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APPLICATION OF ANALYTICAL COMPLEX FUNCTIONS FOR THE INVESTIGATION OF SOME CLASSES OF VORTEX FLOWS

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Regardless of the possibilities of modern electronic computing machines, it is still of considerable interest to find exact solutions of the equations of motion of fluids, although they can be found by means of some more severe limitations. The flows best investigated so far are, most certainly, the potential flows of an incompressible fluid, which, on the other hand, cannot be arrived at because of the viscosity existing in each real fluid. These flows have been best studied thanks partly to the properties of the analytical complex functions, i.e. functions the real and complex parts of which satisfy the Cauchy-Riemann equations. In the papers cited a method is mentioned that was established for the study of vortex flows of the real fluid, this method being based on the application of nonanalytical complex functions, that is complex functions depending upon the complex variable $x+iy$ and its conjugated value $x-iy$. Some new solutions were obtained, while in some already known solutions a physical interpretation was attributed to certain constants.

In this paper the following question was put forward : do there exist some vortex flows, for the investigation of which the analytical complex functions could be used, i.e. complex functions that are exclusively related to the investigation of potential flows of an incompressible fluid? The answer is positive: there exist such functions. One of such classes of vortex flows was related to the analytical complex functions by Konstantin Voronjec, while certain results in this connection were obtained by the author of the present paper. These results were not published so far, and their publication is being undertaken at the present occasion as yet an other appreciation of our great teacher, Professor Konstantin Petrović Voronjec.

If a stationary pseudo two dimensional flow ($v_z=0$) of an incompressible fluid ($\operatorname{div} \vec{v}=0$) is considered, provided $\vec{v} \parallel \operatorname{rot} \vec{v}$, then the dynamical equations are automatically satisfied, and the energy of the fluid flow remains constant in the flow space in terms of time. For the investigation of the kinematics of such flows the following equations are available:

$$
\begin{aligned}
&v_y \left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)=0, \quad v_x \left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)=0, \\
&v_x \frac{\partial v_x}{\partial z}+v_y \frac{\partial v_y}{\partial z}=0, \quad \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0.
\end{aligned} \tag{1}
$$

The third equation of this system shows that the velocity square does not depend on z, because that equation can be rewritten in the form of $\frac{\partial}{\partial z}(v_x{}^2+v_y{}^2)=0$, while the remaining three equations are satisfied by the functions $\varphi=\varphi(x, y, z)$ and $\psi=\psi(x, y, z)$ by means of the relations

$$
v_x=\frac{\partial \varphi}{\partial x}=\frac{\partial \psi}{\partial y}, \qquad v_y=\frac{\partial \varphi}{\partial y}=-\frac{\partial \psi}{\partial x}. \tag{2}
$$

It should be noted that the functions φ and ψ represent neither the velocity potential nor the flow function. In fact, ψ represents flow surfaces, which define flow lines at their intersections with planes z=const. On the other hand, φ represents families of surfaces that are perpendicular to the surfaces ψ=const. Provided the variable z is attributed a role of a parameter, then on the grounds of equations (2), it is possible to conclude that

$$
w(x+iy;\ z)=\varphi(x, y;\ z)+i\psi(x, y;\ z)
$$

is an analytical complex function, and that even its derivative

$$
\frac{dw}{d(x+iy)}=ve^{-i\vartheta}=\bar{v}, \tag{3}
$$

is also an analytical complex function (where v is the velocity magnitude, ϑ the angle between this velocity vector and the x-axis). Also, $\ln \bar{v}=\ln v-i\vartheta$ is an analytical complex function that satisfies the Cauchy-Riemann equations:

$$
\frac{\partial(\ln v)}{\partial x}=\frac{\partial(-\vartheta)}{\partial y}, \qquad \frac{\partial(\ln v)}{\partial y}=-\frac{\partial(-\vartheta)}{\partial x}. \tag{4}
$$

The left hand sides of the equations (4) do not depend upon z, since the velocity square does depend on x, y only. But, because of the conditions (2), the projections v_x and v_y of the velocity v depend on z, this requiring that $\vartheta=\vartheta(x, y, z)$. Because of (4), this function must have the following form:

$$
\vartheta(x, y, z)=\vartheta_1(x, y)+\vartheta_2(z).
$$

Thus, from (3) one obtains

$$w(x, y; z) = e^{-i\vartheta_2} \int v e^{-i\vartheta_1} d(x+iy),$$

since the function w is determined to the additive constant accuracy. The result of the integration is

$$e^{-i\vartheta_2} [f_1(x, y) + i f_2(x, y)] = \varphi + i\psi, \tag{5}$$

or

$$\varphi(x, y, z) = f_1(x, y) \cos \vartheta_2(z) + f_2(x, y) \sin \vartheta_2(z),$$

$$\psi(x, y; z) = -f_1(x, y) \sin \vartheta_2(z) + f_2(x, y) \cos \vartheta_2(z).$$

