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ON A FOUNDATION FOR MATHEMATICS — A VIEW OF MATHEMATICS I*

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1. Introduction

This paper represents the beginning of our final considerations of an approach to the foundations of mathematics initiated by the paper [5]. We shall deal in it with mathematical activity and its goals. We shall start from the assumption that the primary goal of mathematical activity is the creation of certain entities which will comprise in themselves (all) that activity. Such entities, we shall call mathematical entities. Our next assumption is that any performed mathematical activity creates conditions, i.e., makes a groundwork for a new mathematical activity and hence for the creation of new mathematical entities. These new entities are of a higher level with respect to old ones. If we now assume that all mathematical entities constitute an edifice which we shall call the world of mathematics, then we shall have that this world consists of mathematical entities of various sorts and levels.

In the creation of such a world we accept a symbolic form of presentation. Namely, we assume that there is a collection of symbols which stand for mathematical entities of various sorts and levels. Such a collection will be a symbolic frame of the world of mathematics. We shall denote it by $\mathcal{S}$. If we build up the world on this collection, then we shall say that we have a symbolic form of the world of mathematics and of mathematical entities as its constituents. We shall obtain concrete mathematical entities by naming symbols of such a world according to their creative procedures given in the paper.

If we assume that mathematical entities in question are certain organized wholes, which we call spatial wholes, then we might say that the world of mathematics consists of spatial wholes of various sorts and levels. Together with these entities always go some other entities: connectives between them. In such a way we obtain that the world of mathematics consists of two sorts of entities of various levels. It means that for its creation is enough to start from a subcollection $\mathcal{M}$ of $\mathcal{S}$, consisting of two-sort symbols of various levels. Other symbols of $\mathcal{S}$ are then reserved to stand for properties and other things which are relevent for entities of $\mathcal{M}$.

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The collection $\mathcal{M}$ will serve as a framework for the creation of the world of mathematics. We shall specify the fundamental acts necessary for its creation. The central act occupies of course the creation of spatial wholes — objects of the world. We shall define the concept of a spatial whole and point out its main features. Furthermore, we shall give some examples of spatial wholes and establish the link between the creation of particular kinds of spatial wholes and certain standard mathematical conceptions, like formalism and intuitionism. In such a manner we shall show that these conceptions justify our attitude concerning the goal of mathematical activity. Otherwise, all these investigations will serve as a basis for the process of formalization.

2. Species and spatial wholes

This section is devoted to a general discussion of the mathematical world and to main concepts which arise in the creation of this world. These concepts are species and spatial wholes. We shall see which mathematical activity is comprised in their creation.

Before we begin our consideration of the above concepts we would be shortly concerned with the activity of human beings in general and then within such an activity would find the position of the mathematical activity.

Certainly, in a human activity one can always recognize two things: the goal of activity and means which men have at their disposal to attain the goal.When one is provided with these two things then he has still to decide in which manner to realize the activity. It means that he must have a plan — a scheme for its performing; of course, there also must be criteria for deciding on each of these things.

Before all, the goal of activity has to be determined according to our needs and wishes; these two things are otherwise restricted by certain external moments. Since an activity is always realized within a frame which has its own principles, then we must take into account that it should not violate these principles. If the question is about the organization of a society, then we have principles of various kinds, like social, political and many others. All these particular moments are beyond our interest and therefore we shall not be concerned with them here. However, they will find their place in our global considerations of the organization of $\mathcal{S}$; of course, in a form which we shall be able to set up.

We further have that means for attaining the goal of activity are different and determined according to our wishes to have some, in a certain sense, optimal properties of the goal. Independently of concrete goals people deal with discovering general and always new means for performing their activities and then in concrete situations utilize adequate ones according to desired properties of the goals and considered objects by means of which they build them up. Clearly, we cannot apply any means to each collection of concrete objects. Thus when we specify means we decide on their choice according to the regarded collection of objects. At the end of these general considerations of activities of human beings we shall still be concerned in short with plans for performing activities. The plans are to be given in oral or written form and their purpose is to specify and also to memorize performances of activity.

