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REPRESENTATION THEOREM FOR MINIMAL σ-ALGEBRAS

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The purpose of this paper is to state some properties of minimal separating σ-algebras and of σ-compact topological spaces. Original motivation of this work is to consider the problem of existence of a minimal separating σ-algebra without any singleton. This fine problem, comes from a problem of statistics, is proposed by H. Morimoto who communicated me the following elementary but fundamental example of such a σ-algebra which appears in [18]:

Let X be an uncountable set and x be an element of X, then the σ-algebra consisting of subsets A of X with the property that "$x \in A$ and A is co-countable or $x \in A$ and A is countable" is minimal separating and does not contain $\{x\}$.

In statistics, various σ-fields are considered as mathematical expressions of statistical experiments. In some special cases, one of the properties of the σ-fields with statistical relevance called "pairwise sufficiency" reduces to their separating property.

Existence of minimal pairwise sufficient σ-fields is of interest and the σ-field given at the outset of this paper is one such example. It naturally leads to the question as to whether any more examples exists and, further, how they are characterized, and these are exactly the problem treated here.

Considering the structure of the above example, it is natural to imagine that there are many other types of such σ-algebras, and this is realized by considering a natural correspondence between the notions of minimality of σ-algebras and σ-compactness of related topological spaces, and that of σ-complete 2-valued measures and limit points of σ-topological spaces.

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1. Definitions, notions and elementary properties

We begin with the notions and definitions of concepts needed for the descriptions and discussions below. A cardinal number k is called regular if it is not a λ sum of smaller cardinals for all $\lambda<k$. A set B consisting of subsets of X is called a k-algebra over X provided that it is closed under complementation and λ-union for all $\lambda<k$. ω_1-algebra is usually called a σ-algebra, that is, closed under complementation and countable union. k-algebra B is called a separating algebra if for any distinct elements x, y of X, there is a set A of B such that $x\in A$ but $y\notin A$, in other words if

$$\forall A\in B(x\in A\equiv y\in A)\rightarrow x=y.$$

k-algebra B is called minimal if it is minimal in the sense of set inclusion. A subset $\{G_i:i\in I\}$ of k-algebra B is called a generator of B if it is the smallest k-algebra containing the subset. For a k-algebra B the following two properties are equivalent:

(a) B is minimal separating,

(b) $\{G_i:i\in I\}$ is a generator of B if and only if it separates the points of X.

Let $\{G_i:i\in I\}$ be a generator of separating k-algebra B over X, and put

$$G_{i0}=G_i \text{ and } G_{i1}=X-G_i.$$

Then there is a natural correspondence j between X and a subset Y of 2^I which consists of functions with domain I and values in $2=\{0, 1\}$ in such a way that

$$j(x)(i)=k\equiv x\in G_{ik}.$$

By this correspondence j, the set X may be considered as a subset Y of 2^I and B may be considered as a k-algebra over Y with the generators

$$Y_{ik}=\{p\in Y:p(i)=k\},$$

because of the property

$$G_{ik}=j^{-1}(Y_{ik})$$

and inverse image keeps complementation and union. Of course such B is always a separating algebra.

Let a be a subset of I with the cardinality less than k, that is, $\#a<k$, by a neighbourhood of $p\in Y$ of 2^I we mean the set

$$U(p;\ a)=\{q\in Y:\forall i\in a(p(i)=q(i))\}.$$

The k-topology of Y is introduced by the system of neighbourhoods

$$U_p=\{U(p;\ a):a\subset I,\#a<k\}.$$

ω and ω_1-topology are usually called weak and σ-topology, respectively. A k-topological space Y, i.e. the subspace Y of 2^I with k-topology, is called λ-compact if for any function which associates p with its neighbourhood $U(p;\ a_p)$, there is a subset b of Y with $\#b<\lambda$ such that

$$Y\subset \bigcup_{p\in b} U(p;\ a_p).$$

It is well-known that this property is characterized by the following properties:

(a) Let $\{O_j : j \in J\}$ be an open covering of Y, then there is a subset b of J with $\# b < \lambda$ such that

$$Y \subset \bigcup_{j \in b} O_j.$$

A dual form of this expression is:

