Зборник радова Математичког института Нова серија књ. 3 (11), 1979.
Recueil des travaux de l'Institut Mathématique, Nouvelle série, tome 3 (11), 1979,

TORSION OF A RECTANGULAR ROD WITH CRACKS

V. Basilevich

(Communicated October 14, 1977)

The solution of the torsional problems can be reduced to the determination of the stress function

(1) $$\Phi(x, y)$$

which satisfies the differential equation in the region of the cross — section

(2) $$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=-2 G \theta$$

and the condition

(3) $$\Phi_k(x, y)=0$$

on the boundary and along the edges of cracks. [1]

We denote by Φ_1 the value of the function $\Phi(x,y)$ in the region 1—1—2—2 (Fig. 1) and by Φ_2 its value in the region 2—2′—3—3′. These functions must satisfy the differential equation (2) the boundary condition (3) and the condition of continuity on the line 2—2′.

If we take Φ_1 in the form of

(4) $$\Phi_1=G \theta(a^2-x_1{}^2)+\sum_{n=1,3\ldots}^{\infty}\left(A_n Ch \frac{n \pi y}{2 a}+B_n Sh \frac{n \pi y}{2 a}\right) \cos \frac{n \pi x_1}{2 a}=$$
$$=\sum_{n=1,3\ldots}^{\infty}\left[G \theta \frac{32 a^2}{n^3 \pi^3}(-1)^{\frac{1}{2}(n-1)}+A_n Ch \frac{n \pi y}{2 a}+B_n Sh \frac{n \pi y}{2 a}\right] \cos \frac{n \pi x_1}{2 a}.$$

this function satisfies the differential equation (2) and the boundary condition (3) along the edges 1—2. The condition (3) along the edges 1—1 will be satisfied if $\Phi_1=0$ with $y=b$.

