ are eigendistributions of J with eigenvalues $+i$ and $-i$, respectively. Since $\mathfrak{k} + \mathfrak{m}^{10}$ is a subalgebra of $\mathfrak{g}$, J is integrable, see section 2.

Conversely, any invariant complex structure J on $M = G^\tau/K^\tau$ defines a decomposition

$$\mathfrak{m} = \mathfrak{m}^{10} + \mathfrak{m}^{01}, \quad \tau\mathfrak{m}^{10} = \mathfrak{m}^{01},$$

of $\mathfrak{m}$ into a sum of two Ad_{K^τ}-invariant subspaces $\mathfrak{m}^{10}, \mathfrak{m}^{01}$ such that $\mathfrak{k} + \mathfrak{m}^{10}$ is a subalgebra. Denote by Q, resp., Q^- the set of roots associated with root vectors from $\mathfrak{m}^{10}$, resp., $\mathfrak{m}^{01}$. Then $Q^- = -Q$ and $R = R_0 \cup Q \cup (-Q)$. The closed set $R_0 \cup Q$ is parabolic in the sense of Bourbaki, and hence it contains some system of positive roots R^+, such that $R^+ \supset R_0^+$. Then it is clear that $Q \subset R^+$. Denote by Π the system of simple roots from R^+. Then $\Pi \supset \Pi_0$ and the restriction of roots from $\Pi' = \Pi \setminus \Pi_0$ to $\mathfrak{t}$ defines a T-basis. □

COROLLARY. *There exists natural one-to-one correspondence between regular points $t_0 \in \mathfrak{t}$ and invariant Kahler metrics g_{t_0} on flag manifold $M = G^\tau/K^\tau$. The metric g_{t_0} associated with point t_0 is given by $g_{t_0} = \omega_{t_0}(\cdot, J\cdot)$, where ω_{t_0} is the symplectic form associated with t_0 and J is the invariant complex structure, associated with T-chamber $C \ni t_0$.*

Remark. If we change J to complex structure J' associated with other T-chamber, then we get invariant pseudo-Kahler metric g, i.e. a pseudo-Riemannian metric with Levi-Civita connection ∇^g such that $\nabla^g J' = 0$.

IV.6. Invariant Kahler-Einstein metrics. Let $\Pi_0 = \{\phi_1, ..., \phi_k\}$ be fixed basis of the root system R_0 and $\Pi = \Pi_0 \cup \Pi'$, $\Pi' = \{\alpha_1, ..., \alpha_m\}$ is a basis of the root system R, which contains Π_0. We will denote by $\bar{\alpha}_1, ..., \bar{\alpha}_m$ the fundamental weights associated with simple roots $\alpha_1, ..., \alpha_m$ i.e.

$$\langle \bar{\alpha}_i | \alpha_j \rangle := \frac{2(\bar{\alpha}_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}, \quad \langle \bar{\alpha}_i | \phi_j \rangle = 0 \ \forall \phi_j \in \Pi_0.$$

The fundamental weights $\bar{\alpha}_i$ form a basis of the space $\mathfrak{t}^*$ dual to $\mathfrak{t}$ (and isomorphic to $\mathfrak{t}$ via Killing form). The natural isomorphism between $\mathfrak{t}^*$ and the space $\Omega^2(M)^{G^\tau}$ of invariant 2-forms on M is given by

$$\mathfrak{t}^* \ni \xi = \sum k_i \bar{\alpha}_i \leftrightarrow \omega_\xi = \frac{i}{2\pi} d\xi = \frac{i}{4\pi} \sum_{\alpha \in R'_+} \langle \xi | \alpha \rangle \omega^\alpha \wedge \omega^{-\alpha},$$

where ω_α are 1-forms on $\mathfrak{m}$, dual to $E_\alpha, \omega_\alpha(E_\beta) = \delta_{\alpha\beta}$. One can check that ω_ξ is Ad_{K^τ}-invariant 2-form on $\mathfrak{m}^\tau = T_0M$. Hence, it defines an invariant 2-form ω_ξ on $M = G^\tau/K^\tau$. Two-form ω_ξ is integer, i.e. defines an integer cohomological class, iff the coordinates k_i are integers. The 1-form

$$\sigma = \sum_{\alpha \in R'_+} \alpha$$

is called Koszul form. It is a linear combination of weights $\bar{\alpha}_i$ with integer positive coefficients n_i:

$$\sigma = \sum_{i=1}^{m} n_i \bar{\alpha}_i.$$

The numbers n_i are called Koszul numbers.

