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ON SOME ASPECTS OF BUBBLE GROWTH RATES;

Effect of Bubble Shape

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Abstract. This work introduces a new method for evaluating bubble growth rates for heat diffusion controlled growth in a uniformly superheated liquid. The method is applicable to growth problems involving bubbles of arbitrary shape. Based on a simple physical model, an explicit solution for a general case of time varying shape is presented. For a special case of a spherical bubble, the model yields the known asymptotic solution.

Nomenclature

A, bubble dome area
b, a parameter defined by equation (12)
B, a parameter defined by equation (12)
C_p specific heat of liquid
h_{fg} latent heat of evaporation
J_a, Jakobi number
k, thermal conductivity
l, length of a fluid element
m, a parameter defined by equation (12)
m''', a sink strength
P, a parameter defined by equation (12)
q'' interface heat flux
R, bubble radius
r, radial coordinate
s, a variable defined by equation (10),
T, temperature
ΔT, liquid superheat, T_∞ — Tsat
t, time

V, velocity, bubble volume
x, a coordinate direction normal to the interface

Greek Letters

α, thermal diffusivity of liquid
$\beta(t)$, a variable defined by equation (17)
δ, thermal boundary layer thickness
η, self similarity variable; equations (5) and (17) angle, Fig. 3
$\nu(t)$, a variable, equation (C4)
ρ, liquid density
a variable, equation (4)

Subscripts

s.i. semi-infinite case
v, vapor

Introduction

Bubble growth problem for heat transfer controlled growth in a uniformly superheated liquid was considered by a number of investigators, most noteably by Forester and Zuber [1], Plesset and Zwick [2], and most completely by Scriven [3]. Common to all the methods was an assumed spherical shape of the growing bubble.

It is known that bubbles growing near a wall may be of various shapes, tending to be nearly spherical if slowly growing, or oblate if rapidly growing [4]. Recently Cooper et al [5] have reported experiments on single bubbles under simplified low-gravity conditions. The general shapes of such bubbles were shown to change during growth, from nearly hemispherical to nearly spherical, depending broadly on the relative magnitude of the inertial and surface tension stresses. These are approximately proportional to $(1/t)$ and $(1/\sqrt{t})$ respectively. Hence the former predominates for small t and the latter for large t.

At present there is no available method for evaluating evaporation rates at the bubble interface for bubble shapes other than spherical.

The objective of this work is to develop a method for determining the temperature field over the bubble dome, and thus the vapor generation rates at the vapor-liquid interface. The method does not determine the bubble shape or changes of shape during the growth but rather predicts vapor generation rates for a given change of shape.

The work is not concerned with the microlayer contribution. This is not intended to imply that the microlayer contribution is insignificant; the adopted approach simply reflects the fact that the phenomenon considered – effect of bubble shape on evaporation rates – applies mainly to the bubble dome contribution. The microlayer contribution could be added by some of the methods already developed, e.g. ([6].

Model and Analysis

For a growing bubble in a uniformly superheated liquid, the thermal boundary layer around the bubble dome is much smaller than the bubble radius (or the local radius of curvature for a nonspherical bubble). This is because in heat transfer controlled growth one can show that $\delta/R \sim 1/Ja$ and in most practical situations the Jakob number (Ja) would be much larger than one. Therefore, $\delta/R \ll 1$. Based on this argument the temperature field in the vicinity of the interface can be described in a plane geometry system. This is the first component of the model

As the liquid evaporates at the interface, a certain portion of the boundary layer ($\Delta\delta$) is converted into vapor. The effect of this on the temperature field would be negligible if the value of $\Delta\delta/\delta$ is much smaller than one. Since $\Delta\delta/\delta$ is of the order of $Cp\Delta T/h_{fg}$, for most cases of practical interest one would have $\Delta\delta/\delta \ll 1$. Consequently, the effects of liquid removal at the interface on the temperature field can be neglected. This is the second component of the model.

The third component of the model is related to the stretching effects on the boundary layer due to a continuous increase of the interface area during bubble growth. This increase causes both a flow of liquid toward the interface and for a control volume with fixed (undeformable) boundaries, flow of fluid out of the control volume parallel to the interface. A fluid element of length l, normal to the interface, will change its length due to the interface srettching. Conservation of mass requires $l \cdot A$ = constant, and therefore

$$\frac{dl}{dt} = -\frac{l}{A}\frac{dA}{dt} \tag{1}$$

where A is the bubble dome area. Clearly $\left(-\frac{dl}{dt}\right)$ represents the liquid velocity toward the interface. The value of the velocity at the interface ($l=0$) is zero as a consequence of the second component of the model. Thus from relation (1)

$$V_x = -\frac{x}{A}\frac{dA}{dt} \tag{2}$$

This completes the model. Figure 1 summarizes the approach.