Mathematics cannot be set apart from these general activities of human beings. It can deal with them only in abstract forms. According to our views, its goals are, in the main, creations of abstract spatial wholes, means for the purpose are certain constructions — operations and plans of activity are symbolic schemata. In the sequel we shall concern all these concepts.

Now we shall begin with a general description of the mathematical world and main activities for its building. We shall start from a large collection consisting of symbols of different sorts and levels. The levels of starting symbols of $\mathcal{S}$ we denote by -1 and the collection containing all these symbols by $\mathcal{S}_0$. We further have different-sort symbols of the levels 0, 1, . . . and respective collections $\mathcal{S}_1$, $\mathcal{S}_2, \ldots$ which contain them. If we denote the hierarchy of all levels by $\mathcal{J}$, then we can write the above collection as an indexed collection $\langle \mathcal{S}_i \mid i \in \mathcal{J} \rangle$. This collection is so far without any condition being imposed upon it and its elements.

Since $\mathcal{S}$ contains various symbols in itself we therefore have to carry out some systematizations in it in order to make it capable of suiting our purposes. To do this we shall consider nature around us. Two basic concepts in it are real objects of various levels: electrons, atoms, molecules, actual objects we are surrounded by etc. and forces among them. Forces on a level in nature may be of different characters and sources which we shall not discuss here. These two concepts are quite sufficient for building up the real world; forces are otherwise responsible for its existence as a whole, although they are not sufficient for a complete descirption of it: of all phenomena and events in it. Taking into account the former fact, we shall select in $\mathcal{S}$, by the analogy, a collection $\mathcal{M}$ of two-sort symbols of different levels, which will be sufficient for our purposes: building up the mathematical world. One sort of symbols in it will correspond to objects: natural or abstract and the other to connectives between these symbols. The former symbols we shall call *objects* and the latter, *arrows*. Thus, the collection $\mathcal{M}$ consists of objects and arrows of various levels. Such a collection, which is otherwise quite natural, will serve as a framework for building up the mathematical world. Other symbols of $\mathcal{S}$ will only serve for its description.

In what follows we shall make some further specifications in $\mathcal{M}$. Namely, we shall let the possibility to characterize and hence to differentiate the symbols of $\mathcal{M}$. We can do this by adjoining to each level of $\mathcal{M}$ certain new symbols of $\mathcal{S}$ which will become certain integral parts of the symbols of $\mathcal{M}$, characterizing them. These new symbols we assume to be characteristic properties, which mathematical entities can be supposed to possesss. Since we have in $\mathcal{M}$, on each level, two-sort symbols, then the adjoined symbols, to any level of $\mathcal{M}$, have also to be two-sort: the ones for objects and the others for arrows. We here assume that there are some relationships between the properties of objects and arrows: we assume that arrows have properties of carrying information on objects and their properties; information are otherwise to be specified in each concrete case.

By means of symbols representing properties of objects we can make certain selections in $\mathcal{M}$. These selections are our starting acts. What do we do, in fact? We select (all) objects on a level of $\mathcal{M}$, agreeing in some common (attributes) — characteristic property(ies), in particular collections. Such collections we then call species. We shall specify this somewhat more.

Denote by $\mathfrak{P}$ the collection of (all) possible properties which mathematical entities on a level of $\mathcal{M}$ can be supposed to possess. By applying this collection to the considered level of $\mathcal{M}$ we shall select various collections of objects and arrows on it. Let us see in which way. First, we shall concern the question of the selection of collections of objects on the regarded level of $\mathcal{M}$.

Let us consider a many-valued function

$$S: \mathfrak{P}_{ob} \rightarrow \mathcal{M}_{ob},$$

where $\mathfrak{P}_{ob}$ means the collection of properties which mathematical objects on the considered level of $\mathcal{M}$ can be supposed to possess and $\mathcal{M}_{ob}$ means the collection of objects on that level of $\mathcal{M}$. Such a function we shall call the application of $\mathfrak{P}_{ob}$ to $\mathcal{M}_{ob}$. It assigns, to each property $P \in \mathfrak{P}_{ob}$, a collection of objects of the considered level of $\mathcal{M}$ for each of which one can suppose to possess this property. Such a collection we shall call a species. Thus we define a species as follows:

DEFINITION 1. By a *species* on a level of $\mathcal{M}$ we mean the image of a property P under an application S of $\mathfrak{P}_{ob}$ to $\mathcal{M}_{ob}$, which consists of (all) those objects of $\mathcal{M}_{ob}$ for which one can suppose to possess this property.