The equations (2) show that the functions $f_1(x,y)$ and $f_2(x,y)$ satisfy the Cauchy--Riemann conditions:

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}. \tag{6}$$

Therefore, any potential flow that is determined by the complex potential $f_1(x, y) + i f_2(x, y)$, can be used to formulate these classes of the pseudo-plain vortex flows of an incompressible fluid. In literature, these flows are called helicoidal flows. The magnitude of the vortex is

$$2\vec{\omega} = \operatorname{rot} \vec{v} = -\vec{v} \frac{d\vartheta_2}{dz}.$$

It is quite natural to raise the following question: can a real fluid ($\nu \neq 0$, where ν is the kinematic viscosity) flow also under these conditions? This is, in fact, reduced to the question: can the velocity projections

$$v_x = \frac{\partial f_1}{\partial x} \cos \vartheta_2(z) + \frac{\partial f_2}{\partial x} \sin \vartheta_2(z),$$

$$v_y = \frac{\partial f_1}{\partial y} \cos \vartheta_2(z) + \frac{\partial f_2}{\partial y} \sin \vartheta_2(z),$$

satisfy the dynamic equations for a stationary flow of a viscous incompressible fluid:

$$\frac{\partial}{\partial x}\left(\frac{v^2}{2} + \frac{p}{\rho} - U\right) = \nu \Delta v_x,$$

$$\frac{\partial}{\partial y}\left(\frac{v^2}{2} + \frac{p}{\rho} - U\right) = \nu \Delta v_y, \tag{7}$$

$$\frac{\partial}{\partial z}\left(\frac{v^2}{2} + \frac{p}{\rho} - U\right) = \nu \Delta v_z.$$

Here, U is the potential of external forces, p is the pressure and ρ the fluid density. Since $v_z=0$, the last equation of the system (7) shows that in this type of flow, the total energy must not depend on z. The functions f_1 and f_2 are harmonic ones, and, therefore,

$$\Delta v_x=\frac{\partial f_1}{\partial x}\frac{d^2\cos\vartheta_2(z)}{dz^2}+\frac{\partial f_2}{\partial x}\frac{d^2\sin\vartheta_2(z)}{dz^2},$$

$$\Delta v_y=\frac{\partial f_1}{\partial y}\frac{d^2\cos\vartheta_2(z)}{dz^2}+\frac{\partial f_2}{\partial y}\frac{d^2\sin\vartheta_2(z)}{dz^2},$$

and, thus, the solution of the first equations of the system (7) is as follows:

$$\frac{v^2}{2}+\frac{p}{\rho}-U=\nu\left[f_1(x, y)\frac{d^2\cos\vartheta_2(z)}{dz^2}+f_2(x, y)\frac{d^2\sin\vartheta_2(z)}{dz^2}\right]+C.$$

Since, the left-hand side of this system depends on x and y only, we must have

$$\frac{d^2\cos\vartheta_2(z)}{dz^2}=C_1,\quad \frac{d^2\sin\vartheta_2(z)}{dz^2}=C_2.$$

The solutions of these equations are:

$$\cos\vartheta_2(z)=\frac{1}{2}C_1z^2+C_3z+C_5,$$

$$\sin\vartheta_2(z)=\frac{1}{2}C_2z^2+C_4z+C_6.$$

Thus, the problem is solved formally. However, the condition

$$\cos^2\vartheta_2(z)+\sin^2\vartheta_2(z)=1$$

cannot be satisfied since the constants $C_1, C_2, \ldots$ are real. Therefore, the pseudo-plain stationary flow of a viscous incompressible fluid, in which the velocity vector would be colinear with the vorticity rot $\vec{v}$, cannot be established without some special forces.

Finally, yet another class of vortex flows is obtainable by means of the analytical complex functions. For example, let a nonviscous fluid ($\nu=0$) be considered. Then, the compatibility condition for the dynamic equations $(\vec{v}$, grad $2\omega)=0$ indicates that $2\omega=2\omega(\psi)$. By a well known method it is possible to set lines that are orthogonal to the lines ψ. Now, in fact, we have that $\lambda\vec{v}=$grad φ, where $\lambda(x, y)$, for the time being is an arbitrary function of the variables x, y. The last expression together with the continuity equation lead to the following relationships:

$$v_x=\frac{1}{\lambda}\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y},\quad v_y=\frac{1}{\lambda}\frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x},$$

whence it follows that

$$\Delta\varphi=\frac{1}{\lambda}\,(\operatorname{grad}\lambda,\ \operatorname{grad}\varphi)=\frac{\partial(\lambda,\psi)}{\partial(x,y)},$$

$$\Delta\psi=\frac{1}{\lambda}\,(\operatorname{grad}\lambda,\ \operatorname{grad}\psi)=\frac{1}{\lambda^2}\,\frac{\partial(\lambda,\varphi)}{\partial(x,y)}.$$