PROPOSITION. [Al-Per] *The 2-form ω_σ associated with Koszul form σ is the invariant nondegenerate closed 2-form which represents first Chern class of the invariant complex structure J, associated with (Π, Π_0). Moreover, the corresponding metric $g_\sigma = \omega_\sigma(\cdot, J\cdot)$ is an invariant Kahler–Einstein metric.*

IV.7. Painted Dynkin graphs and classification of flag manifolds. *Definition.* 1) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{k}$ a complex reductive Lie algebra which is isomorphic to centralizer of some element $t \in \mathfrak{g}$. The Dynkin graph Γ of the root system R of a $\mathfrak{g}$ with some vertices painted in black is called a painted Dynkin graph of type $(\mathfrak{g}, \mathfrak{k})$, if after deleting black vertices we get the Dynkin graph of the root system R_0 of $\mathfrak{k}$.

2) A bijection $\varepsilon : \Pi \to V(\Gamma)$ between roots of some basis Π of R and vertices $V(\Gamma)$ of Γ such that Γ becomes the Dynkin graph of Π, is called an equipment of the painted Dynkin graph Γ.

We will denote by Π_W and Π_B the subset of white and, respectively, black roots from Π (i.e., the roots associated with white and, respectively, black vertices of Γ). A painted Dynkin graph of type $(\mathfrak{g}, \mathfrak{k})$ with an equipment defines a flag manifold $M = G^\tau/K^\tau$, where G^τ is the compact form of the Lie group G with Lie algebra $\mathfrak{g}$ and K^τ is the subgroup of G^τ associated with the compact form $\mathfrak{k}^\tau$ of the Lie algebra $\mathfrak{k}(\Pi_W) = \mathfrak{h} + \mathfrak{g}([\Pi_W])$.

Remark that the Weyl group W of R acts on the set of equipments of a painted Dynkin graph Γ on the right and equipments ε, $\varepsilon \circ w$, $w \in W$ define G-isomorphic flag manifolds.

Assume that there exist equipments of Γ

$$\varepsilon : \Pi \to V(\Gamma), \quad \varepsilon' : \Pi' \to V(\Gamma),$$

which define nonisomorphic flag manifolds (that is the subalgebras $\mathfrak{k}(\Pi_W)$ and $\mathfrak{k}(\Pi'_W)$ generated by white roots are not conjugated). Changing ε' to $\varepsilon' \circ w$ for appropriate $w \in W$ we may assume, that $\Pi' = \Pi$. Then $\varepsilon' = \alpha \circ \varepsilon$, where α is an automorphism of the underline Dynkin graph $\bar{\Gamma}$ of Γ which is not an automorphism of the painted graph Γ (that is $\alpha\Pi_W \neq \Pi_W$). This is possible only when a connected component of $\bar{\Gamma}$ has type A_l, D_l or E_6. We have

PROPOSITION. *Let Γ be a painted Dynkin graph and $\bar{\Gamma}$ underlined Dynkin graph.*

1) If $\bar{\Gamma}$ has no nontrivial automorphism or any automorphism of $\bar{\Gamma}$ is an automorphism of Γ (i.e. it preserves the set of white vertices), then all equipments of Γ define isomorphic flag manifolds.

2) Let $\varepsilon : \Pi \to V(\Gamma)$ be an equipment of Γ which defines a flag manifold M and α is an automorphism of $\bar{\Gamma}$ which is not an automorphism of Γ. Then the equipment $\varepsilon' = \alpha \circ \varepsilon : \Pi \to V(\Gamma)$ defines a flag manifold nonisomorphic to M if the sets Π_W, Π'_W of white roots associated with ε and $\alpha \circ \varepsilon$ are not conjugated by an element of the Weyl group. Moreover, any flag manifold associated with an equipment of Γ can be obtained by this way.