When a species $S(P)$ is defined, then any mathematical object which has been or might have been generated before $S(P)$ and which satisfies the condition P, is a member of the species $S(P)$. In the sequel we shall deal with the mode of generation of mathematical objects and in such a way shall contribute to the specification of members of species.

Although the study of species is not our main task in the paper, we shall still deal with certain concepts that concern them. At that, all used signs will have the usual meanings. Otherwise, one can find the definitions of these concepts in [11].

A species $S(P)$ is *empty* if, in the application of S, we cannot select any object of $\mathcal{M}_{ob}$ which satisfies the condition P. If the application S is a single-valued function, then we have the case of a *singleton* species. The size of a species is otherwise to be determined by its relating to the species of natural numbers as it is given in [11].

We further have certain relationships between species. These relations arise from the relationships which exist between the properties. If we have, for instance, that there is a relationship $P \rightarrow P'$ between two properties P and P' of $\mathfrak{P}_{ob}$, which means that, if an object has the property P, then it also has the property P', then we shall have that the species $S(P)$ is contained in the species $S(P')$, or that it is a *subspecies* of the species $S(P')$. If the above is also valid conversely, then we shall say that the species $S(P)$ and $S(P')$ are *equal.*

We can now define the concept of splitting up a species. If there is a relation $S(P) = S(P') \cup S(P'')$, where $P' \neq P''$, then we shall say that $S(P)$ is *split up* into species $S(P')$ and $S(P'')$. If $S(P')$ is here a subspecies of the species $S(P)$ and $S(P'')$ the difference $S(P) - S(P')$, then we shall say that $S(P')$ is a *detachable* subspecies of $S(P)$.

One could deal now with further questions concerning species. However, we shall not do this, especially because some of these questions are not essential for this paper and since some of them will arise later in the consideration of species which are endowed with collections of arrows and then with a certain structure. Thus we shall consider that species are specified enough for our further purposes.

Having finished with the selections of collections of objects on particular levels of $\mathcal{M}$, called species, we shall be concerned with the selection of arrows. We shall assume that any species of any level of $\mathcal{M}$ is endowed with a collection of arrows. Let us see in which manner we distribute arrows over species. If we have a species $S(P)$, then we assume that arrows in $S(P)$ are those which naturally belong to it and will do so if they preserve the property P. This property is intrinsic for the objects of species. We shall call the arrows with this property relevant arrows So, their definition is as follows:

DEFINITION 2. By an arrow *relevant* for the species under consideration we understand the arrow which preserves certain intrinsic properties characterizing its objects.

In such a way species of $\mathcal{M}$ are endowed with arrows which carry in themselves information on their objects: their structure and properties. From now on, when we say a species $S(P)$, we shall alwas regard that it is endowed with a collection of relevant arrows. Here $S_{ob}(P)$ will mean the collection of objects of $S(P)$ and $S_{ar}(P)$, the collection of arrows of $S(P)$. Otherwise, if there is no possibility of confusion, we shall denote a species $S(P)$ simply by S, i.e., we shall identify it with the application S.

Now, in order to make the species capable of satisfying our purposes we shall provide them with a certain fundamental structure. We assume here the structure of a (quasi)category*, In the following section we shall explain what this structure means.