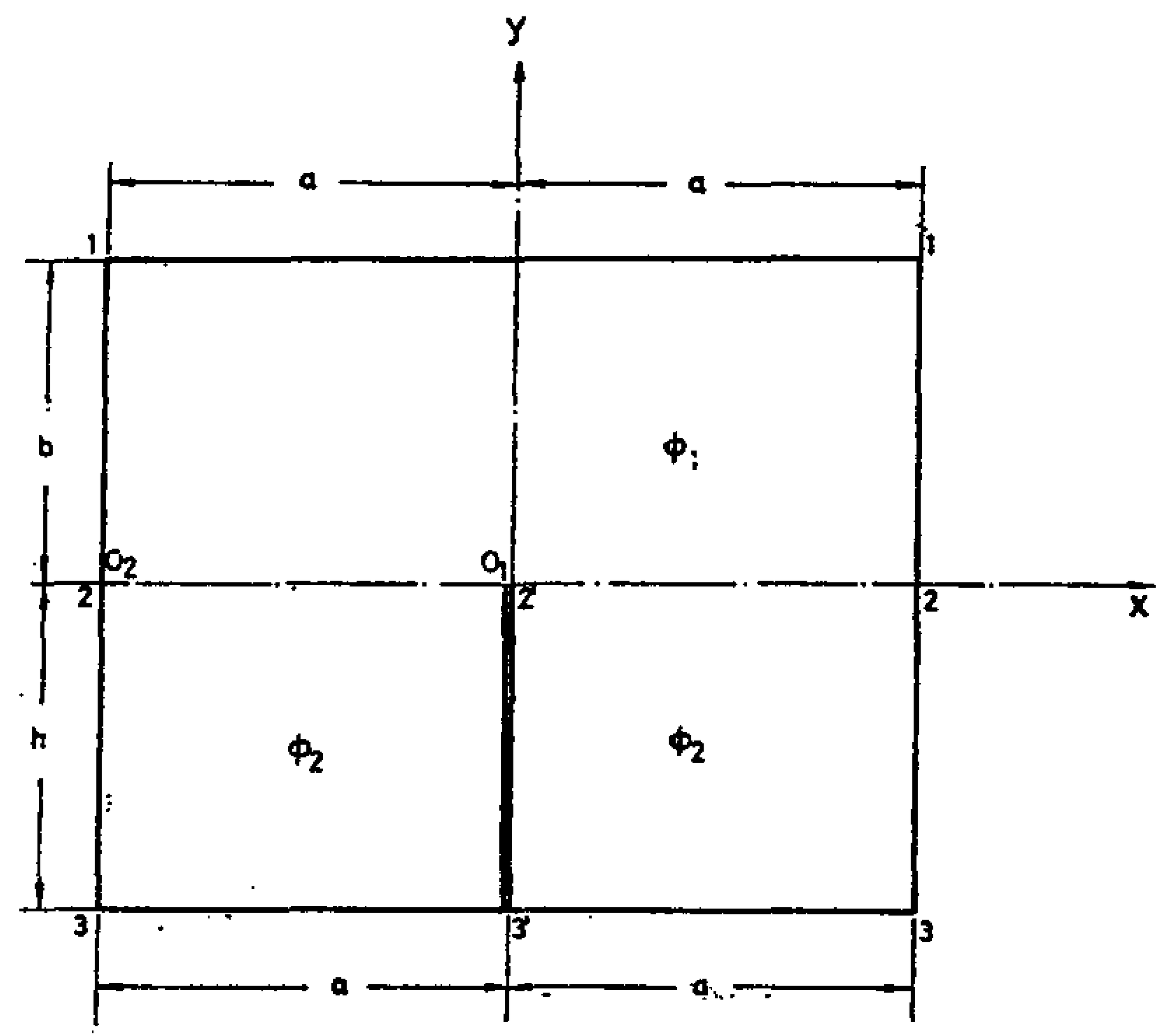

Fig. 1

(5) $$G\theta\frac{32a^2}{n^3\pi^3}(-1)^{\frac{1}{2}(n-1)}+A_n Ch\frac{n\pi b}{2a}+B_n Sh\frac{n\pi b}{2a}=0$$

From the equation (5) it follows: the coefficients A_n and B_n must satisfy the condition

(6) $$A_n=-G\theta\frac{32a^2}{n^3\pi^3}(-1)^{\frac{1}{2}(n-1)}\frac{1}{Ch\frac{n\pi b}{2a}}-B_n Th\frac{n\pi b}{2a}$$

Taking Φ_2 in the form

(7) $$\Phi_2=G\theta(ax_2-x_2^2)+\sum_{m=1,2\ldots}^{\infty}\left(C_m Ch\frac{m\pi y}{a}+D_m Sh\frac{m\pi y}{2a}\right)\sin\frac{m\pi x_2}{a}=$$
$$=\sum_{m=1,2\ldots}^{\infty}\left\{G\theta\frac{4a^2}{m^3\pi^3}[1-(-1)^m]+C_m Ch\frac{m\pi y}{a}+D_m Sh\frac{m\pi y}{a}\right\}\sin\frac{m\pi x_2}{a}$$

this function satisfies the differential equation (2) and the boundary conditions (3) along the edges 2—3 and of crack 2′—3′.

The condition (3) along the edges 3—3′ will be satisfied if $\Phi_2=0$ with $y=-h$.

(8) $$G\theta\frac{4a^2}{m^3\pi^3}[1-(-1)^m]+C_m Ch\frac{m\pi h}{a}-D_m Sh\frac{m\pi h}{a}=0$$

From the equation (8) it follows: the coefficients C_m and D_m must satisfy the condition

(9) $$C_m = -G\theta \frac{4a^2}{m^3\pi^3}[1-(-1)^m]\frac{1}{Ch\frac{m\pi h}{a}} + D_m Th\frac{m\pi h}{a}$$

Along the line 2—2′—2 the value of function Φ_1 must coincide with the value of function Φ_2.

This condition is satisfied if the multiplicator of the function Φ_2 from (7)

$$f_1(x_1) = \begin{cases} +\sin\frac{m\pi x_2}{a} & \text{for} \quad -a < x_1 < 0 \\ -\sin\frac{m\pi x_2}{a} & \text{for} \quad 0 < x_1 < +a \end{cases}$$

is developed in the following Fourier's series

(10) $$f_1(x_1) = -(-1)^m \frac{2m}{\pi} \sum_{n=1,3\dots}^{\infty} \frac{1}{m^2 - \left(\frac{n}{2}\right)^2} \cos\frac{n\pi x_1}{2a}$$

and by substituting (10) in (7).

