Зборник радова Математичког института, Нова серија, књ. 2 (10), 1977.
Recueil des travaux de l'Institut Mathématique, Nouvelle série, tome 2 (10), 1977,

Symposium: Set Theory. Foundations of Mathematics, Beograd, 29. 08. — 2. 09. 1977.

CONTINUUM PROBLEM AT MEASURABLE CARDINALS

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Exposition

Given any set, how to evaluate the cardinal of its power set? The above is known as continuum problem. In ZFC, initial ordinals can be taken to represent cardinals. Thence the problem reads: determine function F, so that for all ordinals α:

(0) $$2^{\omega_\alpha} = \omega_{F(\alpha)}.$$

Cantor has proved that $2^{\omega_\alpha} \geqslant \omega_{\alpha+1}$, for all α. Therefore we can split F so that

(1) $$\omega_{F(\alpha)} = \omega_{\alpha+f(\alpha)}.$$

Putting $f(\alpha)=1$, for $\alpha \in Ord$, we obtain a formulation of generalised continuum hypothesis (GCH).

It is known that

(2) $$\alpha \leqslant \beta \text{ implies } F(\alpha) \leqslant F(\beta)$$

and

(3) $$cf\,\omega_{F(\alpha)} > \omega_\alpha.$$

The (3) is known as König's lemma.

Here we shall first list important recent progress on the matter, assuming the fundamental results of Gödel and Cohen are known.

In [7] Silver has proved the following theorem.

1.1. THEOREM: if ω_α is a singular cardinal of cofinality greater than ω, then:

(4) $$\forall \beta < \alpha\, 2^{\omega_\beta} = \omega_{\beta+1} \text{ implies } 2^{\omega_\alpha} = \omega_{\alpha+1}.$$

However, the problem of all singular cardinals is still unsolved. In J. Stern [8] we found the following hypothesis on singular cardinals, for which the consistency and independence are open questions. HCS: let ω_α be a singular cardinal. Then

(4′) $$\forall \beta < \alpha\, 2^{\omega_\beta} = \omega_{\beta+1} \text{ implies } 2^{\omega_\alpha} = \omega_{\alpha+1}.$$

Jensen in [6] has proved the next theorem.

1.2. THEOREM: if negation of HCS is consistent with ZFC so is the axiom of uncountable measurable cardinals (AM).

For regular cardinals we have the fundamental result of Easton [3]:

1.3. THEOREM: for any function F defined on all ordinals α such that ω_α is a regular cardinal and F satisfies (2) and (3), consistency of ZFC implies the consistency of $ZFC+EA_F$. Here EA_F is the formula

$$\forall\, \alpha \in D_{om}(F)\ \ 2^{\omega_\alpha} = \omega_{F(\alpha)}.$$

Here we note that 1.3. theorem, we found in Jech [5], theorem 37, in a somewhat different notation. There presented formulation is adjusted for the following theorem that we have proved. Let F and f be defined by (Ø) and (1). From Chang and Keisler [1], section 4.2. we know that if there is an uncountable measurable cardinal then there is a normal ultrafilter on it.

1.4. THEOREM: let k be an uncountable measurable cardinal and let D be a normal ultrafilter on it. Then

$$\{\beta < K : 2^{|\beta|} = |\beta|^+\} \in D \text{ implies } 2^k = k^+. \tag{5}$$

$$|f(k)| \leqslant |\prod_D f(\beta)|. \tag{6}$$

Above $|X|$ denotes a cardinal of X, $\prod_D$ is ultraproduct modulo normal filter D. (5) says that if continuum hypothensis is true on a set in D, then it is true at measurable cardinal k. Hence it implies that the value 2^k is determined when continuum hypothesis holds on a set in D. (5) is the special case of (6) which can be read as: the number of cardinals α such that $k < \alpha \leqslant 2^k$, is constrained with the value of $|\prod_D f(\beta)|$. Here $f(\beta)$ is a nonempty subset of k, which enumerates the cardinals from ω_β to 2^{ω_β}.