Let S be a species on a level of $\mathcal{M}$. Endow it with two unary functions $\mathcal{D}_0, \mathcal{D}_1 : S \to S_{ob}$ and a binary function $\mathcal{C} : S^2 \to S$. In that way we obtain a system $\langle S; \mathcal{D}_0, \mathcal{D}_1, \mathcal{C} \rangle$. We have the following meanings in this system: $\mathcal{D}_0(\alpha)=x$ means that the object x is the source of the arrow α; $\mathcal{D}_1(\alpha) = y$ means that the object y is the target of the arrow α and $\mathcal{C}(\alpha,\beta)=\gamma$, which we shall also write as $\mathcal{C}(\alpha,\beta;\gamma)$, means that the arrow γ is the composition of the arrow α, followed by the arrow β.

If we now involve certain laws to specify the functions in the above system, we shall obtain a desired fundamental structure. First, we have a structure called a quasicategory:

DEFINITION 3. By a *quasicategory* we mean a system $\langle S; \mathcal{D}_0, \mathcal{D}_1, \mathcal{C} \rangle$ for which the following two groups of laws hold:

C1. $$\mathcal{D}_n(\mathcal{D}_m(\alpha)) = \mathcal{D}_m(\alpha), \quad n, m = 0, 1,$$
$$\exists \gamma\, \mathcal{C}(\alpha, \beta; \gamma) \Rightarrow \mathcal{D}_1(\alpha) = \mathcal{D}_0(\beta),$$
$$\mathcal{C}(\alpha, \beta; \gamma) \Rightarrow \mathcal{D}_0(\alpha) = \mathcal{D}_0(\gamma) \wedge \mathcal{D}_1(\gamma) = \mathcal{D}_1(\beta),$$
$$\mathcal{C}(\alpha, \beta; \gamma) \wedge \mathcal{C}(\alpha, \beta; \gamma') \Rightarrow \gamma = \gamma';$$

C2. $$\mathcal{C}(\mathcal{D}_0(\alpha), \alpha; \alpha) \wedge \mathcal{C}(\alpha, \mathcal{D}_1(\alpha); \alpha).$$

The symbols $\wedge$, $\Rightarrow$ and $\exists$ have usual meanings: $\wedge$ (and), $\Rightarrow$ (if ..., then ...) and $\exists$ (there exists). If we do not take differently, these and other logical symbols will have only such meanings throughout the paper.

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* In our papers, we have called this term so far a fundamental (quasi)semigroupoid. We think that this term is better because it carries in itself a structural meaning of the concept. Meanwhile, this is only our opinion, And since category theory is a highly developed theory, then there is no reason for changing the names of it and its concepts. Therefore, we accept here the standard name — a (quasi)category.

If we add a new law to the first group of laws

$$\mathcal{D}_1(\alpha)=\mathcal{D}_0(\beta) \Rightarrow \exists\gamma\, \mathcal{C}(\alpha, \beta; \gamma),$$

and also the law

C3. $$\mathcal{C}(\alpha, \beta; \delta)\wedge\mathcal{C}(\beta, \gamma; \zeta)\wedge\mathcal{C}(\alpha, \zeta; \eta)\wedge\mathcal{C}(\delta, \gamma; \xi)\Rightarrow\eta=\xi,$$

which means the associativity of $\mathcal{C}$, then a quasicategory becomes a category [15]

Furthermore, if we add the law

C4. $$\forall\alpha\exists\beta(\mathcal{C}(\alpha, \beta; \mathcal{D}_0(\alpha))\wedge\mathcal{C}(\beta, \alpha; \mathcal{D}_1(\alpha))),$$

then from a quasicategory we obtain a quasisemigroupoid and from a category, a groupoid.

If moreover we take that $\mathcal{D}_0=\mathcal{D}_1$ and that both are constant functions, then a (quasi)semigroupoid is reduced to a (quasi)semigroup and a (quasi)groupoid to a (quasi)group.

Certainly, a morphism between two (quasi) categories is a *functor* [15]; it is a relevant arrow in our sense. We have further morphisms between functors, called *natural transformations*, then morphisms between natural transformations, morphisms between these new morphisms etc. By this process of involving relevant arrows, we could define certain many-valued functors [6] between (quasi)categories possessing various structures as, for instance, the simplicial one, etc.

