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CAPILLARY SOLITARY WAVES AND THEIR DISINTEGRATION ON THE SHELF

Vladan D. Djordjević

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1. Introduction

The interest in studying various problems of nonlinear wave theory in the presence of some kind of inhomogeneity in the direction of the wave propagation has increased very much recently. The source of inhomogeneity in the case of surface waves is mostly the varying depth of liquid, in the case od magneto-acoustic waves in a plasma — the varying density of plasma, in the case of lattice waves — the varying mass etc. It has been shown in all these cases that the problem reduces to one of well known equations of nonlinear wave theory, as for example Korteweg-de Vries (K-dV) equation, nonlinear Schrödinger (NL-S) equation and others, with varying coefficients. K-dV equation with varying coefficients for surface waves was derived by Kakutani (1971a), for magneto-acoustic waves by Kakutani (1971b) and by Asano and Ono (1971) and for lattice waves by Ono (1972). A general form of NL-S equation with varying coefficients was conjectured by Ono (1974) and the concrete equation for surface waves was derived by Djordjević and Redekopp (1977a).

A general theory of these equations was given on the example of K-dV equation by Ono (1972) and by Johnson (1973). The main and the most interesting result of this theory is referred to the evolution of nonperiodic waves of permanent form based on the balance between nonlinearity and dispersion, so-called solitons, between two homogenous regions. These regions in the case of surface waves may be composed of two regions of constant, but different depth to build the shelf. It was shown on that occasion that a soliton can disintegrate by passing from the deeper region onto the shallower one into more solitons whose number n depends on the depth of the shelf, the so-called eigendepth h_1 in the following way (h_0 is the depth in front of the shelf):

$$\frac{h_1}{h_0}=\left[\frac{n(n+1)}{2}\right]^{-4/9}$$

The corresponding result for packets of surface waves, which are, as known, governed by NL-S equation was given by Djordjević and Redekopp (1977a) for the case of the disintegration of a concave E solution:

$$\frac{h_1}{h_0}=\left[\frac{n(n+1)}{2}\right]^{-8/27}.$$

When surface waves are in question, in all afore-mentioned papers only pure gravity waves were treated, i.e. the effect of capillarity was completely ignored, which means that relatively long waves have been had in mind. For relatively short waves capillarity has to be taken into account. It is well known on the other hand that capillarity very often leads to qualitatively new phenomena, as for example second-harmonic resonance, see Leibovich and Seebass (1974) and resonant interaction between long and short waves, Djordjević and Redekopp (1977b). This is the reason that the disintegration of capillary solitary waves on the shelf is considered in this paper. It is shown that the disintegration of a soliton on the shelf is possible also by increasing the depth of liquid. It is shown, too, that for relatively great capillarity only one soliton can emerge. We will call the depth of liquid on the shelf in this case the charateristic eigendepth. It is shown that it can be less as well greater than the depth in front of the shelf.

Since we anticipate here to have relatively short solitary waves and since the existence of solitary waves is possible only in relatively shallow liquid, i.e. in a liquid whose depth is much less than the wave-length, it means that the depth has to be very small, i.e. we will work with a thin film of liquid.

2. K-dV equation with varying coefficients for capillary solitary waves

We will consider the problem following Fig. 1 (z points vertically upwards):