Remark. It is not difficult to give a necessary and sufficient condition that the set Π_W, Π'_W of white roots associated with equipments ε, $\varepsilon' = \alpha \circ \varepsilon$ are not conjugated. For example, for $\bar{\Gamma} = A_l$ this condition is that not all white vertices of Γ are isolated.

Examples. 1) The equipments

$$\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$$

define nonisomorphic flag manifolds: projective space of lines and the dual projective space of hyperplanes.

2) The equipments

define isomorphic flag manifolds $\mathcal{F}_{12\ldots l-2}(\mathbb{C}^{l+1})$.

Definition. Two painted Dynkin graphs Γ, Γ' of the same type $(\mathfrak{g}, \mathfrak{k})$ are called to be equivalent if there exist equipments $\varepsilon : \Pi \to V(\Gamma)$, $\varepsilon' : \Pi' \to V(\Gamma')$ which define isomorphic flag manifolds.

LEMMA. *Two painted Dynkin graphs Γ, Γ' are equivalent iff there exist equipments $\varepsilon : \Pi \to V(\Gamma)$, $\varepsilon' : \Pi' \to V(\Gamma')$ with the same set Π_W of white roots.*

Proof. It is clear that equipments Γ, Γ' with the same set Π_W of white roots define the same stability subalgebra $\mathfrak{k} = \mathfrak{k}(\Pi_W) = \mathfrak{h} + \mathfrak{g}([\Pi_W])$ and hence the same flag manifold.

Conversely, if the equipments $\varepsilon, \varepsilon'$ define conjugated subalgebras $\mathfrak{k}(Pi_W)$, $\mathfrak{k}(\Pi'_W)$, then there exists an element n from the normalizer of the joint Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}(\Pi_W)$ and $\mathfrak{k}(\Pi'_W)$ which transforms Π_W onto Π'_W. Denote by w the corresponding element of the Weyl group $W = N_G(\mathfrak{h})/T$ where T the maximal torus of G with associated Lie algebra $\mathfrak{h}$. Then the equipments $\varepsilon', \varepsilon \circ w$ has the same set of white roots. □

To state the main result which gives a classification of flag manifolds of a given compact semisimple Lie group G^τ, we need the following definition.

Definition. Let G be a painted Dynkin graph of type $(D_l, \mathfrak{k})$ for some $\mathfrak{k}$. We say that Γ has class B (resp., WWB or WWW) if at least one of the right-end vertices α_l, α_{l-1} is black (resp., α_l, α_{l-1} are white, but α_{l-2} is black or, respectively, $\alpha_l, a_{l-1}, \alpha_{l-2}$ are white).

THEOREM. *Two connected painted Dynkin graphs Γ, Γ' are equivalent iff they have the same type $(\mathfrak{g}, \mathfrak{k})$ and, in the case $\mathfrak{g} = D_l$, the same class.*

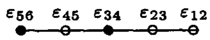
Proof. Due to Lemma, the proof reduces to the construction for any two connected painted Dynkin graphs Γ, Γ' of the same type and class equipments

$$\varepsilon : \Pi \to V(\Gamma), \quad \varepsilon' : \Pi \to V(\Gamma')$$

with the same set $\Pi_W = \Pi'_W$ of white roots. It was done by Nishiyama [Nish]. □

Examples. The painted graph Γ

corresponds to two nonisomorphic flag manifolds of the form $SU_5/T^2 \times SU_2 \times SU_3$. They are defined by the following equipments of Γ:



Painted Dynkin graphs equivalent to Γ are the following:

IV.8. Painted Dynkin graphs, T-chambers and Koszul numbers. Now we fix a flag manifold $M = G^\tau/K^\tau$ and a Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathbb{C}E_\alpha, \quad \mathfrak{k} = \mathfrak{h} + \sum_{\phi \in R_0} \mathbb{C}E_\phi$$

of the corresponding complex Lie algebras $\mathfrak{g}, \mathfrak{k}$. We fix also a basis Π_0 of the root system R_0. Let Γ be a painted Dynkin graph of type $(\mathfrak{g}, \mathfrak{k})$. We say that an equipment $\varepsilon : \Pi \to \Gamma$ is admissible if the white roots of Π form the set Π_0. Let

$$\Pi_B = \Pi \setminus \Pi_0 = \{\alpha_1, ...\alpha_m\}.$$

Denote by $\bar{\alpha}_1, ..., \bar{\alpha}_m$ the corresponding fundamental weights:

$$\langle \bar{\alpha}_i | \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}, \quad \langle \bar{\alpha}_i | \Pi_0 \rangle = 0.$$

PROPOSITION. *There exists natural one-to-one correspondence between T-chambers of a flag manifold $M = G^\tau / K^\tau$ and admissible equipments $\varepsilon : \Pi \to \Gamma$. The T-chamber associated with an equipment $\varepsilon : \Pi \to \Gamma$ is given by*

$$C = \Big\{ t = \sum_{i=1}^{m} x_i B^{-1} \bar{\alpha}_i, \ x_i > 0 \Big\},$$

where B is the Killing form and $\bar{\alpha}_i$ are fundamental weights associated to the black roots $\alpha_i \in \Pi \setminus \Pi_0 = \Pi_B$. The corresponding invariant complex structure J is defined by the decomposition of the complexified tangent space

$$T_0^{\mathbb{C}} M = \mathfrak{m} = \sum_{\alpha \in R' = R \setminus R_0} \mathbb{C} E_\alpha$$

given by $\mathfrak{m} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$, $\mathfrak{m}^{10} = \mathfrak{g}(R'_+)$, $\mathfrak{m}^{01} = \mathfrak{g}(R'_-)$, where $R'_\pm = R' \cap R^\pm$, $R^\pm = [\Pi]_\pm$. The Chern form of J is given by

$$\omega_\delta = \frac{i}{2\pi} \sum_{\alpha \in R'_+} \langle \sigma | \alpha \rangle \omega^\alpha \wedge \omega^{-\alpha},$$

where ω_σ are 1-forms on m, dual to E_α, $\omega_\alpha(E_\beta) = \delta_{\alpha\beta}$, and

$$\sigma = \sum_{i=1}^{m} n_i \bar{\alpha}_i = \sum_{\alpha \in R'_+} \alpha$$

is the Koszul form and n_i are some natural numbers (Koszul numbers).

If $\mathfrak{g}$ is one of the classical Lie algebras A_l, B_l, C_l, D_l, then the Koszul numbers can be calculated as follows [Al-Per]:

$n_i = 2 + b_i$, where b_i is the number of white roots which are connected with the black root α_i in the painted Dynkin graph Γ by a chain of white roots.

In the case $\mathfrak{g} = B_l$, the white long roots of the last right white chain are counted with multiplicity two and last short root is counted with multiplicity one.

In the case $\mathfrak{g} = C_l$, white roots of the last right white chain are counted with multiplicity two.

In the case $\mathfrak{g} = D_l$, the last white chain which define the root system D_k is considered as a chain of length $2(k-1)$. If one of the right end roots is black and the other is white, then the coefficient b_i near the black root is equal $2(k-1)$, where k is the number of white roots connected with the black root. The calculation of

Koszul numbers for flag manifolds of exceptional Lie groups is an open problem. We see that the Koszul numbers of an invariant complex structure do not depend on equipment of a painted Dynkin graph Γ, but only on Γ.

The following proposition gives an explanation of this phenomenon.

PROPOSITION. *Let $\varepsilon : \Pi \to \Gamma$, $\varepsilon' = \Pi' \to \Gamma'$ be admissible equipments of painted Dynkin graphs Γ, Γ' which define a flag manifold $M = G^\tau/K^\tau$. Denote by C, C' (resp., J, J') the associated T-chambers (resp., invariant complex structures on M). Then the following conditions are equivalent:*

i) the painted graphs Γ, Γ' are isomorphic;

ii) T-chambers C, C' are equivalent with respect to T-Weyl group W^{R_0}:

$$C' = wC \text{ for some } w \in W^{R_0};$$

iii) the complex structures J, J' are G-diffeomorphic, i.e. there exists a diffeomorphism f of M which commutes with G such that $f^*J = J'$.