(b) Let $\{C_j : j \in J\}$ be a family of closed subsets of Y with less than λ intersection property, that is

$$\# b < \lambda \rightarrow \bigcap_{j \in b} C_j \neq \varnothing,$$

then their intersection is not empty, namely

$$\bigcap_{j \in J} C_j \neq \varnothing.$$

Let X and Y be k-topological spaces of 2^I and 2^J, respectively. Then a function $f: X \rightarrow Y$ is called uniformely continuous if there is a function

$$g: P_k(J) \rightarrow P_k(I)$$

where $P_k(I) = \{a \subset I : \# a < k\}$ such that for all $b \in P_k(J)$ and $p \in X$

$$U(p; g(b)) \subset U(f(p); b).$$

Let A be a subset of k-topological space Y of 2^I. Then a subset a of with $\# a < k$ is called a support of A if for every $p, q \in Y$,

$$\forall i \in a\,(p(i) = q(i)) \rightarrow p \in A \equiv q \in A.$$

A subset A with support is closed and open, i.e. a clopen set of Y. Let B^* be the set of all such subsets of Y. Then B^* is a k-algebra provided that k is a regular cardinal. It is also clear that B^* is a separating k-algebra including the k-algebra generated by its basic open sets.

Let Y be a k-topological space of 2^I. Then it is called a k-space if for any subset a of I with $\# a < k$, there is a subset b of Y with $\# b < k$ such that

$$Y \subset \bigcup_{p \in b} U(p; a).$$

By this definition, we have that k-compact k-topological space is a k-space.

2. k-compactness and minimality of k-algebras

We begin with an easy property of k-spaces.

LEMMA 1. In k-space Y of 2^I, the k-algebra B^* of sets with support of cardinality less than k coincides with the k-algebra B generated by the basic open sets of Y.

PROOF. Let A be an element of B^*, then there is a subset a of I with $\# a < k$ such that

$$A = \bigcup_{p \in A} U(p; a).$$

Since Y is a k-space, there is a subset b of Y with $\#b<k$ such that

$$\bigcup_{p\in A} U(p;\, a) = \bigcup_{p\in b} U(p;\, a) = \bigcup_{p\in b}\bigcap_{i\in a} Y_{ip(i)}.$$

By the definition of B, it is closed under less than k union and intersection. Therefore we have

$$A = \bigcup_{p\in b}\bigcap_{i\in a} Y_{ip(i)} \in B.$$

This means that $B=B^*$ by the inclusion mentioned above.

Next lemma reveals a property of minimality of k-algebras.

LEMMA 2. Suppose that the k-algebra B generated by all the basic open sets of k-topological space Y of 2^I ia a minimal separating k-algebra. Then Y is k-compact and hence a k-space.

PROOF. Suppose Y is not k-compact, then there is a function

$$p \to U(p;\, a_p)$$

such that for any subset b of Y with $\#b<k$, we have

$$Y - \bigcup_{p\in b} U(p;\, a_p) \neq \varnothing.$$

Let B_1 be the set of all A in B with the following property:

(1) There is a subset b of Y with $\#b<k$ such that for any $q, r\in Y$

$$q, r \notin \bigcup_{p\in b} U(p;\, a_p) \to q\in A \equiv r\in A.$$

Since k is a regular cardinal, we have that B_1 is a k-algebra. Now we shall show that B_1 is separating. Suppose $p\neq q$, then since B is separating, there is a set A of B such that $p\in A$ but $q\notin A$, so we have

$$p\in A\cap U(p;\, a_p) \text{ and } q\notin A\cap U(p;\, a_p).$$

By the definition of B_1, we have $A\cap U(p; a_p)\in B_1$ and so it is separating. By the minimality of B, we have $B=B_1$.