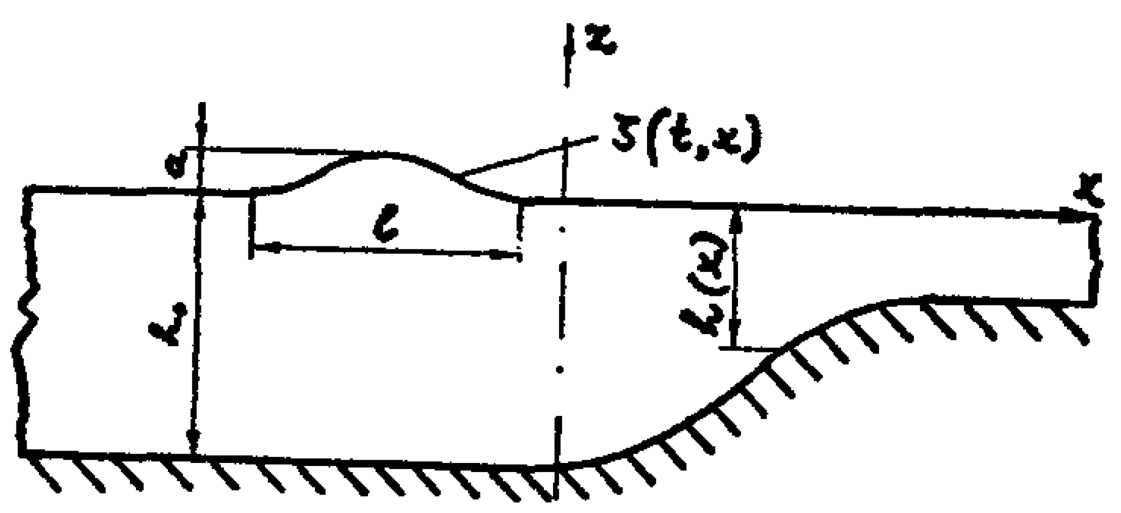

Fig 1.

A solitary wave formed in the region $x<0$ on the free surface of a liquid of constant depth h_0 is propagating in the direction of x. In the region $x>0$ the depth $h(x)$ is changing slowly in a certain way. The evolution of the solitary wave ought to be determined in this region taking into account capillary effects. We will choose the following scales: l for x, h_0 for z, h and ζ, $l/\sqrt{gh_0}$ for t-time (g-acceleration due to gravity), $\sqrt{gh_0}$ for u-velocity in the direction of x, $h_0\sqrt{gh_0}/l$ for w-velocity in the direction of z, $\rho g h_0$ for p-pressure (ρ - density of the fluid) and

$\rho g h^2_0$ for T-surface tension coefficient. The governing nondimensional equations for perturbed quantities and the boundary conditions on the bottom and on the free surface will be:

$$u_t + uu_x + wu_z = -p_x$$

$$w_t + uw_x + ww_z = -p_z$$

$$u_x + w_z = 0$$

$$z = -h(x): \qquad w = -\frac{dh}{dx} u \tag{1}$$

$$z = \zeta(t, x) \cdot \qquad w = \zeta_t + u + \zeta_x$$

$$p = \zeta - \frac{h_0^2}{l^2} T \frac{\zeta_{xx}}{\left(1 + \frac{h_0^2}{l^2} \zeta_x^2\right)^{3/2}}$$

As it is usually the case in nonlinear wave theory, see for example Johnson (1973), we will introduce the far field coordinates in the following way:

$$\tau = \int^x \frac{dx}{c(\xi)} - t, \ \xi = \varepsilon x;$$

where $\varepsilon = a/h_0 \ll 1$ is a small parameter representing the slope of the wave. We obviously allow the speed c of the wave to vary slowly in the direction of the wave propagation due to the presence of the inhomogeneity. We will further assume that:

$$\frac{h_0^2}{l^2} = k\,\varepsilon, \ k = 0(1),$$

with what we actually provide the desired balance between nonlinearity and dispersion necessary for the existence of solitary waves and that the depth is changing slowly in such a way that:

$$\frac{dh}{d\xi} = h' = 0(1).$$

We will now expand all unknown quantities in the following asymptotic series:

$$\begin{Bmatrix} u \\ w \\ p \\ \zeta \end{Bmatrix} = \varepsilon \begin{Bmatrix} u_0 \\ w_0 \\ p_0 \\ \zeta_0 \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} u_1 \\ w_1 \\ p_1 \\ \zeta_1 \end{Bmatrix} + \cdots.$$

and, substituting them into (1), expanding the boundary conditions at $z = \zeta(t, x)$ around the undisturbed free surface $z = 0$ and equating the coefficients of like powers in ε, we will have in the first approximation:

$$
(4)\qquad
\begin{aligned}
& cu_{0\tau} = p_{0\tau} \\
& p_{0z} = 0 \\
& u_{0\tau} + cw_{0z} = 0 \\
z = -h: \quad & w_0 = 0 \\
z = 0: \quad & w_0 = -\zeta_{0\tau} \\
& p_0 = \zeta_0 .
\end{aligned}
$$