Now it is evident that the axiom of uncountable measurable cardinals contradicts the Easton's result given in 1.3. theorem; to check that, let k and D be as in 1.4. theorem. Define F

$$F(\alpha) = \begin{cases} \alpha + 1 & \textit{iff } \alpha \neq k \text{ and } cf\,\omega_\alpha = \omega_\alpha \\ \alpha + 2 & \textit{iff } \alpha = k \end{cases}$$

This F satisfies (2) and (3), so by the conclusion of 1.3. theorem we can take as axiom

$$\forall\, \alpha \in D_{om}(F)\ \ 2^{\omega_\alpha} = \omega_{F(\alpha)}.$$

But the set of all regular cardinals less then k belongs to D. Hence by (5) $2^k = k^+$, contradicting $F(k) = k+2$ which means that $2^k = k^{++}$. Moreover, since (5) is a special case of (6), similiarly to above we see that if F violates the (6) $ZFC+AM+EA_F$ is inconsistent. What with the opposite question? Taking into account Silver's result that the consistency of $ZFC+AM$ implies the consistency of $ZFC+AM+GCH$, we state the conjecture: let F be defined on all α for which ω_α is regular and let F satisfy (2), (3) and (6). Then the consistency of $ZFC+AM$ implies the consistency of $ZFC+AM+EA_F$.

As we have seen above, the continuum problem was separately treated for singular and regular cardinals. But according to (6), may F be such to prevent the existence of measurable cardinals? Then in $ZFC+EA_F$, HCS would become a theorem.

Proof

First we list two D. Scott's results on normal measure, as we found them in the section 4.2. of Chang-Keisler [1].

DEFINITION. A filter D over a measuralbe cardinal k is said to be normal if:

1. D is an k-complete nonprincipal ultrafilter;
2. in the ultrapower $\prod_D \langle K, <\rangle$, the k-th element is the identity function on k.

2.1. THEOREM: let k be an uncountable measurable cardinal. Then there is a normal ultrafilter over it.

2.2. THEOREM: if k is a measurable cardinal and D a normal ultrafilter on it then

$$\langle R(k+1), \in\rangle \cong \prod_D \langle R(\beta+1), \in\rangle.$$

2.3. COROLLARY: let $\varphi(x)$ be a formula. Then

$$\langle R(k+1), \in\rangle \models \varphi(k) \text{ } iff \text{ } \{\beta<k : \langle R(\beta+1), \in\rangle \models \varphi(\beta)\} \in D.$$

As a consequence of the above we note that the set of strongly inaccessible cardinals less than k belongs to D. Also

$$\left|\prod_D R(\beta+1)\right| = 2^k.$$

2.4. THEOREM: let D be an ultrafilter over a cardinal k .Let

$$\mathfrak{A} = \langle A, <_A\rangle = \prod_D \langle k, <\rangle. \text{ If } f \in {}^k k \text{ and } f(\beta) \neq \varnothing$$

when $\beta \in k$, then

$$\left|\prod_D f(\beta)\right| = \left|\{g_D^{\mathfrak{A}} \in \mathfrak{A} : g_D^{\mathfrak{A}} <_A f_D^{\mathfrak{A}}\}\right|.$$

PROOF: let $g \in \prod_{\beta\in k} f(\beta)$. Then $g \in {}^k k$. Define

1. $g_D = \{h \in \prod_{\beta\in k} f(\beta) : \{i<k : g(i) = h(i)\} \in D\}$.
2. $g_D^{\mathfrak{A}} = \{h \in {}^k k : \{i<k : g(i) = h(i)\} \in D\}$.

It is clear that $g_D \subset g_D^{\mathfrak{A}}$. Define $\pi : \prod_D f(\beta) \to A$, by $\pi g_D = g_D^{\mathfrak{A}}$. π is $1-1$. For, if $g_D \neq h_D$ and g_D, $h_D \in \prod_D f(\beta)$, then $g_D \cap h_D = \varnothing$. Suppose that $\pi g_D = \pi h_D$. Then $g_D^{\mathfrak{A}} = h_D^{\mathfrak{A}}$, and hence $\{i<k: g(i)=h(i)\} \in D$. It follows that $h_D = g_D$. Contradiction. Put $F = \{g_D^{\mathfrak{A}} \in \mathfrak{A} : g_D^{\mathfrak{A}} <_A f_D^{\mathfrak{A}}\}$. We shall prove that $\pi(\prod_D f(\beta)) = F$. Let $g_D \in \prod_D f(\beta)$. Then $\{\beta<k: g(\beta)<f(\beta)\} = k \in D$. It follows that $g_D^{\mathfrak{A}} <_A f_D^{\mathfrak{A}}$. Hence $g_D^{\mathfrak{A}} \in F$. Let now $g_D^{\mathfrak{A}} \in F$. Then $x = \{\beta<k: g(\beta)<f(\beta)\} \in D$. Let $\bar{g} \in {}^k k$ be such that