Proof. Let Γ be a painted Dynkin graph and $\varepsilon : \Pi \to V(\Gamma)$, $\varepsilon' : \Pi' \to V(\Gamma')$ two admissible equipments of Γ which define a flag manifold $M = G^\tau/K^\tau$. Then, using case by case study of the root systems R for all simple Lie algebras $\mathfrak{g}$, one can check that there exists an element $w \in W^{R_0}$ such that $\Pi' = w\Pi$. Hence, admissible equipments $\varepsilon, \varepsilon'$ of Γ defines W^{R_0}-equivalent T-chambers C, C':

$$C' = wC.$$

Conversely, let $C, C' = wC$, $w \in W^{R_0}$ be two W^{R_0}-equivalent T-chambers. Denote by $\Pi = \Pi_0 \cup \Pi_B$, $\Pi' = \Pi_0 \cup \Pi'_B$ the corresponding systems of simple roots. Then

$$\Pi' = w\Pi, \quad \Pi'_B = w\Pi_B, \quad w\Pi_0 = \Pi_0.$$

This shows that the corresponding painted Dynkin graphs are isomorphic. It remains to show that W^{R_0}-equivalent T-chambers $C, C' = wC$ correspond to G-diffeomorphic invariant complex structures J, J' on $M = G^\tau/K^\tau$.

It is known that a G-diffeomorphism of M which commutes with G^τ is given by

$$n : gK^\tau \mapsto gK^\tau n = gnK^\tau, \ gK^\tau \in G^\tau/K^\tau,$$

for $n \in N_{G^\tau}(\mathfrak{k}^\tau)$. We may assume that Ad_n preserves the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. This means that the group $N_{G^\tau}(\mathfrak{k}^\tau)/K^\tau$ of G-diffeomorphisms of $M = G^\tau/K^\tau$ is identified with a subgroup of the T-Weyl group

$$W^{R_0} = \{A \in W = N_{G^\tau}(\mathfrak{h}^\tau)/T, AR_0 = R_0\}.$$

Now the equivalency of ii) and iii) follows from the fact that T-Weyl chamber C and invariant complex structure J determine each other. If $\Pi = \Pi_0 \cup \Pi_B$ is the system of simple roots, associated with C, then $\mathfrak{m}^{10} = \mathfrak{g}\left([\Pi_B]\right)$ is the i-eigenspace for the corresponding complex structure J. Conversely, if $\mathfrak{m}^{10}$ is i-eigenspace for an invariant complex structure J, then Π_B is the set of all simple roots from asymmetric closed system $R'_+ = \{\alpha \in R,\ E_\alpha \in \mathfrak{m}^{10}\}$ and $C = \{t \in \mathfrak{t},\ \alpha(t) > 0,\ \forall \alpha \in \Pi_B\}$. □

Remark. It is not difficult to prove that two invariant complex structures J, J' on a flag manifold $M = G^\tau/K^\tau$ are diffeomorphic iff the corresponding T-chamber C, C' are $\mathrm{Aut}(R, R_0)$-equivalent, where $\mathrm{Aut}(R, R_0)$ is the group of all automorphisms of R, which preserves R_0.

IV.9. Some examples. Now we illustrate the general theory by some examples. For simplicity, we will consider only flag manifolds of the group of type A_l.

Any painted Dynkin graph of type A_l is equivalent to a painted graph Γ of the following type:

Besides the left chaine of black vertices of length $m \geq 0$ (which may be absent), all other black vertices are isolated. The lengths of the chains of white vertices are $n_1 \geq n_2 \geq \ldots \geq n_k\rangle,\ k \geq 0$ and $l + 1 = \sum n_i + m$.

Such a graph Γ will be called standard. It defines two nonisomorphic flag manifolds of the type

$$M = SU_{l+1}/U(1)^m \times S\left(U(n_1) \times \ldots \times U(n_k)\right),$$

if Γ has no nontrivial automorphisms or not all white roots are isolated (i.e. $n_1 > 1$). In other cases, Γ define unique flag manifold M.