Since the basic open set Y_{i0} belongs to B for every i, we have a subset b_i of Y with $\#b_i<k$ such that

$$q, r \notin \bigcup_{p\in b_i} U(p;\, a_p) \to (q\in Y_{i0} \equiv r\in Y_{i0}).$$

Hence there is a function $s: I\to 2$ such that

$$Y - \bigcup_{p\in b_i} U(p;\, a_p) \subset Y_{is(i)}.$$

We consider a neighbourhood of s in k-topological space 2^I,

$$W(s,\, a) = \{p\in 2^I\colon\ i\in a\, (p(i) = s(i))\}.$$

For any subset a of I with $\# a<k$, we put

$$b=\bigcup_{i\in a} b_i,$$

then we have $\# b<k$ and

$$\emptyset\neq Y-\bigcup_{p\in b} U(p;\, a_p)\subset\bigcap_{i\in a} Y_{is(i)}=Y\cap W(s;\, a).$$

Now we shall show that s is an element of Y.
So suppose $s\notin Y$ and let p^* be a fixed element of Y. Let B_2 be the set of all A in B with the following property:

(2) There is a subset a of I with $\# a<k$ such that any for $q\in Y$

$$q\in W(s;\, a)\rightarrow(q\in A\equiv p^*\in A).$$

By the relation

$$W(s;\, \bigcup_{i\in c} a_i)=\bigcap_{i\in c} W(s;\, a_i),$$

we see that B_2 is a k-algebra. Now we shall show that B_2 is separating. let p, q be two elements of Y such that $p\neq q$. Then we have $p\neq p^*$ or $q\neq p^*$, so we may assume $p\neq p^*$. Since p^*, q, $s\neq p$, there is a subset a of I with $\# a<k$ such that

$$q,\, p^*\not\subset U(p;\, a),\ U(p;\, a)\cap W(s;\, a)=\emptyset.$$

This means that for any $r\in Y$, we have

$$r\in W(s;\, a)\rightarrow(r\in U(p;\, a)\equiv p^*\in U(p;\, a)).$$

By the definition of B_2, we have $U(p;\, a)\in B_2$ and $q\notin U(p;a)$. This means that B_2 is separating and so by minimality of B, we have $B=B_2$. By $p^*\neq s$, there is a subset a of I with $\# a<k$ such that

$$U(p^*;\, a)\cap W(s;\, a)=\emptyset.$$

Since $U(p^*;\, a)\in B_2$, there is a subset a_1 of I with $\# a_1<k$ such that

$$r\in W(s,\, a_1)\rightarrow(r\in U(p^*;\, a)\equiv p^*\in U(p^*;\, a)).$$

We consider a point

$$q^*\in Y-\bigcup_{p\in c^*} U(p;\, a_p)\subset Y\cap W(s;\, a\cup a_1)$$

where $c^*=\bigcup_{i\in a\cup a_1} b_i$, then by $q^*\in W(s;\, a\cup a_1)\subset W(s;\, a)$, we have

$$q^*\in U(p^*;\, a).$$

This contradicts with

$$U(p^*;\, a)\cap W(s;\, a)=\emptyset.$$

This contradiction shows that $s\in Y$, that is Y is close.

Now we consider the neighbourhood $U(s;\, a_1)$. Then by putting

$$b^*=\bigcup_{i\in a_s} b_i$$

we have $\#b^*<k$ and

$$Y-\bigcup_{p\in b^*}U(p;a_p)\subset U(s;a_s)=Y\cap W(s;a_s).$$

This means that

$$Y\subset\bigcup_{p\in b^*}U(p;a_p)\cup U(s;a_s)$$

which contradicts to the choice of $U(p, a_p)$. Hence Y is k-compact.

LEMMA 3. Let X and Y be k-topological spaces of 2^I and 2^J. If X is k-compact and $f: X\to Y$ is continuous, then f is uniformly continuous and the image $f(X)$ is k-compact.

PROOF. Let $f: X\to Y$ be continuous and a be a subset of J with $\#a<k$. By the continuity of f, there is a subset a_p of I such that

$$f(U(p;a_p))\subset U(f(p);a).$$

By the k-compactness of X, we have a subset $b(a)$ of X with $\#b(a)<k$ such that

$$X\subset\bigcup_{p\in b(a)}U(p;a_p).$$

Now we put

$$a^*=\bigcup_{p\in b(a)}a_p.$$

For $q\in X$, there is $p\in b(a)$ such that $q\in U(p;a_p)$, so we have

$$r\in U(q;a^*)\subset U(p;a_p)\to f(r)\in U(f(p);a).$$

This means that

$$f(U(q;a^*))\subset U(f(p);a)=U(f(q);a),$$

so f is uniformly continuous.