$$\bar{g}(\beta) = g(\beta) \quad \text{if} \quad \beta \in x$$

$$\bar{g}(\beta) = 1 \quad \text{if} \quad \beta \in k \backslash x.$$

Then $\bar{g} \in g_D^{\mathfrak{A}}$. But $\bar{g} \in \prod_{\beta \in k} f(\beta)$ and $\bar{g}_D \in \prod_D f(\beta)$. Therefore $\pi \bar{g}_D = g_D^{\mathfrak{A}}$ and thus π maps $\prod_D f(\beta)$ onto F.

2.5. THEOREM *let k be a measurable cardinal, D a normal ultrafilter over k. Then $\mathfrak{A} = \langle A, <_A \rangle = \Pi_D \langle k, < \rangle$ is well ordered with the relation. $<_A$. Order type of $\mathfrak{A}$ is greater than 2^k.*

PROOF. By lemma 4.2.13. from [1], $<_A$ is a well ordering. Further

$$2^k = \left| \prod_D R(\beta+1) \right| \leqslant \left| \prod_D \langle k, < \rangle \right| \leqslant 2^k.$$

Hence order type ot $\mathfrak{A} \geqslant 2^k$ and obviously ot $\mathfrak{A} < |2^k|^+$; defining b as $b(\beta) = |R(\beta+1)|$, we see that $b \in {}^k k$ and hence $b_D \in \mathfrak{A}$. The proof then follows from 2.4. theorem and the fact that b_D is not the last element in $\mathfrak{A}$.

2.6. COROLLARY for every $f_D \in \mathfrak{A}$ there is an ordinal γ_f so that f_D is the $\gamma_f - th$ element of $\mathfrak{A}$, and $|\Pi_D f(\beta)| = |\gamma_f|$; for every ordinal $\varkappa < ot\, \mathfrak{A}$ there is an $f^\varkappa \in {}^k k$, such that $f_D^\varkappa$ is the $\varkappa$-th element in $\mathfrak{A}$.

Now we can give the proof of 1.4. theorem.

Functions F and f are defined by $(\varnothing)$ and (1); if $\beta < k$ then $cf|\beta| < k$, $\omega_\beta < k$, $F(\beta) < k$, $2^{\omega_\beta} < k$ and $f(\beta) < k$. Hence the restriction $f \restriction_k \in {}^k k$ and $(f \restriction_k)_D \in \prod_D \langle k, < \rangle$. We define

$$G_f = \{g_D \in \mathfrak{A} : g_D <_A f_D\} \text{ and}$$

$$H = \{h_D \in \mathfrak{A} : \{\beta<k : h(\beta) \in [\omega_\beta, \omega_{\beta+f(\beta)}) \cap c_{ard}\} \in D\}.$$

That is, for $h_D \in H$, $h(\beta)$ is a cardinal and $\omega_\beta \leqslant h_{(\beta)} < \omega_{\beta+f(\beta)}$. Hence, for every $h_D \in H$, there is some $g_D \in G_f$ so that

(*) $\{\beta<k : h(\beta) = \omega_{\beta+g(\beta)}\} \in D$. Define $\pi: H \to G_f$ with

$$\pi h_D = g_D \ iff \ (*).$$

It is easy to check that πh_D does not depend on elements of h_D and that π is 1—1. Therefore

$$|H| \leqslant |G_f|.$$

Let $\varkappa$ be a cardinal such that $k \leqslant \varkappa < 2^k$. By the 2.6. corollary there is an $f^{\varkappa} \in {}^k k$, such that $f^{\varkappa}_D$ is the $\varkappa$-*th* ordinal in $\mathfrak{A}$, eg. $\gamma_{f^{\varkappa}} = \varkappa$. From the same corollary

$$\left| \prod_D f^{\varkappa}(\beta) \right| = |G_{f^{\varkappa}}| = |\varkappa| = \varkappa.$$

