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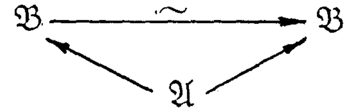
HOMOGENEOUS-UNIVERSAL MODELS OF THEORIES WHICH HAVE MODEL COMPLETIONS

Žarko MIJAJLOVIĆ

1. Introduction

In the present work our attention is turned to those Jónnson classes of models which are classes of models of theories which have model completions. Main reason for that lies in the fact that the class of models of a theory which has the model completion is almost a Jónnson class, therefore that part of model theory which concern model completions may be applied in full power. In such sense this paper is closly related to the works of others as of M. Yasuhara [6], Comfort-Negreponties [3] etc. (relatively complete list of references on the subject can be found in the works just cited). The terminology that is used in this paper is mostly according to [2] and [5], however we repeat some of it, since it is not uniquely determined in general, and also some assumptions and conventions are introduced.

A language is denoted by L, the language of a theory T by $L(T)$ and of a model $\mathfrak{A}$ by $L(\mathfrak{A})$. It is assumed throughout that $L(T)$ is countable and that T has infinite models. Universes of models $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ are denoted by A, B, C respectively, and the cardinal number of A by $|A|$. By $\mathfrak{M}(T)$ is denoted the class of all models of T. As usual $\mathfrak{A} < \mathfrak{B}$ means that $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$ and $\mathfrak{A} <_1 \mathfrak{B}$ states the fact that $\mathfrak{B}$ is an existential extension of $\mathfrak{A}$ (i.e. $\mathfrak{A} \subseteq \mathfrak{B}$ and for every existential formula ψ and valuation $\mathbf{a}$ in A $\mathfrak{A} \models \psi[\mathbf{a}]$) iff $\mathfrak{B} \models \psi[\mathbf{a}]$. Symbol $\bar{a}$ stands for a sequence $a_1, a_2, \ldots, a_n$ if the subscript n is of no importance in the consideration. So if f is a function, then $f\bar{a}$ stands for $fa_1, fa_2, \ldots, fa_n$. The arrow in a diagram $\mathfrak{A} \to \mathfrak{B}$ represents an unnamed embedding $f\colon \mathfrak{A} \to \mathfrak{B}$ and similiarly $\xrightarrow{\sim}$, $\xrightarrow{<}$ represent an (unnamed) isomorphism and an elementary embedding respectively. If an arrow has more then one occurence in a diagram, then each occurrence of the arrow may represent a different embedding. A name of an element $a \in A$ is denoted by $\underline{a}$. A model $\mathfrak{A}$ is an universal model of T if it is a model of T and if for every $\mathfrak{B} \models T$, $|B| \leqslant |A|$, $\mathfrak{B}$ is embeddable into $\mathfrak{A}$. A model $\mathfrak{B}$ of T is a homogeneous model of T if for every $\mathfrak{A} \models T$, $|A| < |B|$, the diagram $\mathfrak{B} \leftarrow \mathfrak{A} \to \mathfrak{B}$ can be



completed to the shown commutative diagram.

A model $\mathfrak{A}$ is an elementary universal model of T if $\mathfrak{A}\models T$ and for every model $\mathfrak{B}$, $\mathfrak{B}\equiv\mathfrak{A}$ and $|A|\leqslant|B|$ implies $\mathfrak{B}\overset{<}{\to}\mathfrak{A}$. A model $\mathfrak{A}$ is an elementary homogeneous model of T if $\mathfrak{A}\models T$ and for every set $X\subseteq A$, $|X|<|A|$, and any map $p:X\to A$ $(\mathfrak{A},x)_{x\in X}\equiv(\mathfrak{A},px)_{x\in X}$ implies the existence of an automorphism $f:\mathfrak{A}\overset{\sim}{\to}\mathfrak{A}$ such that $f\upharpoonright X=p$. With $T_\forall$, $T_{\forall\exists}$ are denoted respectively the sets of universal, universal-existential sentences which are consequences of T. We state well known basic facts which relate theories $T_\forall$, $T_{\forall\exists}$ to the threory T.

THEOREM 1.1. 1° $\mathfrak{A}\models T_\forall$ *iff there is* $\mathfrak{B}\models T$ *such that* $\mathfrak{A}\subseteq\mathfrak{B}$.

2° $\mathfrak{A}\models T_{\forall\exists}$ *iff there is* $\mathfrak{B}\models T$ *such that* $\mathfrak{A}<_1\mathfrak{B}$. $\dashv$

In connection with this theorem, we remark that in general the following holds: $\mathfrak{A}\models T_{\Pi_{n+1}}$ iff there is $\mathfrak{B}\models T$ such that $\mathfrak{A}<_n\mathfrak{B}$, where T_{Π_n} is the set of all Π_n^0 consequences of T and $\mathfrak{A}<_n\mathfrak{B}$ means that $\mathfrak{A}\subseteq\mathfrak{B}$ and for every Π_n^0 formula ψ and assignment $\boldsymbol{a}$ in A $\mathfrak{A}\models\psi[\boldsymbol{a}]$ iff $\mathfrak{B}\models\psi[\boldsymbol{a}]$.