Since we have specified the basic collections of symbols of $\mathcal{M}$, called species, and have involved certain fundamental structures in them, we shall proceed further to make certain organized wholes from them. From now on we shall fix the fundamental structure on species. We assume it to be a (quasi)category: it means a quasicategory or a category, when it is necessary. A species endowed with such a structure, we shall call a *fundamental world.*

In order to make an organized whole fom a fundamental world in question we must claim that it allows some reasonable creations and other activities in itself. In what follows we shall deal with creations and collections on which they ought to be performed.

The basic purpose of creations on a fundamental world is to give us a possibility to construct new objects from the old ones. We dealt in [6] with certain creations on categories. We created certain concepts having certain geometrical shapes: cylinders, cones, etc. Here we shall be concerned with cones and cocones, since wanted constructions are contained in the creation of certain kinds of these. Thus, here cones and cocones are creative concepts. We shall call them simply *creative concepts.* In what follows we shall explain what they mean.

By a *cone* in a fundamental world W we mean a triple $(U, \Phi, \{v\})$, consisting of a subcollection U of W, a collection Φ of arrows of W and a singleton subcollection $\{v\}$ of W, consisting of an object v of W, called the vertex of the cone, such that for any arrow $\alpha\colon u'\to u\in U_{ar}$ there are arrows $\varphi : u\to v$ and $\varphi' : u'\to v$ of Φ so that $\mathcal{C}(\varphi, \alpha; \varphi')$ holds. In future, when it is obvious from the context what is the basis and vertex of the cone, we shall always identify it with the collection of arrows connecting these.

Let $\mathfrak{C}(U)=\{\Phi_i \mid i\in\mathcal{A}\}$ be the collection of all cones over U, endowed with the collection of cone-arrows. The initial object in this collection we shall call the first cone, abbreviated f.c., and denote it by Φ^c. A Φ^c is defined in the following manner: for each $\Phi\in\mathfrak{C}(U)$ there is an arrow $\gamma : v^c\to v$, where v^c is the vertex of Φ^c and v of Φ, such that $\mathcal{C}(\gamma, \varphi^c; \varphi)$ holds, for $\varphi\in\Phi$ and $\varphi^c\in\Phi^c$. In the opposite direction, we have the concept of a *cocone* with concepts of a cobasis and a covertex. The terminal object in the collection of all cocones over a collection in the world W we shall call the last cocone and abbreviate it as l.c.c.

Vertices of f.c. and l.c.c., we called in [6] a *sequent* and a *presequent*, respectively. If the basis of f.c. and the cobasis of l.c.c. contain only single objects, then their sequent and presequent we called a *successor* and a *predecessor*, respectively.

The above objects: sequents and presequents will be constructive objects in our fundamental world. Such unique objects are well-known in category theory as colimits and limits, respectively. In this case, we shall diverge from standard terminology [15] and accept our terms for these objects.

Since we have specified objects which are to be constructed we must now specify collections of the world, which will allow their construction. Moreover, we must specify certain conditions on the collections, which will determine the character of constructed objects. Thus, our basic task is to specify choices of collections on which we perform constructions and to specify certain requirements on them which will determine the peculiarity of constructed objects.

First, we shall specify the concept of a choice in a fundamental world under consideration.

DEFINITION 4. By a *choice* in a fundamental world W we mean an application σ of a fundamental world J to the world W, i.e., a many-valued functor

$$\sigma : J\to W,$$

which assigns, to each object $i\in J$, a collection $\sigma(i)\subset W$ and to each arrow $i\to i'\in J$, a relevant arrow $\sigma(i)\to\sigma(i')$. The choice σ is *lawlike*, if there is a law or a collection of laws, according to which it is to be performed.

A choice σ, determined by the collection of laws Λ, we shall denote by σ_Λ. Thus, if we have the chosen collection $\sigma_\Lambda(i)$, $i\in J$, on W, then it will mean that it satisfies the conditions of Λ.

In order to ensure the possibility of having various choices for single objects of J, we shall involve the concept of parametrized choice.