To each q in $f(X)$, let there correspond b_q, any subset of J with $\#b_q<k$, and consider the function

$$q\to V(q;b_q)$$

defined on $f(X)$. By the continuity of f, there is a similar

$$p\to U(p;a_p)$$

on X such that

$$f(U(p;a_p))\subset V(f(p);b_{f(p)}).$$

Since X is k-compact, we have a subset c of X with $\#c<k$ such that

$$X\subset\bigcup_{p\in c}U(p;a_p).$$

Hence we have

$$f(X)\subset\bigcup_{p\in c}f(U(p;a_p))\subset\bigcup_{p\in c}V(f(p);b_{f(p)}).$$

This means that $f(X)$ is k-compact.

LEMMA 4. Let X and Y be k-topological spaces of 2^I and 2^J. If X is k-compact and $f: X \to Y$ is a 1—1 onto continuous function, then $f^{-1}: Y \to X$ is uniformly continuous.

PROOF. Let A be a closed subset of X. Then A is k-compact as a closed subset of X, so $f(A)$ is k-compact and so a closed subset of Y. This means that the image of a closed set is closed, and since f is 1—1 onto, the image of open set is open. By the relation $f(U)=(f^{-1})^{-1}(U)$, we have that the inverse image of an open set U by f^{-1} is open. This means that the function f^{-1} is contionuous. Since $Y=f(X)$ is k-compact, f^{-1} is uniformly continuous.

LEMMA 5. Let X be a k-compact subset of 2^I with k-topology. Then the k-algebra $\underline{B}$ generated by the basic open sets of X is a minimal k-algebra.

PROOF. Let $\{G_j: j \in J\}$ be a separating subset of B. By B^* we denote the k-algebra generated by $\{G_j: j \in J)$. Now we define a function $f: X \to 2^J$ by the relation

$$f(p)(j)=k \equiv p \in G_{jk}$$

where $G_{j0}=G_j$ and $G_{j1}=X-G_j$. Since X is k-compact and so a k-space, we have that every element of B has a support. This means that the above function f is continuous. Since $\{G_j : j \in J\}$ is separating, f is 1—1. Let Y be $f(X)$, then

$$f: X \to Y$$

is a 1—1 onto continuous function. Hence its inverse

$$f^{-1}: Y \to X$$

is uniformly continuous. This means that for any i of I, there is a subset b_i of J with $\# b_i < k$ such that

$$f^{-1}(V(f(p); b_i)) \subset U(p; \{i\}).$$

By the compactness of Y, there is a subset c of Y with $\# c < k$ such that

$$Y = \bigcup_{q \in c} V(q; b_i).$$

Hence by the relation

$$p \in f^{-1}(V(q; \{k\})) \equiv f(p)(k) = q(k).$$

we have

$$f^{-1}(V(f(p); b_i)) = \bigcap_{k \in b_i} f^{-1}(V(f(p); \{k\})) = \bigcap_{k \in b_i} G_{k f(p)(k)}.$$

Hence by the definition of B^*, we have

$$U(p; \{i\}) = \bigcup_{\substack{f(q) \in c \\ q(i)=p(i)}} \bigcap_{k \in b_i} G_{k f(q)(k)} \in B^*.$$

Since $\{U(p; \{i\} : i \in I\}$ is a generator of B, we have $B=B^*$. This means that every separating subsets of B is a generator of B, hence B is a minimal separating k-algebra.

Combining these lemmas, we have the following

THEOREM. Let X be a subset of 2^I with k-topology. Then the k-algebra generated by the basic open sets of X is minimal separating if and only if X is k-compact.

3. Examples and remarks

Let X be a totally disconnected k-complete topological space, that is, any distinct points of X are separated by a clopen set and the intersection of less than k open sets is again an open set. Let $\{G_i : i \in I\}$ be a separating clopen basis of X. Then X can be considered as a subspace of 2^I with k-topology. In the k-topological space 2^I, the element of $\bar{X}$, the closure of X, means a k-additive 2-valused measure on k-algebra B determined by the basic open sets of X. The cannonical relation of point p of $\bar{X}$ and measure μ_p is

$$p \in \bar{A} \equiv \mu_p(A) = 1.$$

Since in the k-topological space X, we have the relation

$$\overline{(\bigcup_{\nu \in a} A_\nu)} = \bigcup_{\nu \in a} \bar{A}_\nu$$

for every a with $\# a < k$, the additivity condition follows.. And if $A \in B$, then it has a support a with $\# a < k$ and so