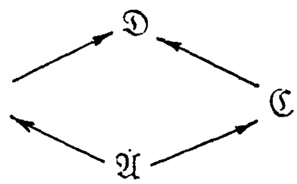
For convenience we repeat the definition of a notion of Jónnson class of models (for basic properties of Jónnson classes see for example [1] and [3]). A class K of models of a language L is a Jónnson class if K satisfies the following conditions:

1° K contains models of arbitrarily large cardinals.

2° K is closed under isomorphic images.

3° K has the joint embedding property (JE): For any $\mathfrak{A},\mathfrak{B}\in K$ there is $\mathfrak{C}\in K$ such that $\mathfrak{A}\to\mathfrak{C}\leftarrow\mathfrak{B}$.

4° K has the amalgamation property (AP): For any $\mathfrak{A},\mathfrak{B},\mathfrak{C}\in K$ diagram $\mathfrak{B}\leftarrow\mathfrak{A}\to\mathfrak{C}$ can be amalgamated to the commutative diagram. In the terminology of M. Yasuhara [6] every $\mathfrak{A}$ is amalgamative in K.



5° K is closed under union of chains of models.

$\mathring{6}_k$ For any $\mathfrak{A}\in K$ and $X\subseteq A$, $|X|<k$, there is $\mathfrak{B}\subseteq A$, $\mathfrak{B}\in K$, $|B|<k$ such that $X\subseteq B$ (k is an infinite cardinal).

Under cited conditions, as B. Jónnson has shown (1960), if $k=k^{\underset{\smile}{k}}$ then K contains an universal-homogeneous model for K. In this paper it is assumed that K is an elementary class i.e. $K=\mathfrak{M}(T)$ for some T. If $\mathfrak{M}(T)$ is a Jónnson class we say simply that T is a Jónnson theory (similar convention is applied to any property P which concern the class $\mathfrak{M}(T)$ By LST (Löwenheim-Skolem-Tarski) theorem, T satisfies 1° and 6° for $k\geqslant\omega_1$. By Chang-Los-Suzko preservation theorem T has property 5° iff T has universal-existential axiomatization. Hence, the really problem that may occur is "Does T have JE and AP?".

The property JE can be syntatically described.

PROPOSITION 1.2. A theory T has *JE* iff the following holds: If θ, ψ are basic formulas (i.e. conjunctions of atomic and negations of atomic formulas) then the consistency of theories $T+\exists\vec{x}\,\theta$, $T+\exists\vec{y}\,\psi$ implies the consistency of of $T+\exists\vec{x}\,\theta+\exists\vec{y}\,\psi$.

PROOF ($\Rightarrow$) Let $\mathfrak{A},\mathfrak{B}\models T$ such that $\mathfrak{A}\models\exists\vec{x}\,\theta$, $\mathfrak{B}\models\exists\vec{y}\,\psi$ where θ,ψ are basic formulas. By *JE* there is $\mathfrak{C}\models T$ such that $\mathfrak{A}\rightarrow\mathfrak{C}\leftarrow\mathfrak{B}$, hence $\mathfrak{C}\models T+\exists\vec{x}\,\theta+\exists\vec{y}\,\psi$.

($\Leftarrow$) Let $\mathfrak{A},\mathfrak{B}\models T$ and assume that there is no $\mathfrak{C}\models T$ such that $\mathfrak{A}\rightarrow\mathfrak{C}\leftarrow\mathfrak{B}$. Then the theory $\Gamma=T+\Delta(\mathfrak{A})+\Delta(\mathfrak{B})$ ($\Delta(\mathfrak{A})$ is the diagram of $\mathfrak{A}$), is inconsistent, hence there are basic formulas $\theta(\vec{x})$, $\theta(\vec{y})$ and $\vec{a}\in A$, $\vec{b}\in B$ such that $\theta(\vec{a})\in\Delta(\mathfrak{A})$ and $\psi(\vec{b})\in\Delta(\mathfrak{B})$ so that $T+\theta(\vec{a})+\psi(\vec{b})$ is inconsistent. Hence $T\vdash\theta(\vec{a})\Rightarrow\neg\psi(\vec{b})$ so $T\vdash\forall\vec{x}\forall\vec{y}\neg(\theta(\vec{x})\wedge\psi(\vec{y}))$, $\{x_1,\dots,x_n\}\cap\{y_1,\dots,y_m\}=\varnothing$. Therefore $T\vdash\neg(\exists\vec{x}\,\theta(\vec{x})\wedge\exists\vec{y}\,\psi(\vec{y}))$ and $\mathfrak{A}\models\exists\vec{x}\,\theta$, $\mathfrak{B}\models\exists\vec{y}\,\psi$, but this contradicts our hypothesis. $\dashv$

COROLLARY 1.3. Assume that any two countable models of T can be embedded into a model of T. Then T has *JE*.