The energy equation for assumed incompressible liquid together with boundary and initial condition is:

$$\begin{aligned} &\rho Cp \frac{\partial T}{\partial t} + \rho Cp V_x \frac{\partial T}{\partial x} - k \frac{\partial^2 T}{\partial x^2} = 0 \\ &t=0 \quad T=T_\infty \\ &x=0 \quad T=T_{sat} \\ &x=\infty \quad T=T_\infty \end{aligned} \tag{3}$$

Substitution of (2) into (3) yields

$$\frac{\partial T}{\partial t} - \zeta \frac{x}{t}\frac{\partial T}{\partial x} - \alpha \frac{\partial^2 T}{\partial x^2} = 0 \tag{3a}$$

where

(4) $$\zeta = \frac{t}{A} \frac{dA}{dt}$$

An alternative derivation to the above formalised simplification from a general form of energy equation, would be to write conservation and energy equations for a control volume with a fixed boundaries. Because of the additional physical insight one would get in this way and because the whole approach involving modeling of this type is new, the alternative derivation of (3) is presented in Appendix A.

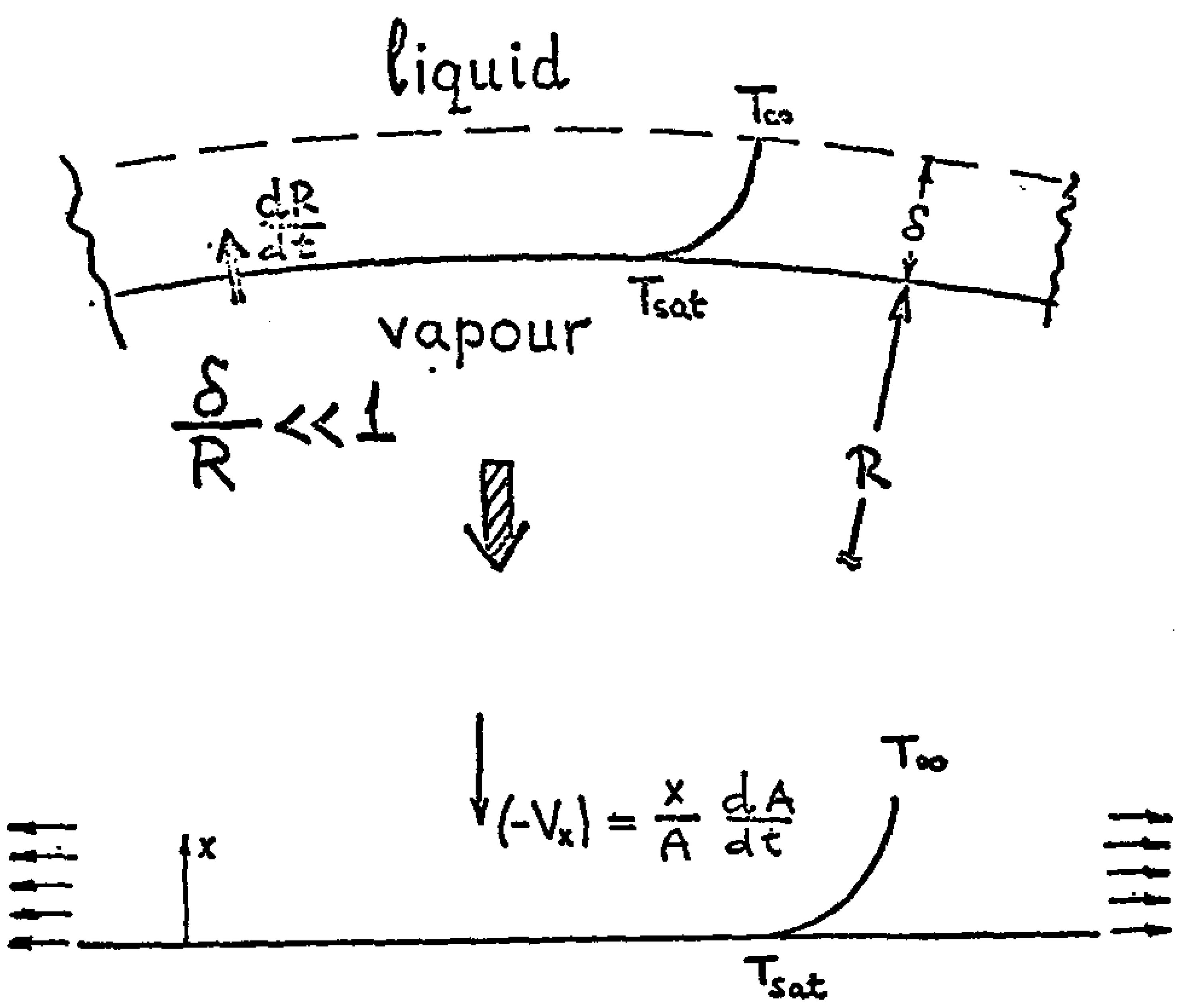

Figure 1. Model

The problem considered above has a similar solution at least for values of ζ =const. With a new variable

(5) $$\eta = (1 + 2\zeta)^{1/2} x / 2\sqrt{\alpha t}$$

relation (3a) transforms into

(6) $$\left.\begin{array}{l} \frac{d^2 T}{d\eta^2} + 2\eta \frac{dT}{d\eta} = 0 \\ \eta = 0 \quad T = T_{sat} \\ \eta = \infty \quad T = T_\infty \end{array}\right|$$

which has the following solution

(7) $$(T - T_{sat}) = (T_\infty - T_{sat}) \text{ erf } [1 + 2\zeta)^{1/2} x / 2\sqrt{\alpha t}]$$

The flux at the interface follows from (7) as

(8) $$q'' = (1+2\zeta)^{1/2} \frac{k\Delta T}{\sqrt{\pi\alpha t}}$$

The volumetric rate of vapor formation over the dome follows from the mass balance at the interface.

$$\frac{dV_D}{dt} = \frac{A q''}{\rho_v h_{fg}}$$

and together with (8)

(9) $$\frac{dV_D}{dt} = A\left(\frac{1+\zeta}{\pi}\right)^{1/2}\left(\frac{\alpha}{t}\right)^{1/2} J_a.$$