$$\bar{A} \cap \overline{X - A} = \varnothing .$$

Conversely any k-additive 2-valued measure $\mu : B \to 2$ determines an element of 2^I by $p(i) = 1 - \mu(G_i)$ which belongs to the closure of X in 2^I. Hence the closure $\bar{X}$ is just the set of all k-additive 2-valued measures on B. An element of X is called a principal or a point measure and an element of $\bar{X} - X$ is a non-principal measure. The notion of k-additive 2-valued measure and k-complete maximal filter or ideal are considered as alternating expressions of the same concept by considering the element of $2 = \{0,1\}$ as quantity 0,1 or as truth value 0=falsity, 1 = truth.

Next, we shall give some examples of k-compact sets by showing the following lemma

LEMMA 6. Let λ be a cardinal number. Then the set

$$X_\lambda = \{f \in 2^I : \# \{i \in I : f(i) = 1\} \leqslant \lambda\}$$

is k-compact in the k-topological space 2^I if and only if

$$\forall \eta < k \ (\eta^\lambda < k).$$

PROOF. First, if there is some η which satisfies

$$\lambda \leqslant \eta < k \leqslant \eta^\lambda$$

then X_λ is not k-compact. Because we can take a subset a of I with $\# a = \eta$, then we have

$$\# \{f \in 2^a : \# \{i \in a : f(i) = 1\} \leqslant \lambda\} = \eta^\lambda \geqslant k.$$

We associate a neighbourhood $U(p; a)$ for every p of X_λ, then no less than k union cover X_λ and so it is not k-compact. The case $k \leqslant \lambda$ is trivial.

Next, we shall show that $\forall \lambda < k (\eta^\lambda < k)$ implies the k-compactness of X_λ. Let $U(p; a_p)$ be a given neighbourhood of p in X_λ. We shall show X_λ can be covered by less than k union of $U(p; a_p)'s$. Let f be a function with the domain in I and values in 2, we denote by f^* the function $f^* : I \to 2$ defined by

$$f^*(i) = \begin{cases} f(i) & \text{if } i \in \mathrm{dom}(f) \\ 0 & \text{if } i \in I - \mathrm{dom}(f) \end{cases}$$

By the induction on ν, we define a subset a_ν of I as follows

$$a_0 = a_{\varnothing^*}$$

where $\varnothing$ is the empty function. For a successor ordinal, we put

$$a_{\nu+1} = \bigcup_{f \in X_\lambda} a_{(f | a_\nu)^*}$$

where $f|a$ is the restriction of f to a. For a limit ordinal, we put

$$a_\nu = \bigcup_{\tau < \nu} a_\tau.$$

We shall show that $\# a_\nu < k$ for all $\nu < \lambda^+$, the smallest cardinal greater than λ. Since the case that ν is a limit ordinal is clear by the regularity of k and $\lambda^+ \leqslant 2^\lambda < k$, we shall show that $\# a_\nu < k$ implies $\# a_{\nu+1} < k$. So we consider the set

$$d_\nu = \{ f | a_\nu : f \in X_\lambda \},$$

then by the assumption of $\# a$ and by the property of λ, we have $\# d_\nu \leqslant \# a_\nu{}^\lambda < k$, so using $\# a_{(f | a_\nu)^*} < k$ for each $f \in X_\lambda$, we have

$$\# a_{\nu+1} \leqslant \sum_{f \in d_\nu} \# a_{f^*} < k.$$

Next, we consider two cases

(1) $$a_{\nu+1} - a_\nu = \varnothing \text{ for some } \nu < \lambda^+,$$

(2) $$a_{\nu+1} - a_\nu = \varnothing \text{ for all } \nu < \lambda^+.$$

The case (1): For every $f \in X_\lambda$, we have

$$a_{(f | a_\nu)^*} \subset a_{\nu+1} = a_\nu.$$

For any $f \in X_\lambda$, consider $(f | a_\nu)^* \in X_\lambda$, then by (1), we have

$$f \in U((f | a_\nu)^*; a_{(f | a_\nu)^*}),$$

so we obtain that

$$X_\lambda \subset \bigcup_{f \in d_\nu} U(f^*; a_{f^*}).$$

This means that X_λ can be covered by less than k union of given neighbourhoods.