PROOF Let θ, ψ be basic formulas and assume that $T+\exists\vec{x}\,\theta$, $T+\exists\vec{y}\,\psi$ are consistent theories. By *LST* theorem there are countable models $\mathfrak{A}$, $\mathfrak{B}$ of T such that $\mathfrak{A}\models\exists\vec{x}\,\theta$, $\mathfrak{B}\models\exists\vec{y}\,\psi$. By *JE* for countable models there is $\mathfrak{C}\models T$ so that $\mathfrak{A}\rightarrow\mathfrak{C}\leftarrow\mathfrak{B}$. Then $\mathfrak{C}\models\exists\vec{x}\,\theta\wedge\exists\vec{y}\,\psi$ so $T+\exists\vec{x}\,\theta+\exists\vec{y}\,\psi$ is a consistent theory. $\dashv$

In some cases properties *JE* and *AP* are transferred from one theory to another. Let us see some examples of such kind.

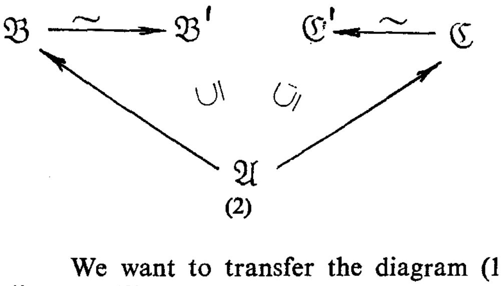
PROPOSITION 1.4. 1° T has *JE* iff $T_\forall$ has *JE*.
2° (M. Yasuhara, [6]) T has *AP* iff $T_{\forall\exists}$ has *AP*.

PROOF 1° ($\Rightarrow$) Assume that T has *JE*, and let $\mathfrak{A},\mathfrak{B}\models T_\forall$. Then there are $\mathfrak{A}',\mathfrak{B}'\models T$ such that $\mathfrak{A}\subseteq\mathfrak{A}'$, $\mathfrak{B}\subseteq\mathfrak{B}'$. T has *JE* so $\mathfrak{A}',\mathfrak{B}'$ can be embedded into a model $\mathfrak{C}\models T$. Since $\mathfrak{C}\models T_\forall$, $T_\forall$ has *JE*. ($\Leftarrow$) Proof is trivial.

2° Assume that T has *AP*. Let $\mathfrak{A},\mathfrak{B},\mathfrak{C}$ be models of $T_{\forall\exists}$ and

(1) $$\mathfrak{B}\supseteq\mathfrak{A}\subseteq\mathfrak{C}.$$

Remark It is sufficient to amalgamate diagrams of the form (1) since every diagram of the sort $\mathfrak{B}\leftarrow\mathfrak{A}\rightarrow\mathfrak{C}$ can be completed to the commutative diagram (2).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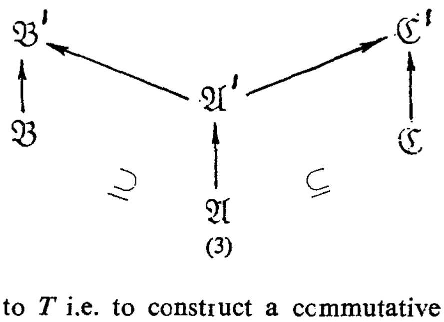

(3)

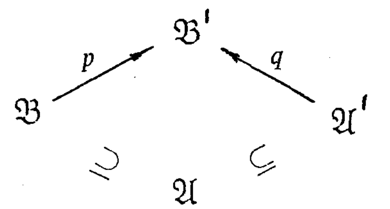
We want to transfer the diagram (1) to T i.e. to construct a commutative diagram (3).

The existence of the model $\mathfrak{A}'$ is provided by T.1.1, moreover it may be taken $\mathfrak{A} <_1 \mathfrak{A}'$. Now we prove that any diagram of the form

(4) $$\mathfrak{B} \supseteq \mathfrak{A} <_1 \mathfrak{A}'$$

can be amalgamated. Consider the theory $\Gamma = T + \Delta(\mathfrak{B}) + \Delta(\mathfrak{A}')$ where $\mathfrak{B} = (\mathfrak{B}, b, a)_{b \in B, a \in A}$, $\mathfrak{A}' = (\mathfrak{A}, a', a)_{a' \in A', a \in A}$. Γ is consistent theory. Assume it is not. In such a case there are basic formulas $\theta(\vec{z}, \vec{x})$, $\psi(\vec{y}, \vec{x})$ so that $\theta(\vec{b}, \vec{a}) \in \Delta(\mathfrak{B})$ and $\psi(\vec{a}'\ \vec{a}) \in \Delta(\mathfrak{A})$ for some $\vec{a} \in A$, $\vec{a}' \in A'$, $\vec{b} \in B$ and $T + \theta(\vec{b}, \vec{a}) + \psi(\vec{a}', \vec{a})$ is inconsistent. Hence $T \vdash \forall \vec{x}\vec{y}\vec{z}\,(\theta(\vec{z}, \vec{x}) \Rightarrow \neg\psi(\vec{y}, \vec{x}))$, so since the formula $\forall xyz\,(\theta(\vec{z}, \vec{x}) \Rightarrow \neg\psi(\vec{y}, \vec{x}))$ is universal, $\mathfrak{B} \models \theta(\vec{b}, \vec{a}) \Rightarrow \forall \vec{y}\,\neg\psi(\vec{y}, \vec{a})$, and thus $\mathfrak{B} \models \forall \vec{y}\,\neg\psi(\vec{y}, \vec{a})$. But $\mathfrak{A} <_1 \mathfrak{A}'$ so $\mathfrak{A}' \models \forall \vec{y}\,\neg\psi(\vec{y}, \vec{a})$ so $\mathfrak{A}' \models \neg\psi(\vec{a}', \vec{a})$, what is contradiction. Hence Γ has a model $\mathfrak{B}' = (\mathfrak{B}, c_b, c_{a'}, c_a)_{a \in A, b \in B, a' \in A'}$ and (4) is amalgamated to the diagram (5) where $p(b) = c_b$, $q(a') = c_{a'}$.