To describe all equipments of the described standard graph and the graph equivalent to its, it is useful to change the notation for the vectors of the standard basis of the space $\mathbb{R}^{l+1}$ as follows. We denote by

$$\delta_1, \ldots, \delta_m, \varepsilon_1^1, \ldots, \varepsilon_{n_1}^1, \ldots, \varepsilon_1^k, \ldots, \varepsilon_{n_k}^k$$

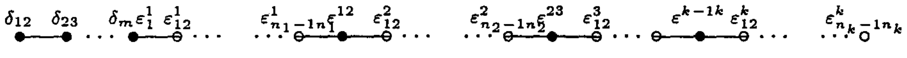
the vectors of the standard orthonormal basis of the space $\mathbb{R}^{l+1}$, restricted to the hyperplane

$$\mathfrak{h}(\mathbb{R}) = \left\{x \in \mathbb{R}^{l+1}, \sum x_i = 0\right\}.$$

Then the root system R of A_l is given by

$$R = \left\{\delta_{ij} = \delta_i - \delta_j, \pm\delta_i\varepsilon_j = \pm(\delta_i - \varepsilon_j^1),\ \varepsilon_{ij}^a = \varepsilon_i^a - \varepsilon_j^1,\ \varepsilon_{ij}^{ab} = \varepsilon_i^a - \varepsilon_j^b\right\}.$$

To simplify the notation, we also set $\varepsilon^{ab} = \varepsilon_{n_a}^a - \varepsilon_1^b$. We define the standard equipment of the standard painted Dynkin diagram Γ as follows.

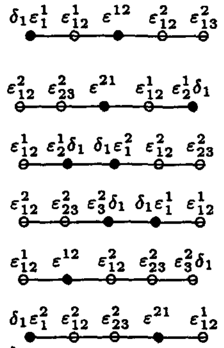


Now we describe admissible equipments of all painted Dynkin graphs, representing the flag manifolds of the types

$$M = SU_6/T^1 \times S\left(U_2 \times U_3\right) \quad \text{and} \quad M = SU_7/T^2 \times S\left(U_2 \times U_3\right).$$

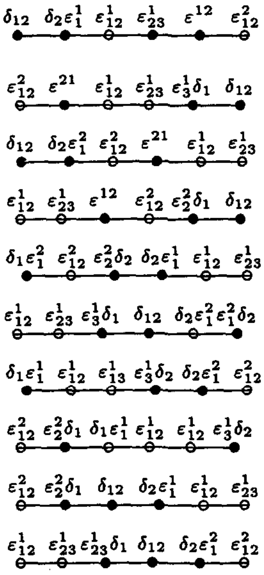
We consider only one of two such manifolds for each of the groups SU_6 and SU_7. The equipments associated with other manifold of such type can be obtained by symmetry of Dynkin graph.

Case of the manifold $M = SU_6/U^1 \times S\left(U_2 \times U_3\right)$; $\mathfrak{g} = A_5$, $\mathfrak{k} = \mathbb{C}^2 + A_1 + A_2$



Hence, on the flag manifold $M = SU_6/U^1 \times S\left(U_2 \times U_3\right)$ there exist 6 distinct invariant complex structures. They are mutually non-G-diffeomorphic, but only three of them are mutually not diffeomorphic.

Case of the manifold $M = SU_7/\left(U^1\right)^2 \times S\left(U_2 \times U_3\right)$; $\mathfrak{g} = A_6$, $\mathfrak{k} = \mathbb{C}^3 + A_1 + A_2$



Other 10 admissible equipments are obtained from described ones by transposition (δ_1, δ_2), which belongs to the T-Weyl group. Hence, we have 5 different painted Dynkin graphs, which corresponds to 5 mutually nondiffeomorphic invariant complex structures. Each of these graphs has 4 admissible equipments. These four equipments determine four different but diffeomorphic invariant complex structures, which is devided into two pairs of G-diffeomorphic complex structures. The G-diffeomorphic complex structures is associated with two equipment of the graph, which are related by the transposition (δ_1, δ_2).

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