The case (2): We put

$$A_\nu=\{f\in X_\lambda : a_{(f\mid a_\nu)^*}-a_\nu\neq\Phi\},$$

and assume that

$$f\in\bigcap_{\nu>\lambda+}A_\nu.$$

Then we have for all $\nu<\lambda^+$,

$$a_{(f\mid a_{\nu+1})^*}-a_{\nu+1}\neq\Phi.$$

Suppose that $f(i)=0$ for all $i\in a_{\nu+1}-a_\nu$, then

$$(f\mid a_{\nu+1})^*=(f\mid a_\nu)^*.$$

Hence we have

$$a_{(f\mid a_{\nu+1})^*}=a_{(f\mid a_\nu)^*}\subset a_{\nu+1}.$$

This contradiction shows that for all $\nu<\lambda^+$, there exists an $i\in a_{\nu+1}-a_\nu$ such that $f(i)=1$. This means that

$$\#\{i\in I:f(i)=1\}>\lambda,$$

which contradicts the assumption $f\in X_\lambda$, and so we have

$$X_\lambda=X_\lambda-\bigcap_{\nu<\lambda+}A_\nu=\bigcup_{\nu<\lambda+}(X_\lambda-A_\nu).$$

Let $f\in X_\lambda-A_\nu$. Then we have that $a_{(f\mid a_\nu)^*}\subset a_\nu$ and so

$$X_\lambda-A_\nu\subset\bigcup_{f\in d_\nu}U(f^*;a_{f^*}).$$

Hence we have

$$X_\lambda\subset\bigcup_{f\in d^*}U(f^*;a_{f^*})$$

where $d^*=\bigcup_{\nu<\lambda+}d_\nu$ and the condition $\#d^*<k$ follows from $\lambda^+\leqslant 2^\lambda<k$. Therefore X_λ can be covered by less than k unicn of given neighbourhoods. Any way, the space X_λ is k-compact.

By the proof of above lemma, we have that if a family D of subsets of I satisfies the condition

(1) $$a\in D,\ b\subset a\rightarrow b\in D,$$

(2) $$\#a<k\rightarrow\#\{b\in D:b\subset a\}<k,$$

then the set of all representing functions of the sets in D is k-compact. For example, if D satisfies the conditions and a partial ordering $\leqslant$ is defined on I, then the set D' of elements of D which is well-ordered or linearly ordered by $\leqslant$, satisfies this condition. Hence if $k=(2^\omega)^+$, then the set of all well-ordered countable subsets of I is k-compact. But of course this set is not ω_1-compact, namely not σ-compact, if I includes a countable increasing sequence.

By using Lemma 6 and the property that the continuous image of a k-compact set is k-compact, we have that

$$X_{f,\lambda}=\{q\in 2^I:\#\{i\in I:q(i)\neq f(i)\}\leqslant\lambda\}$$

is k-compact for any $f: I \to 2$ and λ such that $\forall \eta < k (\eta^\lambda < k)$. Hence any closed subset of less than k union of such $X_{f,\lambda}$ is also k-compact.

Now, we consider the case $k > \omega$, for example $k = \omega_1$, then for any $\eta < k$ and $n < \omega$, we have $\eta^n = \eta < k$. By this, the set

$$X^*_\omega = \{f \in 2^I : \#\{i \in I : f(i) = 1\} < \omega\},$$

being a union of countable k-compact sets, is k-compact. Since each point of this set is not an isolated point, the k-algebra determined by this- k-compact set is an example of a minimal separating k-algebra without a singleton. Since there is no restriction on the cardinality of index set I, each cardinality determines at least one non isomorphic minimal separating k-algebra without a singleton. One may consider, may be pathological, the k-compact space consisting of all finite sets, in which case the minimal separating k-algebra consists of elements which are not sets but classes.

Now we consider, for example, the space ω^I with k-topology. Since each natural number n of ω can be considered as an element of 2^ω by usual binary expansion, we may consider

$$\omega^I \subset (2^\omega)^I = 2^{\omega \times I}$$

So we have that the set

$$X^\#_\omega = \{f \in \omega^I : \#\{i \in I : f(i) \neq 0\} < \omega\}$$

is k-compact, and the k-algebra generated by its basic open sets

$$\{f \in X^\#_\omega : f(i) = n\}$$

is minimal separating and have no singleton.

