Зборник радова Математичкоi института. Нова серија, књ. 3 (11), 1979.
Recueil des traraux de l' Institut Mathématique, Nouvelle série, tome 3 (11) 1979,

A CRITICAL REMARK ON USE OF DRAG COEFFICIENT AT LOW REYNOLDS NUMBERS

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(Communicated June 22, 1977)

Summary. A resistance coefficient is proposed after Lamb's pioneering treatment of the subject. The use of the Reynolds number is not convenient in the range between 0 and 1. The reciprocal value seems more appropriate, and in such a form it is proposed to be called the Oscen number. Thus all slow viscous flows are better represented through the Lamb coefficient and the Oseen number.

1. Introduction

Flow around various bodies at a very small Reynolds number is dominated by viscous forces. The inertia forces are very small and compared with the viscous forces they might be one or two order of magnitudes smaller. This physical fact allowed Stokes (1851) to neglect all inertia terms in the equations of motion and to evaluate the resistance experienced by a slowly moving sphere. The final expression for the resistance of a sphere was obtained in the following form:

$$R=3\pi DU\mu, \tag{1}$$

where D is diameter, U is free stream velocity and μ, the dynamic viscosity of fluid. In a similar way Lamb (1911) extended Stokes' solution to a circular cylinder by taking partially into account the inertia-terms after the manner of Oseen (1910). The final expression for the resistance of a circular cylinder of unit length was found as

$$R/L=\frac{4\pi\mu U}{1/2-\gamma-\log\left(\frac{Re}{8}\right)} \tag{2}$$

where L is length of cylinder, γ is Euler constant = 0.57721 and Re is the Reynolds number based on the diameter of the cylinder. Lamb gave two numerical examples for Re=0.4 and 0.2; the resistances per unit length were 4.31 μU,

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and $3.48\,\mu U$, respectively. The two numerical values correspond to a resistance coefficient defined in the following way:

$$C_R = \frac{R}{L\mu U} \tag{3}$$

Lamb's resistance coefficient is plotted versus the Reynolds number in Fig. 1. with linear scales. It is evident that as Re tends to 0 the C_R^1 also tends to zero.

2. Further development

The historical development did not follow the path indicated by Lamb and shown in Fig. 1. Wieselsberger (1921) was the first who collected all the data available on drag of a cylinder and reduced them to a drag coefficient. The latter was defined as:

$$C_D = \frac{R}{1/2\,\rho U^2 LD} \tag{4}$$

where ρ is the density of the fluid. Hence the drag coefficient is the ratio of the measured drag and the inertia-forces.

Fig. 2 was reproduced from the original Wieselsberger's (1921) plot. The drag coefficient showed small variations from $Re=10$ up to 2×10^5. Contrary to that for Reynolds numbers less than 10 there is a steep increase of the drag coefficient with the decrease of Reynolds number. For example for $Re=0.1$; $C_D=58$.