DEFINITION 5. By a choice in the fundamental world W, *parametrized* by means of a fundamental world $\mathcal{S}$, we mean a collection of choices $\mathfrak{S} = \{\sigma^s \mid s\in\mathcal{S}_{ob}\}$ such that, if there is an arrow $s\to s'\in\mathcal{S}$, then there is also a natural transformation $\sigma^s\to\sigma^{s'}$.

A natural transformaton η between two many-valued functors σ and τ, symbolically $\eta : \sigma\to\tau$, is defined in the same way as the transformation between single-valued ones, but with a difference: instead of an arrow, we now have the concept of a (co)cylinder [6]. Thus, while σ and τ are many-valued functors, $\eta : \sigma\to\tau$ is a (co)cylinder with the lower (co)basis σ and the upper one in τ. If one of these functors is single-valued, then we shall obtain concepts of a cocone and a cone, respectively.

We can represent a parametrized choice $\mathfrak{S}$ as a collection of functors $\sigma: \mathscr{S} \rightarrow \mathrm{Fun}_{mv}(\mathrm{J}, \mathrm{W})$ such that σ^s, $s \in \mathscr{S}_{ob}$ are many-valued functors of J to W and σ^f, for an $\mathscr{S}$-arrow f, are natural transformations; it means that, if $f: s \rightarrow s'$ is an $\mathscr{S}$-arrow, then $\sigma^f: \sigma^s \rightarrow \sigma^{s'}$ is a (co)cylinder.

According to the definition, $\mathfrak{S}$ constitutes a fundamental world: its objects are chosen collections of the fundamental world under consideration and arrows are cylinder-arrows [6].

Since our aim is not to have arbitrary choices but choices determined by certain conditions, then we shall impose these upon them. We shall assume that each choice $\sigma^s \in \mathfrak{S}$ satisfies a collection $\Lambda_s, s \in \mathscr{S}_{ob}$, of conditions. Such a choice, we shall denote by $\sigma^s_{\Lambda_s}$. Hence we have that $\mathfrak{S}$ consists of choices $\sigma^s_{\Lambda_s}$, $s \in \mathscr{S}_{ob}$. If Λ means the collection of collections $\Lambda_s, s \in \mathscr{S}_{ob}$, then we shall write $\mathfrak{S}$ by $\mathfrak{S}_{(\Lambda)}$. Thus $\mathfrak{S}_{(\Lambda)}$ will mean a collection of lawlike choices $\sigma^s_{\Lambda_s}$, $s \in \mathscr{S}_{ob}$. We shall call it the *lawlike parametrized choice*. Such a choice will constitute a fundamental world if the existence of an $\mathscr{S}$-arrow $s \rightarrow s'$ implies the existence of a relevant arrow between collections Λ_s and $\Lambda_{s'}$ and a natural transformation $\eta^{s,s'}: \sigma^s_{\Lambda_s} \rightarrow \sigma^{s'}_{\Lambda_{s'}}$. A relevant arrow $\varphi: \Lambda_s \rightarrow \Lambda_{s'}$ together with a natural transformation $\eta^{s,s'}: \sigma^s_{\Lambda_s} \rightarrow \sigma^{s'}_{\Lambda_{s'}}$ will be a relevant arrow between elements of $\mathfrak{S}_{(\Lambda)}$ which we shall simply call a *choice-arrow*.

Certainly, a lawlike parametrized choice $\mathfrak{S}_{(\Lambda)}$ will be specified when we specify its objects and these when we specify the collection $\Lambda = \{\Lambda_s \mid s \in \mathscr{S}_{ob}\}$. Thus in order to specify the choice $\mathfrak{S}_{(\Lambda)}$ we have to specify collections Λ_s and their connectives. In what follows we shall be concerned, but only in general, with this question.

There are two moments which we have to differentiate in each collection Λ_s of Λ : effective procedures by means of which we choose subcollections of the world under consideration and conditions which chosen collections have to satisfy, such as size, ordering, constructive properties, etc. First of all, we could specify various algorithms for choosing mentioned collections of objects and arrows of the world in question. Among them, however, we shall accept only those which ensure certain necessary properties of chosen collections and hence wanted peculiarities of constructed objects. Of course, if we want to have peculiarities of the whole choice $\mathfrak{S}_{(\Lambda)}$ we have moreover to specify connectives between its members. We could assume an example in which choices are sequences of objects and arrows between them, chosen by a collection of conditions, and choice-arrows are relevant arrows between these sequences. Throughout the paper, we shall deal with the further specification of the collection Λ. We shall also give some concrete examples.