Variable ζ as defined by (4) can be expressed in terms of parameters relating the bubble dome area to the total bubble volume (V). Writing the area as

(10) $$A = s\,V^{2/3}$$

where s is a time varying bubble shape factor.

One has for ζ from the above and (4)

(11) $$\zeta = \frac{2}{3}\,\frac{t}{V}\,\frac{dV}{dt} + \frac{t}{s}\,\frac{ds}{dt}$$

The value of ζ would be independent of time if V and s could be expressed respectively as

(12) $$\left.\begin{aligned} V &= bt^{m} \\ s &= B\,t^{p} \end{aligned}\right\}$$

where b, B, m and p are constant. For this case

(13) $$\zeta = \frac{2}{3}m + p$$

If the bubble grows due to the evaporation over the dome only, one could relate m to p (defined by equation (12) as follows.
From (9)

$$\frac{dV}{dt} \sim \frac{A}{t^{1/2}} = \frac{sV^{2/3}}{t^{1/2}} \sim \frac{V^{2/3}}{t^{1/2-p}}$$

from which

$$V \sim t^{3(1/2+p)}$$

Thus $m = \frac{3}{2} + 3p$ and relation (9) changes for this case into

(14) $$\frac{dV}{dt} = A\left(\frac{3+6p}{\pi}\right)^{1/2} J_a \left(\frac{\alpha}{t}\right)^{1/2}$$

For a bubble growth for which $(A/V^{2/3})=s=\text{const.}$, $p=0$. If, in addition teh bubble is spherical, then relation (14)k, after integration, yields the known asymptotic solution for the bubble radius, [2], [3].

$$R=2\left(\frac{3}{\pi}\right)^{1/2}\alpha^{1/2}J_a\,t^{1/2} \tag{15}$$

In Appendix B the same results were obtained by a method based only on a simple physical argument dealing with changes in an effective thermal boundary layer thickness due to the interface stretching. Moreover the method introduced in the appendix is not restricted to the case when $\zeta=\frac{t}{A}\frac{dA}{dt}=\text{const.}$ For a general case when $\zeta=\zeta(t)$ it is shown that the flux at the interface can be expressed as

$$q''=\left(\frac{t\cdot\exp\left[\int\frac{2\zeta}{t}dt\right]}{\int\limits_0^t\exp\left[\int\frac{2\zeta}{t}dt\right]\cdot dt}\right)^{1/2}\frac{k\Delta T}{\sqrt{\pi\alpha t}} \tag{16}$$