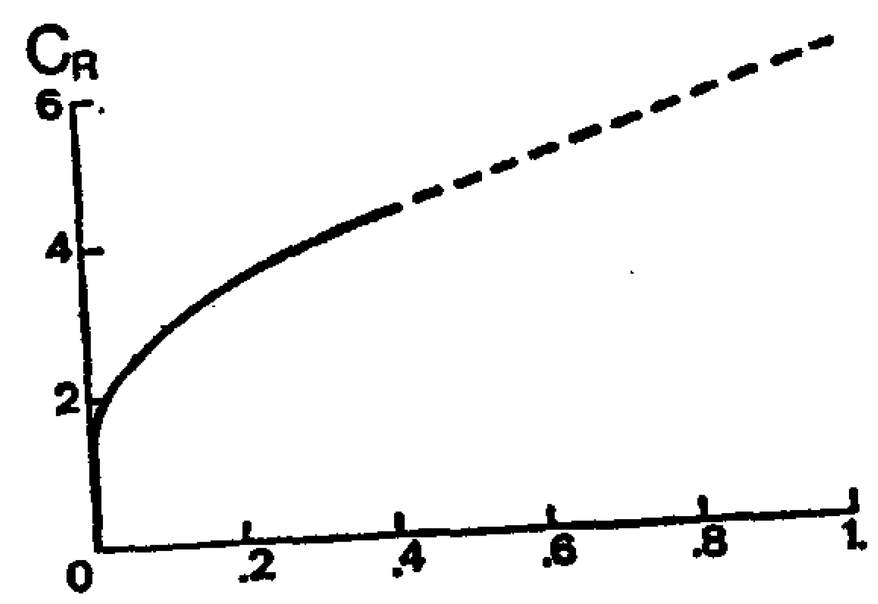

Fig. 1 Resistance coefficient versus Reynolds number.

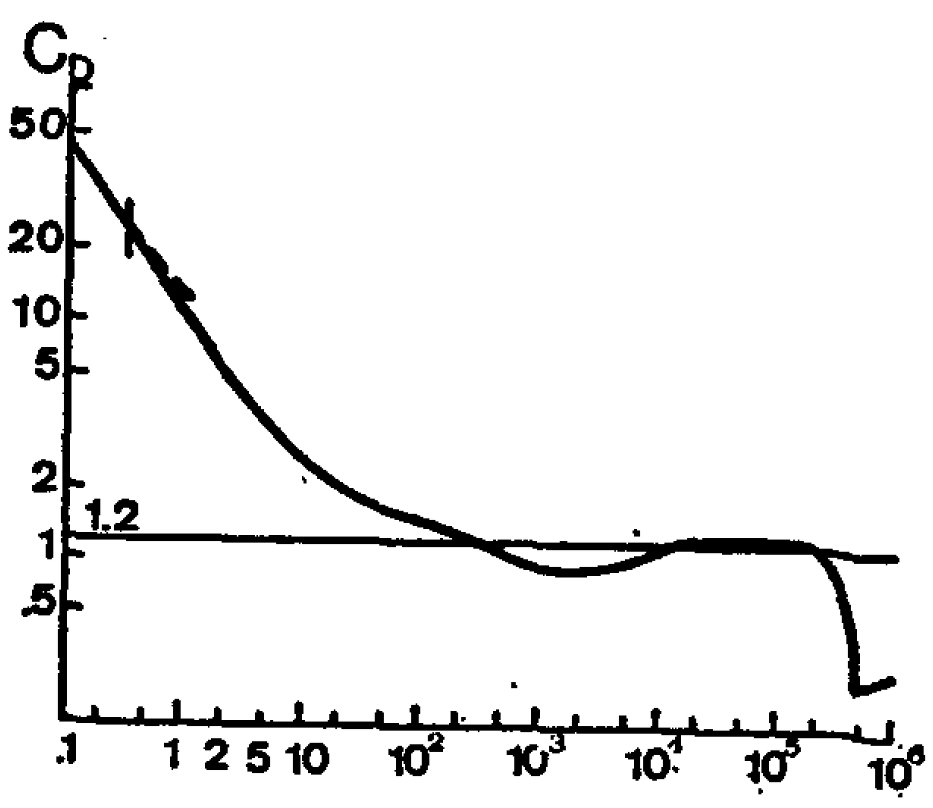

Fig. 2 Drag Coefficient for circular cylinders after Wieselsberger (1921).

This increase of drag coefficient is caused by the decrease of the inertia term; there is no longer proportionality between the drag and velocity square. Wieselsberger (1921) re-wrote Lamb's equation (2) and presented it in the following form:

$$C_D = \frac{8\pi}{Re\,(2.002 - \log Re)} \tag{5}$$

Another variation of Lamb's equation is given in Oseen's book (1927).

$$C_D = \frac{4\pi}{1/2 - \log 0.226\,Re} \tag{6}$$

It is interesting to point out that Oseen called this section: "§ 17.1 Die Lambsche Lösung" (Lamb's solution) but in the final expression (6) on page 179 it became as shown above Eq. (6). A comparison of all these expressions will be given soon.