The solution of this system is easily obtained to be:

$$
(3)\qquad u_0 = \zeta_0 / c, \; w_0 = -\frac{z+h}{h}\zeta_{0\tau}, \; p_0 = \zeta_0,
$$

with $c^2 = h$, while ζ_0 remains undetermined so far. If the procedure is continued until the next approximation, we shall use (3):

$$
\begin{aligned}
& cu_{1\tau} - p_{1\tau} = \frac{1}{h}\zeta_0\zeta_{0\tau} + c\zeta_{0\xi} \\
& p_{1z} = -k\frac{z+h}{h}\zeta_{0\tau\tau} \\
& u_{1\tau} + cw_{0z} = -c\left(\frac{\zeta_0}{c}\right)_\xi \\
z = -h: \quad & w_1 = -\frac{h'}{c}\zeta_0 \\
z = 0: \quad & w_1 = -\zeta_{1\tau} + \frac{2\zeta_0\zeta_{0\tau}}{h} \\
& p_1 = \zeta_1 - \frac{kT}{h}\zeta_{0\tau\tau} .
\end{aligned}
$$

The solution of this system can be obtained in a similar way. ζ_1 remains undertermined, but an equation for ζ_0 in form of a secularity condition emerges:

$$
c'\zeta_0 + \frac{2h}{c}\zeta_{0\xi} + \frac{3}{h}\zeta_0\zeta_{0\tau} + \frac{kh}{3}\left(1 - \frac{3T}{h^2}\right)\zeta_{0\tau\tau\tau} = 0,
$$

that just represents the desired K-dV equation, describing the propagation of a solitary wave over an uneven bottom. With $c = \sqrt{h}$ and introducing $\sigma = 3T/h^2$ we will have:

$$
(4)\qquad \frac{h'}{2\sqrt{h}}\zeta_0 + 2\sqrt{h}\zeta_{0\xi} + \frac{3}{h}\zeta_0\zeta_{0\tau} + \frac{kh}{3}(1-\sigma)\zeta_{0\tau\tau\tau} = 0.
$$

It is obvious that $\sigma = 3/W_e$, where, W_e is a local value of the Weber number. For $\sigma = 0$, i.e. in absence of capillarity, the equation (4) reduces to the known equation by Johnson (1973). In contrast to the case $\sigma = 0$, however, the coefficient of the dispersion term $\zeta_{0\tau\tau\tau}$ of the equation (4) can be positive or negative depending on $\sigma \lesseqgtr 1$, which means that both convex and concave solitary waves are possible. For example, for h=const. say $h=1$, $\sigma=\sigma_0=3T$ and one-solitary wave solution of the equation (4) will be:

$$\zeta_0 = \pm \alpha_0 \operatorname{sech}^2 \sqrt{\frac{3\alpha_0}{4k|1-\sigma_0|}} \left(\tau \mp \frac{\alpha_0}{2}\xi\right), \quad \sigma_0 \lesseqgtr 1. \tag{5}$$

where $\alpha_0=0(1)>0$ is the amplitude of the wave. It can be easily shown that nonlinearity affects the speed of a concave ($\sigma_0>1$) solitary wave in such a way that it decreases with the amplitude.

For $\sigma=1$ the influence of dispersion is lost and nonlinearity prevails, leading to the breaking of the wave.

3. **Disintegration on the shelf**

As mentioned in the introduction, the theory by Johnson (1973) will be applied here in order to study the disintegration of capillary solitary waves on the shelf, i.e. at their propagation between two regions of constant depths. It is assumed thereby that the change in the depth is in some sense rapid, i.e. the change occurs over a short distance. By the substitution:

$$\zeta_0 = \frac{H}{\sqrt[4]{h}}$$

the equation (4) goes into:

$$2\sqrt{h}\, H_\xi + \frac{3}{h\sqrt[4]{h}} HH_\tau + \frac{kh}{3}(1-\sigma) H_{\tau\tau\tau} = 0, \tag{6}$$

with what the derivative of the varying depth of liquid, which may be very great in the region of the sudden change in depth, is removed from the coefficients of the equation (4). It is noticed that this is a highly nontrivial step whose importance has been emphasized by Djordjević and Redekopp (1976c). It was shown by Johnson (1973) that the flow on the shelf can now be described by means of the equation (6) with $h=h_1$=const., where h_1 is the depth of liquid on the shelf. One-solitary wave solution in the region in front of the shelf, where $h=1$ is assumed, which follows from (5) for $\xi=0$ will serve as the boundary condition. Therefore, we will have:

$$2\sqrt{h_1}\, H_\xi + \frac{3}{h_1\sqrt[4]{h_1}} HH_\tau + \frac{kh_1}{3}(1-\sigma_1) H_{\tau\tau\tau} = 0, \tag{7}$$

where $\sigma_1=\sigma_0/h_1^2$ with the boundary condition:

$$\xi = 0: \qquad H = \pm \alpha_0 \operatorname{sech}^2 \tau \sqrt{\frac{3\alpha_0}{4k|1-\sigma_0|}}.$$

Hence, the problem is reduced to a K-dV equation with constant coefficients and consequently is simplified very much, because the inverse scattering method can be employed, Miura (1968). In order to use directly the result of this theory, referring to the disintegration of solitons on the shelf, it is necessary to transform a little the equation (7). Introducing:

$$\overset{*}{\xi}=\frac{k\sqrt{h_1}(1-\sigma_1)}{6}\,\xi \text{ and } H=\frac{2kh_1^2\sqrt[4]{h_1}(1-\sigma_1)}{3}\overset{*}{H}.$$

one can obtain:

$$\overset{*}{H}_{\overset{*}{\xi}}+6\overset{*}{H}\overset{*}{H}_\tau+\overset{*}{H}_{\tau\tau\tau}=0$$

with the boundary condition:

$$\overset{*}{\xi}=0:\qquad \overset{*}{H}=\pm\frac{3\,\alpha_0}{2\,kh_1^{9/4}(1-\sigma_1)}\,\text{sech}^2\,\tau\sqrt{\frac{3\,\alpha_0}{4\,k\,|\,1-\sigma_0\,|}}\,.$$

A formula for the eigendepth follows now simply:

$$\frac{1-\sigma_0}{h_1^{9/4}(1-\sigma_1)}=\frac{n\,(n+1)}{2}$$

where n is the number of solitons, or:

$$\frac{1-\sigma_0}{h_1^{1/4}(h_1^2-\sigma_0)}=\frac{n\,(n+1)}{2}\,.$$

Following conclusions can now be drawn.

a) $\sigma_1 \lesseqgtr 1$ depending on $\sigma_0 \lesseqgtr 1$, i.e. a convex (concave) capillary solitary wave can disintegrate only onto a sequence of convex (concave) capillary solitary waves — transition from a convex solition to a concave one and vice versa is not possible on the shelf. At the head of a sequence of convex (concave) solitons, a soliton with the biggest (smallest) amplitude will march, because it propagates with the biggest speed.

b) For $\sigma_0<1$ the disintegration of a soliton on the shelf is possible only by decreasing the depth of liquid (Fig. 2). The number of solitions increases infinitely when $h_1\to\sqrt{\sigma_0}$. Of course, for $\sigma_0=0$ the result by Ono (1972) and Johnson (1973) cited in the introduction is obtained.

c) For $\sigma_0>1$, i.e. for concave solitary waves, the situation is essentially different because the disintegration of a soliton is possible by decreasing (Fig. 3) as well as by increasing (Fig. 2) the depth of liquid. In addition, one soliton in front of the shelf can emerge into only one soliton on the shelf. We will call the corresponding eigendepth the characteristic eigendepth and denote it by h_{1k}. For $1<\sigma_0<9$, $h_{1k}<1$ and for $\sigma_0>9$, $h_{1k}>1$, see the curve denoted by $n=1$ in Fig. 3. Occurrence of the characteristic eigendepth can be explained in the following manner. On the shelf only depth of liquid is changing suddenly, not the flow parameters. Therefore, a solitary wave formed in the region $x<0$ transforms only partially by passing over the shelf. For $n=2,3,\ldots$ it represents for K-dV on the shelf (7) an arbitrary perturbation on the free surface, which eventually emerges into 2,3,... solitons, while for $n=1$ it fits exactly into the one-solitary wave solution of the equation (7). It is noticed (Fig. 3) that the eigendepth for $n=2,3,\ldots$ in the region $h_1<1$

is extremelly small, because it is bounded by the vertical asymptot $h_1 = \left[\frac{2}{n(n+1)}\right]^4$ of the curve $\sigma_0=\sigma_0(h_1; n)$. For example, for $n=2$ and $\sigma_0=5$, $h_1=0{,}005$ and consequently this region is without any practical importance. The number of solitons increases infinitely when $h_1 \to 0$ and $h_1 \to \sqrt{\sigma_0}$.