Since the above result was obtained without any additional physical interpretation of the process over and above those already used in the analytical model introduced in this section, one would expect that the main problem stated by relation (3a) with $\zeta=\zeta(t)$ should also have a similar solution. And, indeed the introduction of a new variable η defined as

$$\eta=\beta(t)\,x/2\sqrt{\alpha t}$$

where

$$\beta(t)=\left(\frac{t\exp\left[\int\frac{2\zeta}{t}dt\right].}{\int\limits_0^t\exp\left[\int\frac{2\zeta}{t}dt\right]\cdot dt}\right)^{1/2}=\left(\frac{t\cdot A^2}{\int\limits_0^t A^2\,dt}\right) \tag{17}$$

transforms relation (3a), after long but straightforward algebraic manipulations, into an ordinary differential equation identical with (6). Thus the general solution of the problem considered here is

$$(T-T_s)=(T_\infty-T_s)\ \text{erf}\ [\beta(t)\,x/2\sqrt{\alpha t}] \tag{18}$$

The flux at the liquid-vapor interface is given by equation (16) or in terms of the surface area of the dome

$$q''=\left(\frac{A^2 t}{\int\limits_0^t A^2\,dt}\right)^{1/2}\left(\frac{k\,\Delta T}{\sqrt{\pi\,\alpha\,t}}\right) \tag{19}$$

The vapor generation rate follows from (19) as

$$\frac{dV_D}{dt} = A\left(\frac{A^2 t}{\int_0^t A^2\,dt}\right)^{1/2} J_a\left(\frac{\alpha}{\pi t}\right)^{1/2}$$

For a constant bubble shape, i.e. a constant ration of $A/v^{2/3}$ one has

$$\left(\frac{A^2 t}{\int_0^t A^2\,dt}\right)^{1/2} = \sqrt{3}$$

since for this case $V \sim t^{3/2}$, as can be deduced from (9).

Discussion

The results of the analysis presented in the previous section show that when an arbitrary shaped bubble grows without changing shape, i.e. when in expression (11) s =const. or in (13), $p=0$, the flux at the interface is given by the spherical bubble relationship:

$$q'' = \sqrt{3}\frac{k\Delta T}{\sqrt{\pi \alpha t}}$$

This, however, is not the case for bubbles which change their shape during the growth process. Referring to equation (19), it is seen that fluxes either higher or lower than those for a spherically shaped bubble are possible. A simple analysis shows that the result depends on the area volume relation. For the relationship given by equation (12) one can show that when $\frac{ds}{dt}>0$, the fluxes at the interface would be higher than for a spherically shaped bubble. The opposite would be true for $\frac{ds}{dt}<0$. This conclusion is consistent with the physical picture: larger increases in surface area for the same change of bubble volume would cause higher stretching effects thus producing a thinner thermal boundary layer.

Concerning the overall approach adopted here we would like to make two comments.

Firstly, the method, as presented, is based on an average stretching of the boundary layer over the dome area. For shapes other than spherical this is an approximation. The consequence of the approximation is to minimize the evaporation rate per unit area of the dome. Thus, as compared to the present model, the deviation of the actual vapor generation rate from the rate predicted by the spherical bubble is higher for $\frac{ds}{dt}>0$ and lower for $\frac{ds}{dt}<0$. This conclusion is based on the following physical argument. The vapor generation rate over the dome area is proportional to $k\Delta T\int_A \frac{dA}{\delta}$, where δ is an effective thermal boundary layer thickness. The present method replaces the above with $\frac{k\Delta TA}{\bar{\delta}}$, where $\bar{\delta}$ is an average boundary layer thickness over the dome area; $\bar{\delta} = \frac{1}{A}\int_A \delta\, dA$.

Since the average value of $\left(\frac{1}{\delta}\right)$, i.e. $\frac{1}{A}\int \frac{dA}{\delta}$ is always greater than or equal to $1/\bar{\delta}$, the predicted evaporation rate is either equal to or lower than the actual one. For most cases of practical interest one would expect that the presented method is adequate. Nevertheless, in principle the method could be modified in order to remove the assumption of uniform stretching of the thermal boundary layer over the dome area. In this approach one could interpret relation (1) as a condition applied locally. The quantity $\frac{t}{A}\frac{dA}{dt}$ would be treated as a variable around the bubble interface. The local fluxes would still be given by relation (19) with an understanding that the quantity $\left(\frac{A^2 t}{\int_0^t A^2\, dt}\right)^{1/2}$ has local values over the dome. Integration of the interface fluxes over the dome area would then yield the vapor generation rates. Appendix C presents this procedure along with the general results for this case.