(5)



In similar way a model $\mathfrak{C}'$ is obtained with the required property and therefore the diagram (3). T has AP so $\mathfrak{B}' \leftarrow \mathfrak{A}' \rightarrow \mathfrak{C}'$ can be amalgamated and therefore $\mathfrak{B} \subseteq \mathfrak{A} \supseteq \mathfrak{C}$ can too.

($\Leftarrow$) Trivally holds. ⊣

COROLLARY 1.5. If T has JE and AP then $T_{\forall\exists}$ is a Jónnson theory. ⊣

If T has AP, it is not necessarily that T has too. For example, this case occur whenever T is model complete, but not submodel complete.

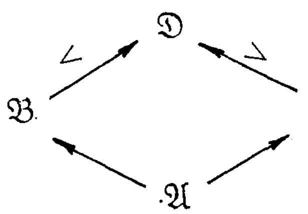
2. Full models

Now we consider those theories T which have model completion T^*. Hence, it is assumed (here and throughout) that T has a model completion. For convenience we repeat the definition of the notion of model completion (it was introduced by A. Robinskon, see [4], [5]). A theory T^* is model completion of T if the following holds:

1° Every model of T^* is a model of T.

2° Every model of T is a submodel of a model of T^*.

3° Any diagram $\mathfrak{B} \leftarrow \mathfrak{A} \rightarrow \mathfrak{C}$, $\mathfrak{A} \models T$, $\mathfrak{B}, \mathfrak{C} \models T^*$ can be amalgamated to the commutative diagram:



Some of basic properties of this notion are: If T has a model completion, then it is unique (up to logical equivalence). T^* is model complete and has universal-existential axiomatization.

It turns out that T and T^* have in common properties JE and AP.

THEOREM 2.1. 1° *T has JE iff T^* has JE.*

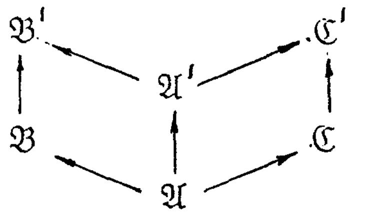
2° *T and T^* have AP.*

PROOF 1°($\Rightarrow$) Let $\mathfrak{A}, \mathfrak{B} \models T^*$ and assume that T has *JE*. Since $T^* \models T$ it follows $\mathfrak{A}, \mathfrak{B} \models T$ so there is $\mathfrak{C} \models T$ such that $\mathfrak{A} \to \mathfrak{C} \leftarrow \mathfrak{B}$. Since T^* is model completion of T, it can be chosen $\mathfrak{C} \models T^*$.

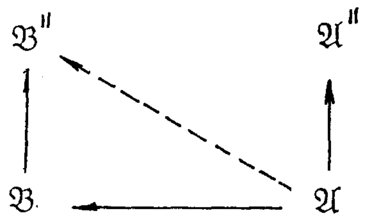
($\Leftarrow$) It follows immediately since every model of T is a submodel of T^*.

2° According to the property 3° of model completion and since every model of T^* is a model of T, it follows that T^* has *AP*.

Now, let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be models of T and assume that $\mathfrak{B} \leftarrow \mathfrak{A} \to \mathfrak{C}$. This diagram can be transferred into a diagram in T^* i.e. there is a commutative diagram (1). Existence of $\mathfrak{A}'$ is provided by property 2° of model completion. Further, there is a model $\mathfrak{B}''$ of T^* such that $\mathfrak{B} \to \mathfrak{B}''$. According to the property 3° diagram (2) can be amalgamated to the commutative diagram (3). In the similar way the model $\mathfrak{C}'$ is obtained, and so the diagram (1) exists. The diagram $\mathfrak{B}' \leftarrow \mathfrak{A}' \to \mathfrak{C}'$ can be amalgamated, so we have obtained commutative diagrams (4) and (5). ⊣



(1) $\mathfrak{A}', \mathfrak{B}', \mathfrak{C}' \models T^*$



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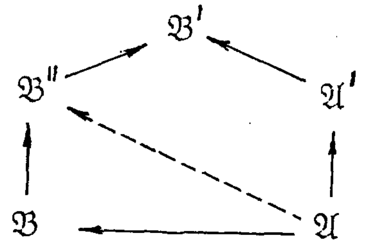

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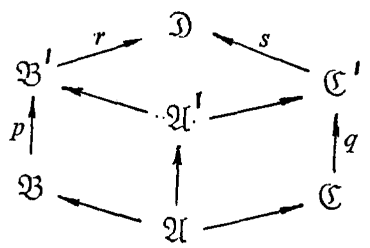

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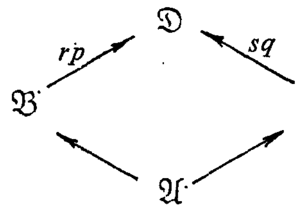


$\mathfrak{D} \models T^*$

(5)

COROLLARY 2.2. T^* *is the model completion of* $T_{\forall\exists}$. ⊣

It should be remarked that T^* in general is not a model completion of $T_\forall$ (but it is the model companion of $T_\forall$).