One intuitive example of minimal separating σ-algebra would be as follows: Suppose there are at most countably many particles and their states, the family X of all positions and states of finite particles in, for example, n-dimensional Euclidean space forms a σ-compact set, and the σ-algebra determined by this topological space is minimal separating σ-algebra without singleton.

We consider the property

$$(*) \qquad \forall \eta < k (\eta^\lambda < k).$$

If $\lambda < k$ and k is regular, then (*) implies

$$k^\lambda = \sum_{\eta < k} \eta^\lambda < k^2 = k < k^+$$

hence k^+ also satisfies the property (*). On the other hand, if $cf(k)$, the cofinality of k, satisfies $cf(k) \leq \lambda$, then $cf(k^\lambda) > \lambda$, by König's theorem, so we have $k^+ \leq k^\lambda$ and so k^+ does not satisfy the property (*). The least cardinal greater than η_0 satisfying (*) is defined by $k = (\eta_0{}^\lambda)^+$, because $(\eta_0{}^\lambda)^\lambda = \eta_0{}^\lambda < k$.

If for example the continuum hypothesis $2^\omega = \omega_1$ is true, then

$$\{f \in 2^I : \#\{i \in I : f(i) = 1\} \leq \omega\}$$

is k-compact for $k = \omega_2, \omega_3, \ldots$ but not for $k = \omega_1, \omega_{\omega+1}, \ldots$.

Interesting problem concerning this is the problem of implication

$$\forall n<\omega\,(2^{\omega}=\omega_{n+1})\rightarrow 2^{\omega_\omega}=\omega_{\omega+1}$$

which is proposed by R. M. Solovay, and is called the singular cardinals problem And this problem is equivalent to $\omega_{\omega+2}$-compactness of above set X_ω under the assumption of

$$\forall n<\omega\,(2^{\omega_n}=\omega_{n+1})$$

Another interesting problem is explicit characterization of ω_1 or σ-compact sets, for example the existence of σ-compact set which is not included in the continuous image of the set of the form $X_\omega^{\#}$. And the characterization of the structure of complete Boolean algebra determined by closed subsets of 2^I divided by the ideal of k-compact sets.

The case $k=\omega$ is well-known case of weak topology, by Tihonov theorem the topological space 2^I is compact, hence a subset is compact if and only if it is closed. This means that the ω-algebra (Boolean algebra of clopen sets) B generated by basic open sets of X is minimal separating if and only if X is closed. There is natural correspondence between the closure $\bar{X}$ of X and the set B^* of all maximal filters (or ideals) of B.

We have already mentioned that every k-compact k-topological space X of 2^I is a k-space. Now we consider the problem of converse implication. That is, whether every closed k-space in k-topological space in 2^I is k-compact or not.

When $I=k$, this property is known as tree property. To explain about this, we define the notion of binary tree, here we say simply a tree. A subset T of $P=\bigcup_{\nu<k} 2^\nu$ is called a k-tree if the following conditions are satisfied:

(a) $$f\in T,\ g\in P,\ g\subset f\rightarrow g\in T,$$

(b) $$0<\#(T|\nu)<k \text{ where } T|\nu=\{f\in T:\mathrm{dom}(f)=\nu\} \text{ and } \nu<k.$$

A function $f:k\rightarrow 2$ is called a total branch of T if

$$\forall\nu<k\,(f|\nu\in T).$$

We say that a cardinal k have the tree property if every k-tree has a total branch.

LEMMA 7. k has tree property if and only if every closed k-space in the k-topological space 2^k is k-compact.