The second comment is related to the objectives and scope of the present work. This investigation does not attempt a complete bubble growth solution for which microlayer contributions and inertia effects would also have to be considered. Even within the scope of its applicability – heat diffusion controlled growth in a uniformly superheated liquid – the method requires as an input information on the bubble shape variations during the growth process. In the present form the work is intended mainly to indicate the consequences of changes in bubble shape on the interface fluxes and hence, on the vapor generation rates. Apart from this, the work introduces a new method whose applicability could be extended beyond the specific problem considered here; for example, to the problems of bubble growth with inclusion of inertia forces. bubble growth in a non-uniform temperature field, and bubble collapse.

Conclusions

The work presents a new approach for evaluation of bubble growth rates in a uniformly superheated liquid. The method is applicable to problems involving bubble shapes other than spherical The results show that for a bubble for which

$(A/V)^{2/3}$ remains constant during growth, where A and V are the bubble dome area and the bubble volume respectively, the interface flux is given as

$$q'' = \left(\frac{3}{\pi}\right)^{1/2} \frac{k\Delta T}{\sqrt{\alpha t}}$$

For a general case when $(A/V)^{2/3}$ varies during bubble growth, the interface flux can be expressed as

$$q'' = \left(\frac{A^2 t}{\int_0^t A^2 dt}\right)^{1/2} \frac{k\Delta T}{\sqrt{\pi\alpha t}}$$

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Appendix A

For a control volume with fixed boundaries, Fig. 2a, the flow rate out of the control volume in the direction parallel to the interface is

$$-\rho \frac{\partial V_x}{\partial x} \Delta x \Delta A$$

or per unit volume

(A1) $$m''' = -\rho \frac{\partial V_x}{\partial x} = \rho \frac{1}{A} \frac{dA}{dt}$$

The conservation of energy applied to the control volume, Fig. 2b, yields,

(A2) $$\rho C_p \frac{\partial T}{\partial t} + \rho C_p \frac{\partial}{\partial x}(V_x T) + C_p m''' T - k \frac{\partial^2 T}{\partial x^2} = 0$$

The third term in the above relation represents the enthalpy outflow from the control volume resulting from term (A1). Clearly, (A2) with (A1) reduce to relation (3). It is of some interest to note here that the complete effect of stretching on the development of the thermal boundary layer around the growing bubble could be compensated by the introduction of an effective mass sink of strength