COROLLARY 2.3. (Test for a class to be a Jónnson class). Assume that a theory T has a model completion T^*, universal-existential axiomatization and a prime model. Then T is a Jónnson theory.

PROOF Closure of T under union of chains of models of T follows from universal-existential axiomatization and AP from the previous theorem. Since T has a prime model $\mathfrak{A}$ (i.e. $\mathfrak{A}$ is embeddable into every model of T), for any $\mathfrak{B}, \mathfrak{C} \models T$ a diagram $\mathfrak{B} \leftarrow \mathfrak{A} \rightarrow \mathfrak{C}$ exists and by AP it can be amalgamated, so T has JE. $\dashv$

Since T^* has universal-existential axiomatization and AP, it may lack only JE in order to be a Jónnson theory. Model complete theory T is model completion of itself, so if T has a prime model, then it is a Jónnson theory. We have assumed that T has model completion T^*, so AP is provided for T but JE is not in general. However the question of JE can be removed if the following relation $\simeq_T$ is introduced in $\mathfrak{M}(T)$.

DEFINITION 2.4. Models $\mathfrak{A}$, $\mathfrak{B}$ of T are *compatible* in T, $\mathfrak{A} \simeq_T \mathfrak{B}$, iff there is a model $\mathfrak{C}$ of T such that $\mathfrak{A} \rightarrow \mathfrak{C} \leftarrow \mathfrak{B}$. (Often the subscript T will be omitted in $\simeq_T$).

A model $\mathfrak{A}$ of T is a *semiuniversal* model of T if for any model $\mathfrak{B} \models T$ $\mathfrak{A} \simeq \mathfrak{B}$ and $|B| \leqslant |A|$ implies $\mathfrak{B} \rightarrow \mathfrak{A}$, that is, $\mathfrak{A}$ is an universal model in the class of all models compatible with $\mathfrak{A}$. A model $\mathfrak{A}$ of T is a *full* model of T if $\mathfrak{A}$ is semiuniversal and homogeneous model of T. A model $\mathfrak{A}$ of T is a *semiprime* model of T if it is prime in the class of all models of T compatible with $\mathfrak{A}$.

EXAMPLE 2.5. If T is the theory of fields, then the Galois field Z_p is semiprime model of T. Every algebraically closed field F of infinite transcedental degree over Z_p is semiuniversal and in fact a full model of T.

In the following proposition the basic properties of the relation $\simeq$ are given.

PROPOSITION 2.6. 1° $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{A} \simeq_T \mathfrak{B}$ for any theory T which has $\mathfrak{A}$, $\mathfrak{B}$ as models.

2° $\mathfrak{A} \rightarrow \mathfrak{B}$ implies $\mathfrak{A} \simeq \mathfrak{B}$.

3° The relation $\simeq$ is an equvalence relation in $\mathfrak{M}(T)$.

4° Assume that $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \models T$. If $\mathfrak{A} \equiv \mathfrak{B}$ and $\mathfrak{B} \simeq \mathfrak{C}$ then $\mathfrak{A} \simeq \mathfrak{C}$.

5° Let $\mathfrak{A}, \mathfrak{B}$ be models of T^*. Then $\mathfrak{A} \equiv \mathfrak{B}$ is equivalent to $\mathfrak{A} \simeq \mathfrak{B}$.

6° If T has a prime model, then every semiprime model of T is prime and every semiuniversal model is universal.

Proofs of these assertions are simple so they are omitted.

THEOREM 2.7. *Let $\mathfrak{A}$ be a model of T and $C(\mathfrak{A})$ the class of all models of T compatible with $\mathfrak{A}$. Then $C(\mathfrak{A})$ is an elementary class of models with JE and AP. If T has an universal-existential axiomatization, then $T_{\mathfrak{A}} = Th(C(\mathfrak{A}))$ is a Jónnson theory. If $C^*(\mathfrak{A})$ is the class of all models of T^* in which models of $C(\mathfrak{A})$ are embeddable, then $T^*_{\mathfrak{A}} = Th(C^*(\mathfrak{A}))$ is a complete theory and the model completion of $T_{\mathfrak{A}}$. Also, $C^*(\mathfrak{A})$ is a class of equivalence under $\simeq_{T^*}$.*