For the function g with the domain k, define the function

$$|g| = \langle |g(\beta)| : \beta < k \rangle.$$

We have

$$\left| \prod_D |f^{\varkappa}(\beta)| \right| = \left| \prod_D f^{\varkappa}(\beta) \right| = \varkappa.$$

That implies

$$|G_{|f^{\varkappa}|}| = \varkappa \text{ and } \gamma_{|f^{\varkappa}|} \geqslant \varkappa,$$

which means that $|f^{\varkappa}|$ is at least $\varkappa$-th element in $\mathfrak{A}$. Since $|f^{\varkappa}|_D \leqslant_A f^{\varkappa}_D$ $(\{\beta < k : |f^{\varkappa}(\beta)| \leqslant f^{\varkappa}(\beta)\} \in D)$, by choice of $f^{\varkappa}$ must be $f^{\varkappa} =_D |f^{\varkappa}|$ and hence

$$X = \{\beta < k : f^{\varkappa}(\beta) \text{ is a cardinal}\} \in D.$$

Since $\gamma_{f^{\varkappa}} = \varkappa \geqslant k$ and D is normal, we have

$$\{\beta < k : f^{\varkappa}(\beta) \geqslant \beta\} \in D.$$

Let $Sinac(k)$ be the set of strongly inaccessible cardinals less than k. As we noticed, $Sinac(k) \in D$. Now we have

$$\text{either } \{\beta < k : f^{\varkappa}(\beta) \geqslant \omega_{\beta + f(\beta)}\} \in D$$

$$\text{or } \{\beta < k : f^{\varkappa}(\beta) < \omega_{\beta + f(\beta)}\} \in D.$$

In the first case we would have

$$\{\beta \in k \cap Sinac(k) : f^{\varkappa}(\beta) \geqslant \omega_{\beta + f(\beta)} = b(\beta)\} \in D,$$

which would imply

$$2^k \leqslant \left| \prod_{D \cap S(Sinac(k))} f^{\varkappa}(\beta) \right| = \left| \prod_D f^{\varkappa}(\beta) \right|.$$

Hence $\gamma_{f^{\varkappa}} \geqslant 2^k$, contradicting assumption for $\varkappa$.
Thus $\{\beta < k : f^{\varkappa}(\beta) < \omega_{\beta + f(\beta)}\} \in D$.
Since $\varkappa \geqslant k$ and $f^{\varkappa} =_D |f^{\varkappa}|$ we have

$$\{\beta < k : f^{\varkappa}(\beta) \in [\omega_\beta, \omega_{\beta + f(\beta)}) \cap \dot{C}ard\} \in D.$$

It follows that there is some $h_D \in H$, so that $f^{\varkappa} \in h_D$, or equally $f^{\varkappa}_D \in H$. Since $\varkappa \neq \varkappa'$ implies $f^{\varkappa}_D \neq f^{\varkappa'}_D$, we have

$$|[k, 2^k) \cap Card| = |(k, 2^k] \cap Card| = |f(k)| \leqslant |H| \leqslant |G_f| = \left| \prod_D f(\beta) \right|,$$

thus completing the proof of (6). Now let

$$X = \{\beta < k : 2^{|\beta|} = |\beta|^+\} \in D.$$

This means that $f(\beta)=1$, when $\beta\in X$. But from (6) we get

$$|f(k)|\leqslant\Big|\prod_{D\cap S(x)} f(\beta)\Big|=1. \text{ Hence } 2^k=k^+.$$

NOTE: in the above proof we had $f\restriction_k$ defined on all $\beta<k$; to apply the *Easton's* argument we need $f\restriction_k$ to be defined on $y=\{\beta<k:\omega_\beta \text{ is regular}\}$. Since $y\in D$, such a difficulty can easily be avoided.

From above it follows that actually

$$2^k\leqslant\omega_{k+ot(\prod_D\langle f(\beta),<\rangle)}.$$

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