The subsequent researchers measured drag at lower Reynolds numbers. Finn (1953) found $C_D=80$ for $Re=0.08$. Jones and Knudsen (1961) found $C_D=300$ for $Re=0.006$. Taneda (1964) measured in glycerine $C_D=500{,}000$ for $Re=0.000025$. Such enormous values gave a wrong impression that the resistance at low Reynolds number is also very large.

Contrary to this trend White (1946) presented his results through the resistance coefficient*. The original plot is reproduced in Fig. 3. He called Lamb's equation the following expression:

$$C_D = \frac{5.46}{\log_{10} \dfrac{7 \cdot 4}{Re}} \tag{7}$$

The remarkable achievement by using Lamb's resistance coefficient instead of drag coefficient was that all his experimental data shown down to $Re=0.00008$ were confined between $2<C_R<8$. There is a strong effect of wall blockage even when the cylinder diameter is 500 times smaller than the tank width, see Fig. 3.

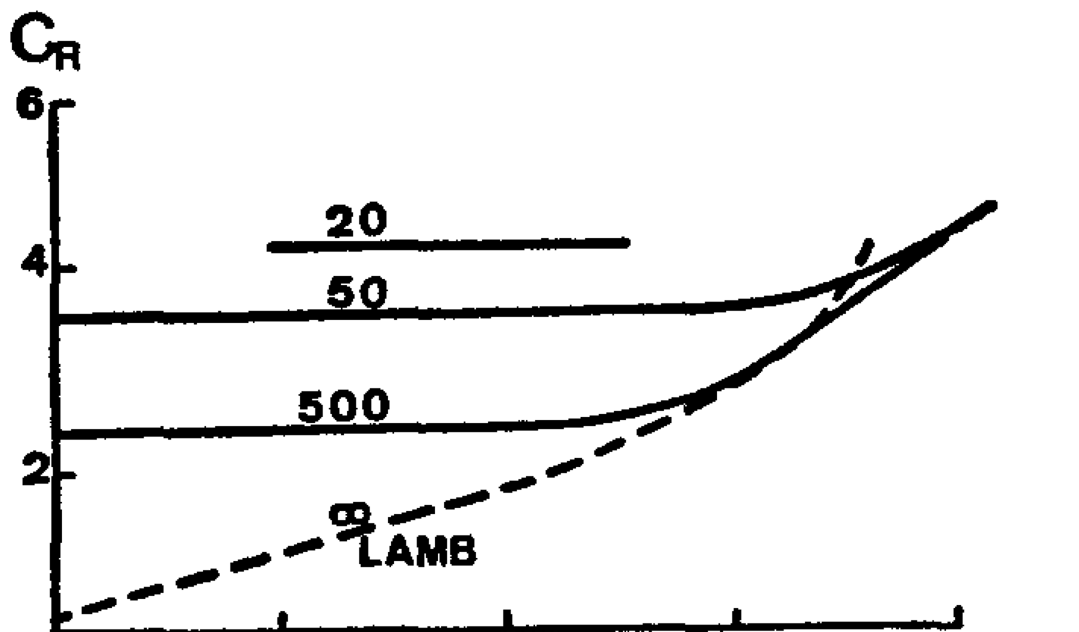

Fig. 3 Resistance coefficient for various rations of tank to cylinder diameter, after White (1946)

3. Check of various drag formulae

It is worthwhile to check the accuracy of various approximations of the original Lamb's equation (2). Table 1 shows the comparison of resistance coefficients and drag coefficients for various Reynolds number between 1 and 0.00001. It is evident that White's approximation (7) is in very good agreement, Oseen's equation (6) is in agreement only for very low Reynolds numbers, whilst Wieselsberger's expression (5) is unsatisfactory in the whole range except for $Re=1$. Hence it seems that only White's approximation is recommendable for usage.

* White called it drag coefficient but denoted it with α.