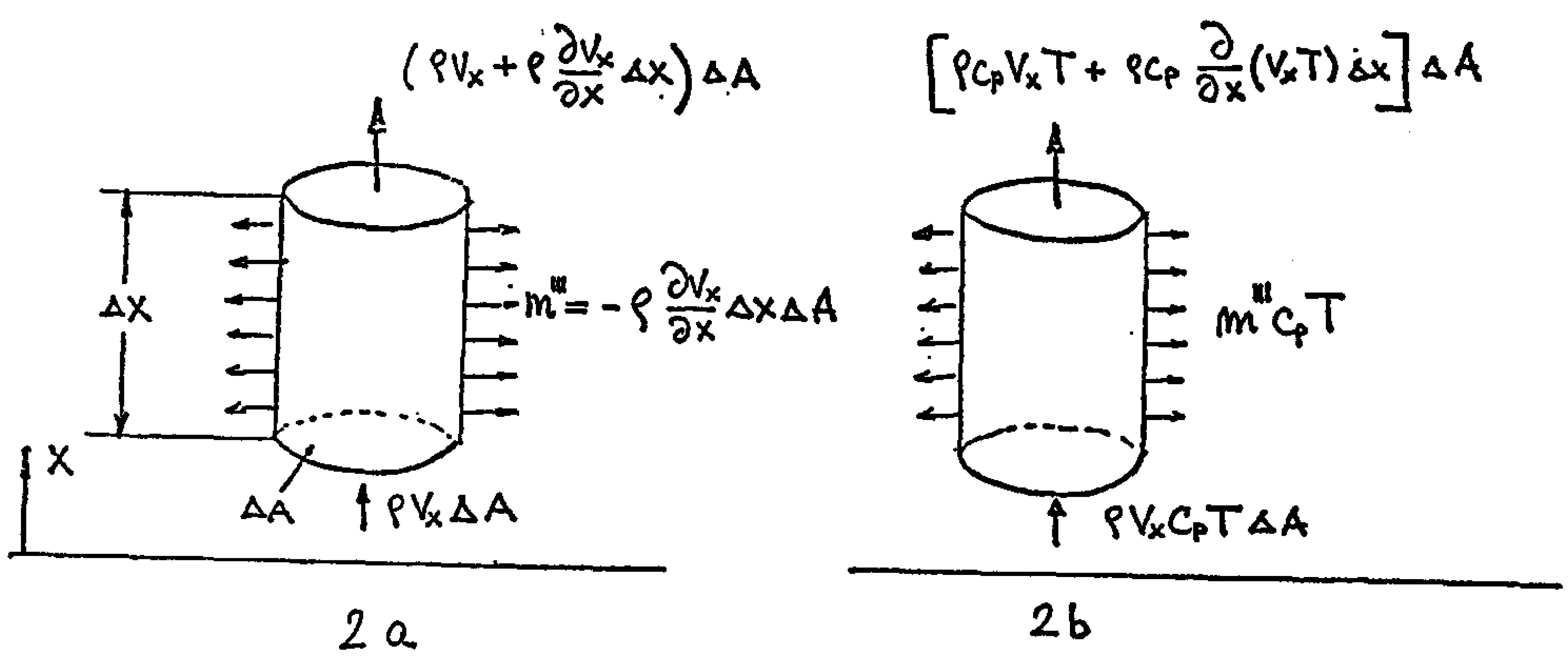

Figure 2.

m''', equation (A1). The sink is uniformly distributed for $x \geq 0$ where $x=0$ represents the interface boundary. It is evident that the sink would generate the identical fluid flow toward the interface as the stretching effect. Also in the energy equation, (A2), the third term could be viewed as the enthalpy removal per unit volume due to the uniformly distributed sink (m'''). Thus the problem of thermal boundary layer development around a growing bubble in a uniformly superheated liquid is equivalent to the development of the layer in a semi-infinite medium containing uniformly distributed sink (m''') whose strength is related to the interface stretching, (A1).

Appendix B

The heat flux for a simple case of semi-infinite body initially at a uniform temperature, T_∞, with $T=T_{sat}$ for $t>0$ at $x=0$ is given as

(B1)
$$q''_{s.i.} = \frac{k\Delta T}{\sqrt{\pi\alpha t}}$$

One could interpret the quantity $\sqrt{\pi\alpha t}$ as an effective thickness of the thermal boundary layer. The growth of the layer with time (which is caused by heat diffusion only) can be expressed as

(B2)
$$\frac{d\delta}{dt}_{s.i} = \frac{1}{2}\frac{(\pi\alpha)^{1/2}}{t^{1/2}} = \frac{\pi\alpha}{2\delta}$$

For the case of a bubble growth one would expect that the fluxes could be expressed by a relation similar to (B1). Specifically, we assume,

$$q''=\frac{k\Delta t}{\lambda\sqrt{\pi\alpha t}} \tag{B3}$$

For this case the effective boundary layer thickness is

$$\delta=\lambda\sqrt{\pi\alpha t} \tag{B4}$$

The difference between the two boundary layers (reflected by a presence of factor λ in (B4) is caused by the stretching effects at the bubble interface. For the stretching effect alone one has $\delta\ A$=const. or

$$\frac{\partial\delta}{\partial A}=-\frac{\delta}{A} \tag{B5}$$

The total change of the effective boundary layer thickness can be expressed as

$$\frac{d\delta}{dt}=\frac{\partial\delta}{\partial t}+\frac{\partial\delta}{\partial A}\frac{dA}{dt} \tag{B6}$$

The first term on the right-hand side represents the growth due to heat diffusion only. We interpret the result given by relation (B2) as the growth rate by heat diffusion only in terms of the instantaneous real thickness, i.e.

$$\frac{\partial\delta}{\partial t}=\left(\frac{d\delta}{dt}\right)_{\text{s.i.}}=\frac{\pi\alpha}{2\delta}$$