PROOF In order to prove that $C(\mathfrak{A})$ is an elementary class we use the theorem (Frayne, Morel, Scott) which states that a class of models is elementary if it is closed under elementary equivalence and ultraproducts. So let $\mathfrak{B}\in C(\mathfrak{A})$ and $\mathfrak{C}\equiv\mathfrak{B}$. Then $\mathfrak{C}\models T$ and $\mathfrak{C}\simeq\mathfrak{B}$ so $\mathfrak{C}\in C(\mathfrak{A})$. Further, assume that $\mathfrak{A}_i\in C(\mathfrak{A})$, $i\in I$. Hence there are models $\mathfrak{B}_i$ so that $\mathfrak{A}_i\to\mathfrak{B}_i\leftarrow\mathfrak{A}$. Let U be an ultrafilter over I. Then $\mathfrak{A}$ and $\mathfrak{A}'=\prod_{i\in C}\mathfrak{A}_i/U$ are embedded into $\prod_{i\in I}\mathfrak{B}_i/U$. Since $\mathfrak{A}'$ is a model of T, it follows that $\mathfrak{A}'\simeq_T\mathfrak{A}$, and therefore $C(\mathfrak{A})$ is an elementary class. That $T_{\mathfrak{A}}$ satisfies JE and AP it is obvious. So assume that T is closed under union of chains of models and let us prove that $T_{\mathfrak{A}}$ is too. Since $\mathfrak{M}(T_{\mathfrak{A}})$ is an elementary class it suffices to prove that $T_{\mathfrak{A}}$ is closed under countable chains of models. Therefore let $\mathfrak{A}_1\subseteq\mathfrak{A}_2\subseteq\ldots$ where $\mathfrak{A}_n\in C(\mathfrak{A})$, $n\in\omega$, and $\mathfrak{A}'=\bigcup_n\mathfrak{A}_n$. Then $\mathfrak{A}'\models T$. Further consider models $\underset{\sim}{\mathfrak{A}_1}=(\mathfrak{A}_1,a^1)_{a^1\in A_1}$, $\underset{\sim}{\mathfrak{A}_2}=(\mathfrak{A}_2,a^1,a^2)_{a^1\in A_1,\,a^2\in A_2},\ldots$ and $\Gamma=T_{\mathfrak{A}}+\Delta(\underset{\sim}{\mathfrak{A}_1})+\Delta(\underset{\sim}{\mathfrak{A}_2})+\cdots$. The theory Γ is finitelly consistent, hence there is a model $\underset{\sim}{\mathfrak{B}}\models\Gamma$ i.e. $\mathfrak{B}\models T$ and $\mathfrak{A}'\to\mathfrak{B}$. Thus $\mathfrak{B}\simeq\mathfrak{A}$ and $\mathfrak{A}'\simeq\mathfrak{B}$, so $\mathfrak{A}'\simeq\mathfrak{A}$. Now we prove that $T^*_{\mathfrak{A}}$ is a complete theory and model completion of $T_{\mathfrak{A}}$. That $C^*(\mathfrak{A})$ is an elementary class it can be proved as it was done for $C(\mathfrak{A})$. Assume that $\mathfrak{B},\mathfrak{C}\in C^*(\mathfrak{A})$. Then there are $\mathfrak{B}',\mathfrak{C}'\in C(\mathfrak{A})$ so that $\mathfrak{B}'\subseteq\mathfrak{B}$ and $\mathfrak{C}'\subseteq\mathfrak{C}$. Since $\mathfrak{B}'\simeq\mathfrak{C}'$ it follows $\mathfrak{B}\simeq\mathfrak{C}$ and therefore $\mathfrak{B}\equiv\mathfrak{C}$ because $\mathfrak{B}$, $\mathfrak{C}$ are models of T^*. Hence, $T^*_{\mathfrak{A}}$ is a complete theory. The last two statements are easy to prove. $\dashv$

Now we proceed to description of saturated models of T^*.

THEOREM 2.8. 1° *If $\mathfrak{C}$ is an infinite saturated model of T^* then $\mathfrak{C}$ is a full model of T.*

2° If $\mathfrak{C}$ is a full model of T of cardinality $\alpha\geqslant\omega_1$ then $\mathfrak{C}$ is a saturated model of T^*.

PROOF. During this proof we shall use the theorem which states that a model $\mathfrak{C}$ is saturated iff it is elementary universal and elementary homogeneous.

1° Assume that $\mathfrak{C}$ is a saturated model of T^*.

CLAIM. *$\mathfrak{C}$ is a semiuniversal model of T.* For that let $\mathfrak{A}\simeq\mathfrak{C}$ and $|A|\leqslant|C|$. Further, there is $\mathfrak{B}\models T^*$ such that $\mathfrak{A}\to\mathfrak{B}$ and by LST theorem it may be assumed that $|B|=\max(|A|,\omega)$. Then $\mathfrak{B}\simeq\mathfrak{C}$, so $\mathfrak{B}\equiv\mathfrak{C}$ and by universality of $\mathfrak{C}$ it follows $\mathfrak{B}\to\mathfrak{C}$ and therefore $\mathfrak{A}\to\mathfrak{C}$.

CLAIM. *$\mathfrak{C}$ is a homogeneous model of T.* Let $\mathfrak{C}\overset{f}{\leftarrow}\mathfrak{A}\overset{g}{\to}\mathfrak{C}$ and $|A|<|C|$. Define a partial isomorphism p on $\mathfrak{C}$ by $pfa=ga$, $a\in A$. Since T^* is the model completion of T, it follows $(\mathfrak{C},fa)_{a\in A}\equiv(\mathfrak{C},pfa)_{a\in A}$ so there is an automorphism $h:\mathfrak{C}\overset{\sim}{\to}\mathfrak{C}$ such that $p\subseteq h$.