It is noted that $2\,\omega=-\Delta\,\psi=0$, when $\lambda=\lambda(\varphi)$. Thus, for the case of a vortex flow λ must depend on φ and ψ, or on ψ, only. It will be assumed that $\lambda=f'(\psi)$. Then, $\Delta\,\varphi=0$, and

$$\frac{\partial\varphi}{\partial x}=\frac{\partial f}{\partial y},\quad \frac{\partial\varphi}{\partial y}=-\frac{\partial f}{\partial x}.$$

This means that $\varphi+if$ is an analytical complex function.
Since,

$$\frac{\partial f}{\partial x}=f'(\psi)\,\frac{\partial\psi}{\partial x},\quad \frac{\partial f}{\partial y}=f'(\psi)\,\frac{\partial\psi}{\partial y},$$

there follows that

$$\Delta\,\psi+\frac{f''(\psi)}{f'(\psi)}\,v^2=0.$$

Since $\Delta\,\psi=-2\,\omega\,(\psi)$, it also follows that $v^2=v^2(\psi)$. It can be readily shown that in this case

$$2\,\omega\,(\psi)=-\frac{d}{d\psi}\left(\frac{v^2}{2}+\frac{p}{\rho}-U\right).$$

Finally, for the determination of the function $\varphi+if$, we shall use the plain of the complex velocity

$$\lambda\,\bar{v}=\lambda\,v\,e^{-i\vartheta}.$$

From this equation we have

$$\ln(\lambda\,\bar{v})=\ln(\lambda\,v)-i\,\vartheta,$$

and, also,

$$\frac{\partial(\ln\lambda\,v)}{\partial x}=\frac{\partial(-\vartheta)}{\partial y},\quad \frac{\partial(\ln\lambda\,v)}{\partial y}=-\frac{\partial(-\vartheta)}{\partial x}.$$

Since both λ and v depend only on ψ, it is convenient to consider φ and ψ as variables, that is to operate with φ and $f(\psi)$. It that case, the Cauchy-Riemann equations lead to the following expressions:

$$-\frac{d(\ln\lambda\,v)}{df}=-\frac{\partial\vartheta}{\partial\varphi}-\frac{v_x}{v_y}\,\frac{\partial\vartheta}{\partial f},$$

$$\frac{d(\ln\lambda\,v)}{df}=\frac{\partial\vartheta}{\partial\varphi}-\frac{v_y}{v_x}\,\frac{\partial\vartheta}{\partial f}.$$

The summation of these expressions will yield

$$-\frac{\partial \vartheta}{\partial f}\left(\frac{v_x}{v_y}+\frac{v_y}{v_x}\right)=0,$$

whence it follows that ϑ does not depend on f, i.e. that $\vartheta = \vartheta(\varphi)$. Thus, both preceding equations are reduced to a single one,

$$\frac{d(\ln \lambda v)}{df}=\frac{d\vartheta}{d\varphi}=\text{const}=k,$$

since the left-hand side of the last equation depends on ψ, while the right-hand side depends on φ. Therefore, the solutions are as follows:

$$\ln(\lambda v)=kf, \quad \vartheta = k\varphi,$$

and, we have

$$\frac{d(\varphi+if)}{dz}=e^{-ik(\varphi+i\psi)},$$

or, finally,

$$\varphi+if=-\frac{i}{k}\ln z.$$

By separating this function into its real and imaginary part, one can obtain the functions φ and f. Each function $\psi(f)$ determines a flow field that satisfies all conditions set.

References

[1] Šašić, M., *The application of nonanalytic complex functions and monogen quaternions in fluid mechanics*, Ph. D. Thesis, Belgrade 1966.

[2] Voronjec, K. and Šašić, M., *Sur quelques applications des fonctions non analytiques dans le mouvement plan d'un fluide incompressible*, Publ. Inst. Math. t. 7 (21), Beograd 1967.

[3] Šašić, M., *O jednoj metodi za rešavanje Oseen-ovih jednačina*. Mat. vesnik, 4(19), sveska 1, Beograd 1967.

[4] Šašić, M., *Jedan slučaj vrtložnog strujanja savršenog nestišljivog fluida*, Zbornik radova sa X Jug. kongresa za racionalnu i primenjenu mehaniku, Baško Polje 1970.