In both cases b) and c) the eigendepth increases with σ_0.

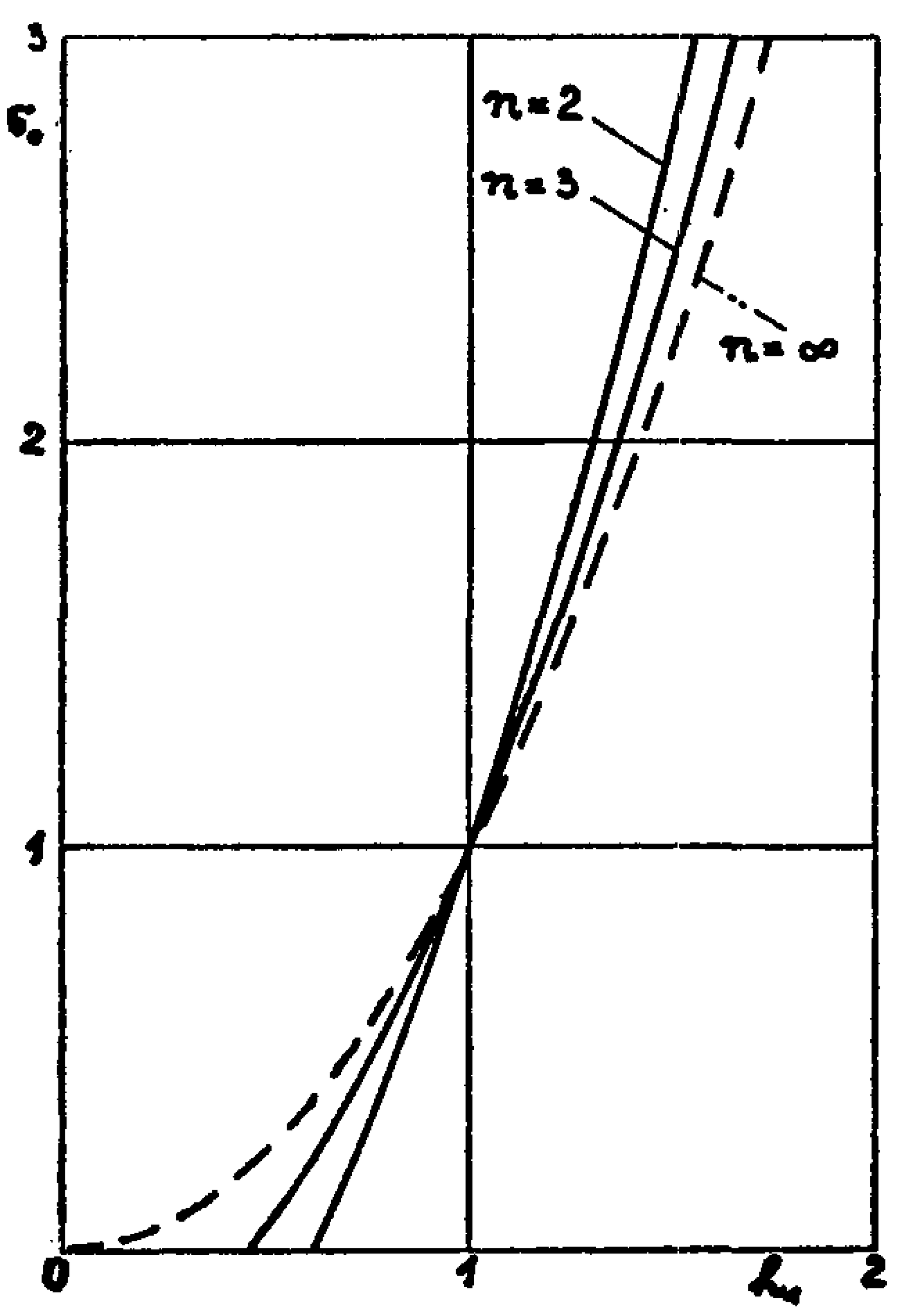

Fig. 2