Besides conditions which are imposed upon objects of $\mathfrak{S}_{(\Lambda)}$, we might also impose crtain conditions upon $\mathfrak{S}_{(\Lambda)}$ as a whole. Namely, we might claim that $\mathfrak{S}_{(\Lambda)}$ as a whole obeys certain conditions: to be directed for instance, to have some creative properties, etc. If Ω is such a collection of conditions on $\mathfrak{S}_{(\Lambda)}$, then we shall emphasize this by writing $\mathfrak{S}^{\Omega}_{(\Lambda)}$ instead of $\mathfrak{S}_{(\Lambda)}$. In the collection Ω, there may be reflected properties of the world $\mathscr{S}$ by means of which the choice $\mathfrak{S}_{(\Lambda)}$ is parametrized. If we suppose that the collections of conditions Λ and Ω are completely specified, then so is the choice $\mathfrak{S}_{(\Lambda)}^{\Omega}$. Otherwise, the collections Λ and Ω may be independent or that the collection Ω contains some further specifications of the collection Λ.

From now on, a choice $\mathfrak{S}_{(\Lambda)}$ on W, parametrized by the world $\mathcal{S}$, we shall regard as a many-valued functor of J into W which assigns, to each object i of J, the fundamental world $\mathfrak{S}_{(\Lambda)}^{\Omega}(i)$, objects of which are collections $\sigma_{\Lambda_s}^{s}(i)$, $s \in \mathcal{S}_{ob}$, of objects and arrows of W determined by rules of Λ_s and relevant arrows of which are choice-arrows $\sigma_{\Lambda_s}^{s}(i) \to \sigma_{s'}^{s'}(i)$, $s, s' \in \mathcal{S}_{ob}$ and which possesses the conditions of Ω, and to each arrow $i \to i'$ of J, a relevant functor $\mathfrak{S}_{(\Lambda)}^{\Omega}(i) \to \mathfrak{S}_{(\Lambda)}^{\Omega}(i')$, i.e., a functor which preserves intrinsic properties of the world $\mathfrak{S}_{(\Lambda)}^{\Omega}$.

Let $\mathfrak{C}(W) = \left\{ \mathfrak{S}_{(\Lambda_\beta)}^{\Omega_\alpha} \mid \alpha \in \mathcal{A} \wedge \beta \in \mathcal{B} \right\}$ be the collection of lawlike parametrized choices on the fundamental world W; at this we assume that there is a collection $\mathcal{S} = \{\mathcal{S}_\alpha \mid \alpha \in \mathcal{A}\}$ of parameter worlds. If $\mathfrak{S}_{(\Lambda_\beta)}^{\Omega_\alpha}$ and $\mathfrak{S}_{(\Lambda_{\beta'})}^{\Omega_{\alpha'}}$ are two members of the collection $\mathfrak{C}(W)$, then we can define a relevant arrow between them. It is a functor $R: \mathfrak{S}_{(\Lambda_\beta)}^{\Omega_\alpha} \to \mathfrak{S}_{(\Lambda_{\beta'})}^{\Omega_{\alpha'}}$ which assigns, to each choice $\sigma_{\Lambda_{s_\alpha \beta}}^{s_\alpha} \in \mathfrak{S}_{(\Lambda_\beta)}^{\Omega_\alpha}$, a choice $R\left(\sigma_{\Lambda_{s_\alpha \beta}}^{s_\alpha}\right) \in \mathfrak{S}_{(\Lambda_{\beta'})}^{\Omega_{\alpha'}}$ and, to each choice-arrow $\varphi \in \mathfrak{S}_{(\Lambda_\beta)}^{\Omega_\alpha}$, a choice-arrow $R(\varphi) \in \mathfrak{S}_{(\Lambda_{\beta'})}^{\Omega_{\alpha'}}$ and moreover preserves the conditions of Ω. Provided with such arrows it becomes a category. We shall return later to some further questions concerning the collection $\mathfrak{C}(W)$ and collections created on it.