2° Assume that $\mathfrak{C}$ is a full model of T.

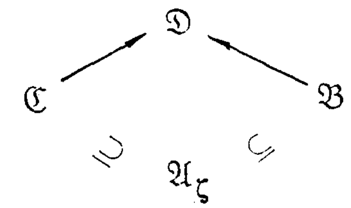
CLAIM. $\mathfrak{C}$ *is a model of* T^*. For that let $\beta = cf\alpha$. Then there is a sequence of sets X_ξ, $\xi<\beta$ so that (1) if $\xi<\zeta$ then $X_\xi \subseteq X_\zeta$, (2) $|X_\xi|<\alpha$, (3) $C=\bigcup_{\xi<\beta} X_\xi$. By transfinite induction we define a sequence of models $\mathfrak{A}_\xi$, $\mathfrak{B}_\xi$, $\xi<\beta$ so that the following hold: (4) For all $\zeta<\xi$ $\mathfrak{A}_\zeta \subseteq \mathfrak{B}_\xi$ (5) If $\xi \geqslant 1$ then $\mathfrak{B}_\xi \subseteq \mathfrak{A}_\xi$ (6) $X_\xi \subseteq A_\xi$, (7) $|A_\xi|$, $|B_\xi|<\alpha$ and (8) $\mathfrak{B}_\xi \models T^*$.

Let $\mathfrak{A}_0$ be such that $\mathfrak{A}_0 < \mathfrak{C}$, $X_0 \subseteq A_0$ and $|A_0| \leqslant \omega$, its existence is provided by *LST* theorem. Assume $\xi \geqslant 1$ and $\mathfrak{B}_\xi$ has been defined. By the inductive hypothesis $|B_\xi|<\alpha$. Therefore, since $|X_\xi|<\alpha$, $|B_\xi \cup X_\xi|<\alpha$. Hence by *LST* theorem there is $\mathfrak{A}_\xi < \mathfrak{C}$ so that $B_\xi \cup X_\xi \subseteq A_\xi$ and $|A_\xi| = |B_\xi \cup X_\xi|$. Thus $|A_\xi|<\alpha$, $\mathfrak{B}_\xi \subseteq \mathfrak{A}_\xi$, and $X_\xi \subseteq A_\xi$.

Models $\mathfrak{B}_\xi$ are defined in the following way.

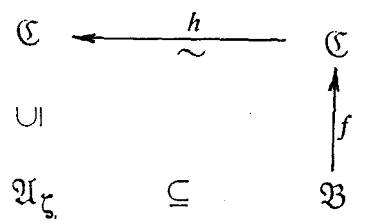
If $\xi<\alpha$ is a limit ordinal, $\xi \neq 0$, then $\mathfrak{B}_\xi = \bigcup_{\zeta<\xi} \mathfrak{B}_\zeta$. The theory T^* is closed under union of chains of models, hence $\mathfrak{B}_\xi \models T^*$. Now assume that $\xi = \zeta+1$. By the induction hypothesis $\mathfrak{A}_\zeta \subseteq \mathfrak{C}$, $|A_\zeta|<\alpha$. Further, there is $\mathfrak{B} \models T^*$ so that $\mathfrak{A}_\zeta \subseteq \mathfrak{B}$ and by *LST* theorem it may be taken $|B| = |A_\zeta|$ i.e. $|B|<\alpha$. T^* is amalgamative, therefore the diagram $\mathfrak{C} \supseteq \mathfrak{A}_\zeta \subseteq \mathfrak{B}$ is completed to the amalgam (9)

(9)



Hence $\mathfrak{B} \simeq \mathfrak{C}$. Since $\mathfrak{C}$ is a semiuniversal model, there is $f\colon \mathfrak{B} \to \mathfrak{C}$. Also, $\mathfrak{C}$ is α-homogeneous model, so there is an automorphism h of $\mathfrak{C}$ so that the diagram (10) commutes.

(10)

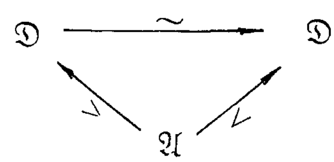


Let $\mathfrak{B}_\xi = hf(\mathfrak{B})$. Then $\mathfrak{B}_\xi \models T^*$, $\mathfrak{A}_\zeta \subseteq \mathfrak{B}_\xi$ and $|B_\xi|<\alpha$. At the end we set $\mathfrak{B}_0 = \mathfrak{B}_1$. It should be observed that β is a limit ordinal, so for all $\xi<\beta$ $\mathfrak{B}_{\xi+1}$ is defined, hence $\mathfrak{A}_\xi \subseteq \mathfrak{B}_{\xi+1}$ and $X_\xi \subseteq B_{\xi+1}$. Therefore $\mathfrak{C} = \bigcup_{\xi<\beta} \mathfrak{B}_\xi$. Since T^* is closed under union of chains of models, it follows that $\mathfrak{C} \models T^*$.