Thus, substituting (B2), (B4) and (B5) into (B6) and assuming that λ is independent of time, one gets,

$$\frac{\lambda\sqrt{\pi\alpha}}{2t^{1/2}}=\frac{\pi\alpha}{2\lambda\sqrt{\pi\alpha t}}-\left(\frac{t}{A}\frac{dA}{dt}\right)\frac{\lambda\sqrt{\pi\alpha t}}{t} \tag{B7}$$

solving for λ this results:

$$\lambda=1/\left[1+2\left(\frac{t}{A}\frac{dA}{dt}\right)\right]^{1/2} \tag{B8}$$

Relation (B8) is valid when $\frac{t}{A}\frac{dA}{dt}$=const., consistent with the assumption that λ is independent of time. For a spherical bubble:

$$\frac{t}{A}\frac{dA}{dt}=1 \text{ and } \lambda=1/\sqrt{3}$$

When $\frac{t}{A}\frac{dA}{dt}$ is not constant, one proceeds as above starting with (B6) but when diffirentiating δ using (B4), λ must be treated as a function of time. This yields an extra term on the left-hand side of equation (B7). The resulting expression is the following differential equation

(B9)
$$\frac{d\lambda^2}{dt}+\frac{(1+2\zeta)}{t}\lambda^2-\frac{1}{t}=0$$

where

(B10)
$$\zeta=\frac{t}{A}\frac{dA}{dt}$$

From (B9) and the condition that λ is finite at $t=0$ there follows:

(B11)
$$\lambda=\left\{\frac{\int_0^t \exp\left[\int \frac{2\zeta}{t}dt\right]dt}{t\exp\left[\int \frac{2\zeta}{t}dt\right]}\right\}^{1/2}$$

Or in terms of the dome area, using identity (A10).

(B12)
$$\lambda=\left(\frac{\int_0^t A^2\,dt}{A^2 t}\right)^{1/2}$$

For ζ=const. (B11) and (B12) reduce to (B8).
The interface flux for the general case follows from (B12) and (B3) as

(B13)
$$q''=\left(\frac{A^2 t}{\int_0^t A^2\,dt}\right)^{1/2}\frac{k\Delta T}{\sqrt{\pi\alpha t}}$$

Appendix C

In case of non-uniform stretching of the thermal boundary layer over the bubble interface, one can evaluate the interface fluxes in terms of local values of the boundary layer thickness and then integrate it over the whole dome in order to find the vapor generation rate.

The interface flux in this case is still given by relation (19) i.e.

(C1)
$$q''=\left[A^2 t/\int_0^t A^2\,dt\right]^{1/2}\frac{k\Delta T}{\sqrt{\pi\alpha t}}$$

where the quantity $[A^2 t/\int_0^t A^2 dt]$ has local values over the dome.

Referring to Figure 3.

$$dA = 2\pi r \cos\Phi\, ds$$

$$ds = r\, d/\cos(\vec{r}, \vec{n})$$

Thus

(C2)
$$dA = 2\pi r^2 \frac{\cos\Phi}{\cos(\vec{r}, \vec{n})} d$$

For a fixed Φ one gets

(C3)
$$\left[\frac{A^2 t}{\int_0^t A^2 dt}\right]^{1/2} = \left(\frac{r^2 \cdot t}{\cos^2(\vec{r}, \vec{n}) \int_0^t \frac{r^4 d}{\cos^2(\vec{r}, \vec{n})}}\right)^{1/2}$$

The evaporation rate over the dome follows from

$$\frac{dV_D}{dt} = \frac{1}{\rho_v h_{fg}} \int_A q''\, dA$$

which together with (Cl) to (C3) yields

(C4)
$$\frac{dV_D}{dt} = 2\pi \nu(t) \left(\frac{\alpha}{\pi t}\right)^{1/2} J_a$$

where

$$\nu(t) = \int_{-\Phi_0}^{\pi/2} \frac{r^2 \cos\varphi}{\cos(\vec{r}, \vec{n})} \left(\frac{r^4 t}{\cos^2(\vec{r}, \vec{n}) \int_0^t \frac{r^4 dt}{\cos^2(\vec{r}, \vec{n})}}\right)^{1/2} d\varphi$$

The above could be simplified when changes in $\cos(r, n)$ at any particular location are much smaller than changes in r. For this case

$$\left(\frac{A^2 t}{\int_0^t A^2 dt}\right)^{1/2} \simeq \left(\frac{r^4 t}{\int_0^t r^4 dt}\right)^{1/2}$$

The choice of Φ_0 in (C4) depends on the bubble geometry and the position of the center 0. The latter should be located on the axis of symmetry at the height where $r \cdot \cos(\vec{r}, \vec{n})$ has a maximum (c.f. Fig. 3).