The coincidence of the functions $(\Phi_1)_{y=0}$ and $(\Phi_2)_{y=0}$ along the line $2-2'-2$ gives the following equation

(11) $$\sum_{n=1,3\dots}^{\infty}\left[G\theta\frac{32a^2}{n^3\pi^3}(-1)^{\frac{1}{2}(n-1)} + A_n\right]\cos\frac{n\pi x_1}{2a} =$$
$$= -\sum_{m=1,2\dots}^{\infty}\left\{G\theta\frac{4a^2}{m^3\pi^3}[1-(-1)^m] + C_m\right\}(-1)^m\frac{2m}{\pi}\sum_{n=1,3\dots}^{\infty}\frac{1}{m^2-\left(\frac{n}{2}\right)^2}\cos\frac{n\pi x_1}{2a}$$

From the equation (11) it follows: the coefficients A_n and C_m must satisfy the condition

(12) $$A_n = G\theta\frac{32a^2}{\pi^3}\left\{\frac{1}{2\pi}\sum_{m=1,3\dots}^{\infty}\frac{1}{m^2\left[m^2-\left(\frac{n}{2}\right)^2\right]} - \frac{(-1)^{\frac{1}{2}(n-1)}}{n^3}\right\} -$$
$$-\frac{2}{\pi}\sum_{m=1,2\dots}^{\infty}\frac{(-1)^m m}{m^2-\left(\frac{n}{2}\right)^2}C_m$$

Equally on the line 2—2′—2 the value of the derivative of Φ_1 must concide with the value of the derivative Φ_2.

This condition is satisfied if the multiplicator of the function Φ_2 from (4).

$$f_2(x_2) = \cos \frac{n\pi x_1}{2a} \quad \text{for} \quad 0 < x_1 < +a$$

be developed in the following Fourier's series

$$f_2(x_2) = -\frac{2}{\pi} \sum_{m=1,2\ldots}^{\infty} \frac{(-1)^m m}{m^2 - \left(\frac{n}{2}\right)^2} \sin \frac{m\pi x_2}{a} \tag{13}$$

and by substituting (13) in (4).

The coicidence of the derivative $\left(\frac{\partial \Phi_1}{\partial y}\right)_{y=0}$ and $\left(\frac{\partial \Phi_2}{\partial y}\right)_{y=0}$ along the line 2—2′—2 gives the following equation

$$-\sum_{n=1,3\ldots}^{\infty} \frac{n\pi}{2a} B_n \frac{2}{\pi} \sum_{m=1,2\ldots}^{\infty} \frac{(-1)^m m}{m^2 - \left(\frac{n}{2}\right)^2} \sin \frac{m\pi x_2}{a} = \tag{14}$$

$$= \sum_{m=1,2\ldots}^{\infty} \frac{m\pi}{a} D_m \sin \frac{m\pi x_2}{a}$$

From the equation (14) it follows: the coefficients D_m and B_n must satisfy the condition

$$D_m = -\frac{(-1)^m}{\pi} \sum_{n=1,3\ldots}^{\infty} \frac{n}{m^2 - \left(\frac{n}{2}\right)^2} B_n \tag{15}$$

Replacing in the expression (15) the index n by i and vice versa, and introducing (15) in (9) after (6) and (9) in (12), we obtain the following system of linear equations

$$B_n + \frac{2}{\pi^2} \operatorname{Cth} \frac{n\pi b}{2a} \sum_{i=1,3\ldots}^{\infty} i B_i \sum_{m=1,2\ldots}^{\infty} \frac{m \operatorname{Th} \frac{m\pi h}{a}}{\left[m^2 - \left(\frac{n}{2}\right)^2\right]\left[m^2 - \left(\frac{i}{2}\right)^2\right]} =$$

$$= -G\theta \frac{32a^2}{\pi^3} \left\{ \frac{1}{2\pi} \operatorname{Cth} \frac{n\pi b}{2a} \sum_{m=1,3\ldots}^{\infty} \frac{1}{m^2\left[m^2 - \left(\frac{n}{2}\right)^2\right]} \left(1 - \frac{1}{\operatorname{Ch} \frac{m\pi h}{a}}\right) - \right.$$

$$\left. -\frac{(-1)^{\frac{1}{2}(n-1)}}{n^3} \frac{\left(\operatorname{Ch} \frac{n\pi b}{2a} - 1\right)}{\operatorname{Sh} \frac{n\pi b}{2a}} \right\}. \tag{16}$$

The coefficients B_n can be calculated from this system of linear equations, after which equations (6), (15) and (9) will determine explicitely the values of A_n, D_m and C_m.