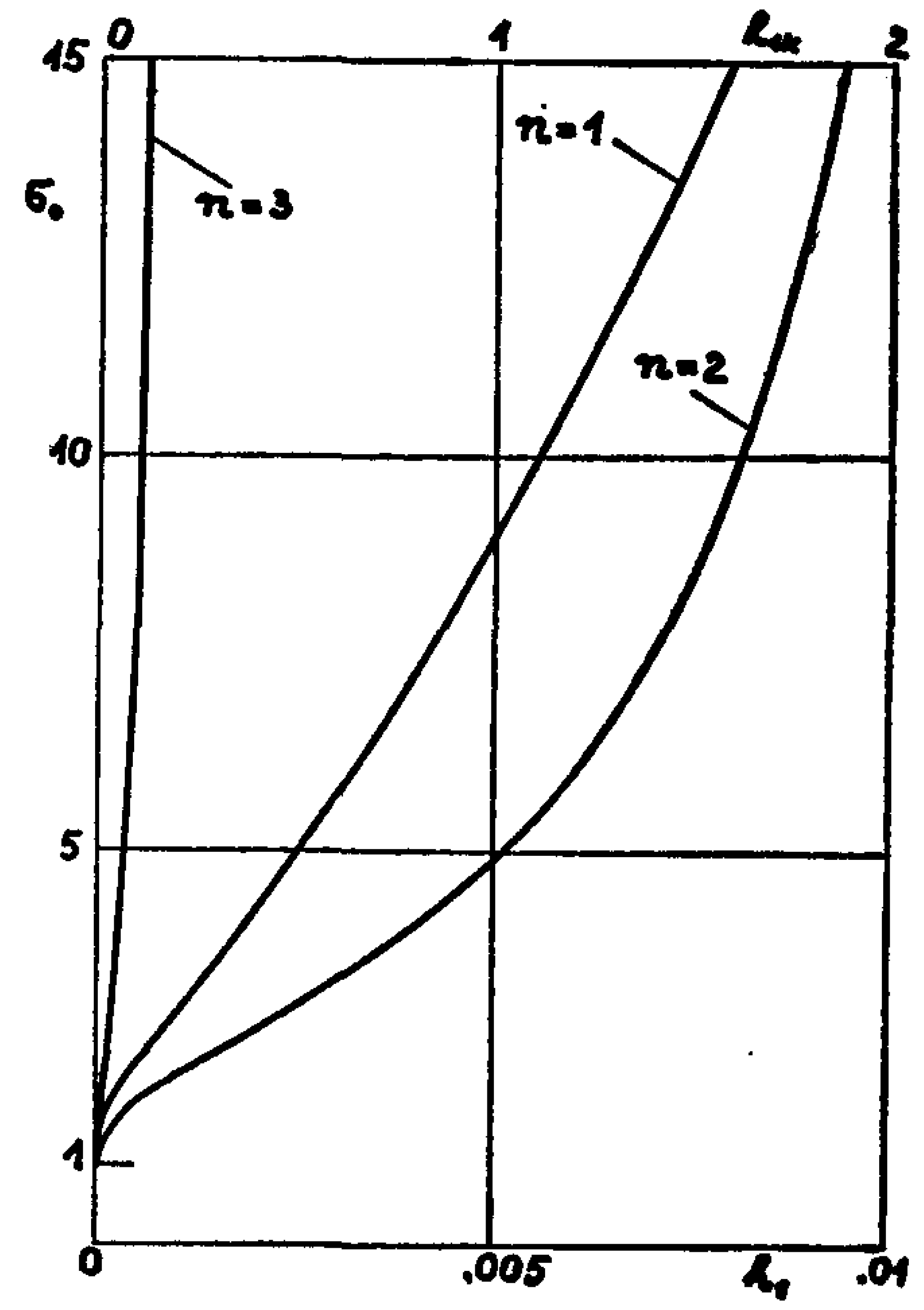

Fig. 3

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