We should like to mention that the above approach is still an approximation. A more rigorous method would be to integrate along stream lines around the bubble but this would require a complete solution for the flow field.

$$dA = 2\pi r \cos\varphi \, ds$$

$$ds = \frac{r\,d\varphi}{\cos(\vec{r}, \vec{n})}$$

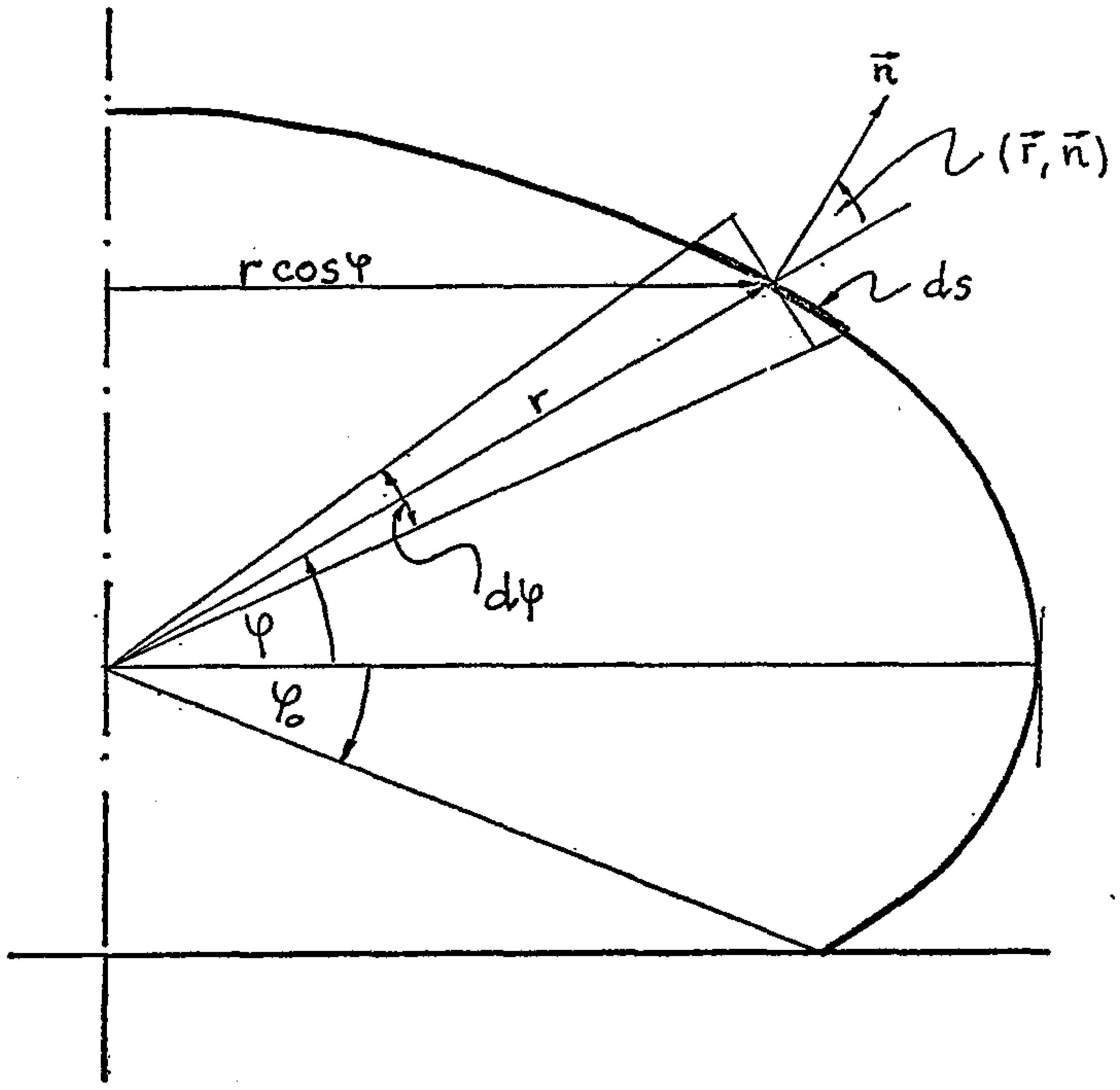

Figure 3.