By this the wanted stress function Φ_1 and Φ_2 are determined and consequently also the stress shate as result of torsion in the cross section of the rod excepting in the region of point 2′ of the crack in which the stress function has singularity.

The entire computation is illustrated by the following example:

Let us take: $a=4$, $b=4$; $h=4$

The system of linear equations becomes

$$+1{,}44B_1-0{,}44B_3-0{,}35B_5-0{,}29B_7-0{,}25B_9=+7{,}29\ G\theta$$

$$-0{,}14B_1+1{,}85B_3-0{,}14B_5-0{,}13B_7-0{,}12B_9=+1{,}27\ G\theta$$

$$-0{,}06B_1-0{,}08B_3+1{,}91B_5-0{,}09B_7-0{,}09B_9=+0{,}48\ G\theta$$

$$-0{,}04B_1-0{,}06B_3-0{,}06B_5+1{,}93B_7-0{,}07B_9=+0{,}25\ G\theta$$

$$-0{,}03B_1-0{,}04B_3-0{,}05B_5-0{,}05B_7+1{,}94B_9+=+0{,}15\ G\theta$$

By solving with successive approximation we obtain the values of the coefficients

$B_1=+5{,}65\ G\theta$; $C_1=+0{,}90\ G\theta$; $D_1=+1{,}26\ G\theta$;

$B_3=+1{,}17\ G\theta$; $C_2=-0{,}64\ G\theta$; $D_2=-0{,}64\ G\theta$;

$B_5=+0{,}51\ G\theta$; $C_3=+0{,}41\ G\theta$; $D_3=+0{,}41\ G\theta$;

$B_7=+0{,}30\ G\theta$; $C_4=-0{,}32\ G\theta$; $D_4=-0{,}32\ G\theta$;

$B_9=+0{,}20\ G\theta$; $C_5=+0{,}33\ G\theta$; $D_5=+0{,}33\ G\theta$;

$A_1=-11{,}75\ G\theta$; $C_6=-0{,}17\ G\theta$; $D_6=-0{,}17\ G\theta$;

$A_3=-1{,}16\ G\theta$; $C_7=+0{,}12\ G\theta$; $D_7=+0{,}12\ G\theta$;

$A_5=-0{,}51\ G\theta$; $C_8=-0{,}09\ G\theta$; $D_8=-0{,}09\ G\theta$;

$A_7=-0{,}30\ G\theta$; $C_9=+0{,}07\ G\theta$; $D_9=+0{,}07\ G\theta$;

$A_9=-0{,}20\ G\theta$; $C_{10}=-0{,}05\ G\theta$; $D_{10}=-0{,}05\ G\theta$.