PROOF. Let T be a k-tree without any total branch. For any $f\in T$, we associate a function $f^*:k\rightarrow 3$ defined by

$$f^*(\nu)=\begin{cases} f(\nu) \text{ if } \nu\in\mathrm{dom}(f)\\ 2 \text{ if } \nu\in k\text{-}\mathrm{dom}(f).\end{cases}$$

Then by the inclusion $3\subset 2^2$, we may consider f^* as an element of 2^k by $3^k\subset(2^2)^k=2^{2\times k}=2^k$. Now we consider a subset T^* of 2^k defined by

$$T^*=\{f^*\in 2^k:f\in T\}.$$

Then T^* is a k-space. Since T has no total branch, T^* is a closed subset of 2^k and each point of which is an isolated point. Let $U(p; a_p)$ be a neighbourhood of p in T^* with

$$U(p; a_p) \cap T^* = \{p\}.$$

Then by $\# T^* = k$, we have that T^* cannot be covered by less than k union of such neighbourhoods. This means that T^* is a closed k-space which is not k-compact.

Next, suppose k has the tree property and X be a closed k-space which is not k-compact. Let $\{U(p; \nu_p) : p \in X\}$ be a covering of X by which X cannot be covered by less than k union of the sets. Let T be the set of functions defined by

$T = \{f | \nu : f \in X$, T cannot be covered by $<k$ of $U(p; \nu_p)$'s$\}$. Since X is a k-space which is not k-compact, T is a k-tree. Hence, by tree property of k, T has a total branch $f : k \to 2$. But since X is closed, we have $f \in X$. This means that $f \in U(f; \nu_f)$ and so $f | \nu_f \notin T$, which is a contradiction.

Followings are known examples about this notion:

(1) ω has tree property. This is known as König's infinity lemma or Brower's fan theorem and is a special case of Tihonov's compactness theorem.

(2) ω_1 does not have tree property. Such an example is known as Aronszajn tree

(3) (Specker) if a regular cardinal k satisfies $\forall \nu < k (2^\nu \leqslant k)$, then k^+ does not have the tree property.

(4) (J. Silver) if k is a real valued measurable then k has tree property.

It is known, by R. M. Solovay, that the consistency of existence of 2-valued measurable cardinal and that of real-valued measurable cardinal are equivalent under *ZFC*, Zermelo-Fraenkel set theory with axiom of choice. And every real--valued measurable cardinal is weakly inaccessible cardinal less than or equal to 2^ω, every 2-valued measurable cardinal is strongly inaccessible, that is, k is regular and $\forall \nu < k (2^\nu < k)$.

In the case k is strongly inaccessible, every subset X of k-topological space 2^I is always a k-space, and the property

$$\forall \nu < k (\#(T | \nu) < k)$$

is always satisfied. In this case k is called weakly compact. That, is, a cardinal k is weakly compact if

$$2^k \text{ with } k\text{-topology is } k\text{-compact.}$$

Followings are known about this notion:

(1) the first strongly inaccessible, the first Mahlo cardinal is not weakly compact. More generally the first cardinal satisfying $\pi_1{}^1$ property is not weakly compact.

(2) every measurable cardinal is weakly compact and it is a limit of weakly compact cardinals.

J. Silver proved that the consistency of existence of weakly compact cardinal implies the consistency of

"ω_2 as tree property"

with the exioms of set theory *ZFC*.

More general case is considered and it is called strongly compact, or simply compact, cardinal if for any set I

$$2^I \text{ with } k\text{-topology is } k\text{-compact.}$$

This notion is also described by using tree like structures. A subset T of

$$P=\{f|a: a\subset I,\ \# a<k\}$$

is called a k-function tree if the following conditions are satisfied:

(a) $$f\in T,\ g\in P,\ g\subset f\rightarrow g\in T,$$

(b) $$0<\#\{f\in T: \mathrm{dom}(f)=a\}<k \text{ for } \nu<k \text{ and } \# a<k.$$

A function $f: I\rightarrow 2$ is called a total function of T if

$$\forall a\subset I(\# a<k\rightarrow f|a\in T).$$

We say that a cardinal k has the k-function tree property if every k-function tree has a total function. For strongly inaccessible cardinals, strong compactness is equivalent to function tree property. For example, we know the followings;

(1) every strongly compact cardinal is measurable.

(2) (Vopenka-Hrbacek) if strongly compact cardinal exists then $V\neq L(a)$ for every set a.

(3) (R. Solovay) $2^\lambda=\lambda^+$ for every singular strong limit cardinal greater than a compact cardinal.

(4) if there exists a strongly compact cardinal, then the first strongly compact cardinal can be the first measurable cardinal.

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