CLAIM. *$\mathfrak{C}$ is an elementary universal model of T^*.* For that, let $\mathfrak{B}$ be a model of T^*, $|B| \leqslant \alpha$ and $\mathfrak{B} \equiv \mathfrak{C}$. Then $\mathfrak{B} \simeq \mathfrak{C}$ and $\mathfrak{B} \models T$ so there is $f\colon \mathfrak{B} \to \mathfrak{C}$. Since T^* is model complete, f is elementary in fact.

CLAIM. *$\mathfrak{C}$ is an elementary homogeneous model.* In order to prove this assertion we need the following...

DEFINITION 2.9. *A model $\mathfrak{D}$ is a weak homogeneous model if every diagram of the sort $\mathfrak{D} \xleftarrow{\geq} \mathfrak{A} \xrightarrow{\leq} \mathfrak{D}$, $|A| < |D|$, can be completed in the commutative diagram:*



(The following question can be stated: Does the weak homogeneity implies elementary homogeneity?)

It is obvious that $\mathfrak{C}$ is a weak homogeneous model. That $\mathfrak{C}$ is elementary homogeneous follows directly from the previous claim and the following...

LEMMA (Morly-Vaught) If $\mathfrak{C}$ is an elementary universal model then $\mathfrak{C}$ is weak homogenous iff it is elementary homogeneous.

For the proof see [3; 11.14]. $\dashv$

There are several results similar to the previous theorem. We would like to mention two theorems of such kind. One is in [3; 11.19] and it is connected with the notion of conservative enlargement L of a class of models K. This theorem aserts that α homogenous-universal models of K and L coincide. However, in this theorem uniformity in assigment of models of class L to models of class K is assumed, what is not the case in our theorem. The second one is the theorem of H. Simmons (6; 3.4.1) which states that if a given theory has the model companion, then all its k-objective (in the sense of M. Yasuhara [6]) models are k-saturated.

3. Full models of a theory with a dense ordering

In some cases it is possible to say exactly in which cardinals a theory T has full models, and according to the theorem 2.8., its model completion has saturated models.

THEOREM 3.1. *Let $\mathfrak{A}$ be a saturated model of cardinality α and assume that it (or its definable expansion) contains a nontrivial dense partial ordering, i.e. in $\mathfrak{A}$ holds $\forall xy \exists z (x<y \Rightarrow x<z<y)$. Then an η_α set can be embedded into $\mathfrak{A}$ and therefore $\alpha = \alpha^{\underset{\smile}{\alpha}}$.*

PROOF Let g be a maximal chain without endpoints and $X, Y \subseteq g$ so that $X < Y$ (i.e. for all $u \in X$, all $v \in Y$, $u < v$), $|X \cup Y| < \alpha$. The set $\Sigma(x) = \{u < x \mid u \in X\} \cup \{x < v \mid v \in Y\}$ is finitely consistent with $Th(\mathfrak{A}_{X \cup Y})$, hence $\Sigma(x)$ is realized in $\mathfrak{A}$, i.e. there is $a \in A$ so that $\mathfrak{A} \models \Sigma(a)$. Therefore $X < a < Y$. Assume that $a \notin g$. Let $b \in g$. Then there are the following possibilities:

1° For some $u \in X$ $b \leqslant u$, so $b \leqslant a$.

2° For some $v \in Y$ $v \leqslant b$, so $a \leqslant b$.

3° $X < b < Y$.

If 3° does not hold for any $b \in g$, then by 1° and 2° $g \cup \{a\}$ is linearly ordered, so by maximality of g $a \in g$, but this contradicts to our assumption. Hence $a \in g$ or there is $b \in g$ so that $X < b < Y$, in any case there is $c \in g$ so that $X < c < Y$. Thus, g is an η_α set so $|g| \geqslant \alpha$. But $g \subseteq A$, hence $|g| = \alpha$. Hence g is an η_α set of cardinality α so (Gillman, cf. [3]) $\alpha = \alpha^{\underset{\smile}{\alpha}}$. $\dashv$

Assume that T is a Jónnson theory. According to the theory of Jónnson classes, if $\alpha > \omega$ and $\alpha = \alpha^{\underset{\smile}{\alpha}}$ then there is a homogeneous-universal model of T of cardinality α. By the previous theorem we have the following...

COROLLARY 3.2. Assume that T contains a nontrivial partial dense ordering, and let α be a cardinal, $\alpha > \omega$. Then T has a full model and T^* has a saturated model of cardinality α iff $\alpha = \alpha^{\underset{\smile}{\alpha}}$. $\dashv$

We list several examples of theories with ordering on which previous theorems can be applied.

T	T^*
1. Theory of linear ordering.	Theory of linear dense ordering without endpoints.
2. Theory of linearly ordered Abelian groups.	Theory of linearly ordered Abelian divisible groups.
3. Theory of Boolean algebras.	Theory of atomless Boolean algebras
4. Theory of distributive lattices with endpoints.	Theory of distributive, complementary, dense lattices with endpoints
5. Theory of ordered fields.	Theory of o. dered real closed fields.

Depending on a theory several names are connected with the theory in two sense: 1° In proof that an appropriate theory T^* is a model completion of T, 2° That the class of models of T is a Jónnson class. For informations of that kind one may consult [2], [3] and [4].

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Faculty of Natural and Mathematical Sciences,
Department of Mathematics, Beograd