The Fig. 2 gives the stress function

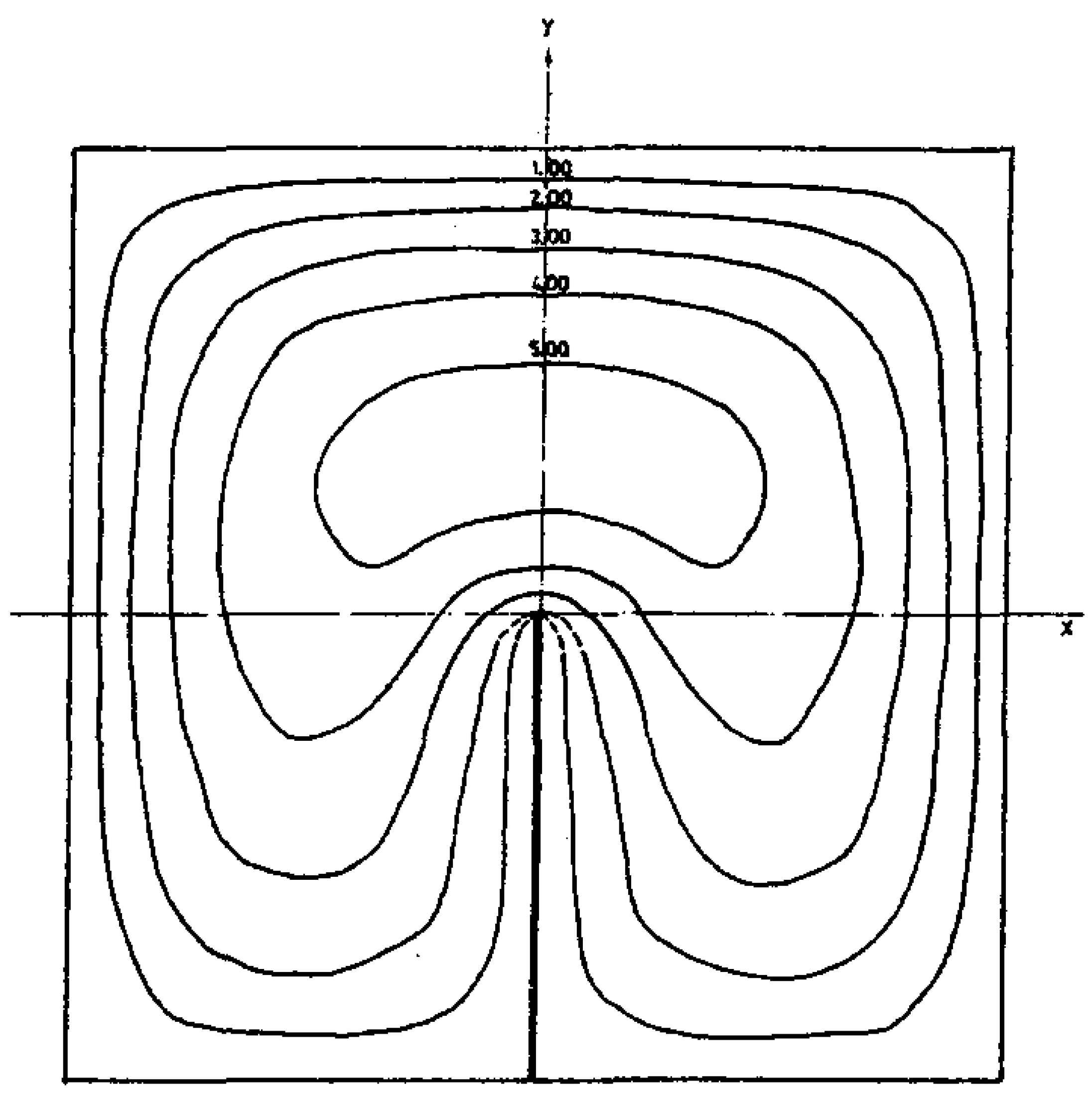

Fig. 2

The torsional moment $\mathfrak{M}_T = 2 \int\limits_{-a}^{+a} \int\limits_{0}^{+b} \Phi_1 \, dx \, dy + 4 \int\limits_{0}^{+a} \int\limits_{-h}^{0} \Phi_2 \, dx \, dy.$

$$\mathfrak{M}_T = \frac{8}{3} G\theta a^3 b + \frac{16a^2}{\pi^2} \cdot \sum_{n=1,3\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(n-1)}}{n^2} \left[A_n \operatorname{Sh} \frac{n\pi b}{2a} + B_n \left(\operatorname{Ch} \frac{n\pi b}{2a} - 1 \right) \right] +$$

$$+ \frac{2}{3} G\theta a^3 h + \frac{8a^2}{\pi^2} \cdot \sum_{m=1,3\ldots}^{\infty} \frac{1}{m^2} \left[C_m \operatorname{Sh} \frac{m\pi h}{a} + D_m \left(1 - \operatorname{Ch} \frac{m\pi h}{a} \right) \right] = 349 \, G\theta.$$

The torsional moment for cross section with oul crack

$$\mathfrak{M}_T = 0{,}1406 \, G\theta (2a)^4 = 596 \, G\theta$$

The reduction of rigidity

$$\frac{349 \, G\theta}{596 \, G\theta} = 0{,}59.$$

REFERENCES

[1] Witold Nowacki, *Teorija sprežystesci*, Panstwowe Wydawnictwe Naukowe, Warszawa 1970.