4. Resistance coefficent versus Oseen number

The Lamb resistance coefficient as presented in Fig. 1 has one disadvantage, namely the Reynolds number range is not spread. In microbiology, for example it is likely that for the small microbs and their flagellas the Reynolds number may be lower than 10^{-6}. In order to allow for extremely low Reynolds numbers it is better to plot Lamb's resistance coefficient, versus the reciprocal value of Reynolds number. This has been done and the result is shown in Fig. 4. The upper Reynolds number range is limited by 1, as it should be due to the approximations used to obtain equation (2). The right-hand side is now unlimited to accommodate for all possible sizes of micro organisms. The reciprocal value of Reynolds number was denoted by Os after Professor Oseen from Upsala, Sweden, who extended Stokes' solution for a sphere in 1910. Lamb applied that method to a circular cylinder in 1911. So it seems that both names are appropriate to mark this subject for future.

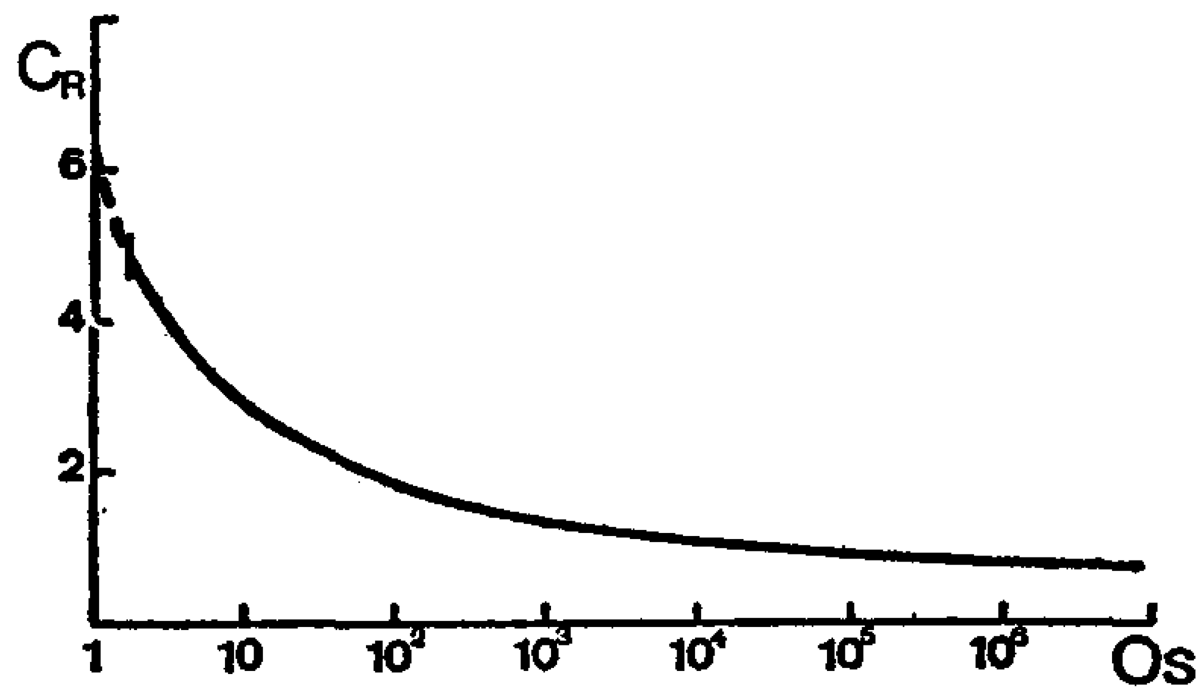

Fig. 4 Resistance coefficient versus Oseen number.

Acknowledgement. The author would like to mention a stimulating influence made by Professor Lighthill's recent lecture on Hydrodynamics of Flagella given at the Imperial College which led to a second thought on the drag coefficient.

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Table 1. Resistance and drag coefficients

	Re	1	0.1	.01	.001	.0001	.00001
	O_S	1	10	100	1000	10000	100000
C_R	(Eq. (2)	6.26	2.91	1.89	1.41	1.12	0.93
	(Eq. (4)	6.26	41.7	4.81	2.55	1.73	1.31
	(Eq. (6)	12.70	3.81	2.24	1.58	1.23	1.00
	(Eq. (7)	6.28	2.92	1.90	1.41	1.12	0.93
C_D	(Eq. (2)	12.52	58.2	378	2820	22400	186000