Since we have finished with a general consideration of choices on W, we shall be concerned with the concept of spatial whole. We have already said that this concept arise from certain constructive activities on the world under consideration. Since we have done all preparations for such activities, we shall proceed to specify them.

Let $\mathfrak{S}_{(\Lambda)}^{\Omega}$ be a lawlike choice functor of J to the world W, parametrized by the world $\mathcal{S}$. As we have already seen, this functor assigns, to each object $i \in J$, a fundamental world $\mathfrak{S}_{(\Lambda)}^{\Omega}(i)$ on W consisting of choices $\sigma_{\Lambda_s}^{s}(i)$, $s \in \mathcal{S}_{ob}$, and of choice-arrows as connectives between them. By means of this functor is specified the choice activity on the considered fundamental world. Our ultimate aim, however, is not such an activity, but the constructive activity. We shall ensure this if we claim that chosen collections in the world allow some creations; we here decided on cone and cocone creations. In that way, the choice activity on a fundamental world becomes a preparatory activity for the creative activity.

If we have, for instance, a choice $\sigma_{\Lambda_s}^{s}$, $s \in \mathcal{S}_{ob}$ in W and a cone as the creative concept on it, then we can express this as a requirement that there is a single-valued functor $F^s : J \to W$ and a natural transformation $\eta^s : \sigma_{\Lambda_s}^{s} \to F^s$. Certainly, the triple $(\sigma_{\Lambda_s}^{s}, \eta^s, F^s)$ is a cone with the vertex F^s; we could say that the functor F^s is a *creative functor* for the choice functor $\sigma_{\Lambda_s}^{s}$. Hence we have that $(\sigma_{\Lambda_s}^{s}, \eta^s, F^s)(i)$, $i \in J$, is a cone with the vertex $F^s(i)$ in the world W. We shall denote a cone over

$\sigma^{s}_{\Lambda_s}$ by ${}_c\sigma^{s}_{\Lambda_s}$. If all choices $\sigma^{s}_{\Lambda_s} \in \mathfrak{S}^{\Omega}_{(\Lambda)}$ allow creations of cones, i.e., if for each $s \in \mathcal{S}_{ob}$ there is a single-valued functor F^s together with a natural transformation $\eta^s : \sigma^{s}_{\Lambda_s} \to F^s$, then we shall emphasize this by ${}_{(c)}\mathfrak{S}^{\Omega}_{(\Lambda)}$. If it is the world about cocones, then we shall accept the denotation ${}_{(cc)}\mathfrak{S}^{\Omega}_{(\Lambda)}$. However, in future, we shall simply write ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ considering that this means that each choice of $\mathfrak{S}^{\Omega}_{(\Lambda)}$ allows a $*$-creation which may be a cone or a cocone, or even both them. These concepts, as we have already said, are *creative concepts* in the paper with a common denotation $*$; if its (co)basis is known, σ for instance, then we shall write it by ${}_*\sigma$. To mention that we could decide on broader kinds of creative concepts such as cylinders and cocylinders. However, we shall only deal with accepted concepts; it means, cones and cocones. Otherwise, these cencepts, as we shall see later, are able to incorporate in themselves logical concepts of production (derivation) with vertices as produced — created objects, peculiarities of which are determined by conditions being imposed on choices. If each choice of $\mathfrak{S}^{\Omega}_{(\Lambda)}$ allows the creation of the concept $*$, then we shall say that $\mathfrak{S}^{\Omega}_{(\Lambda)}$ is *$*$-completed* in itself or outside, depending if the concept $*$ belongs to $\mathfrak{S}^{\Omega}_{(\Lambda)}$ or not; of course, its (co)basis belongs to it. As we know, the number of creative concepts of ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ and connectives between these are determined by means of the world $\mathcal{S}$.

It is clear that for each $s \in \mathcal{S}_{ob}$ there may exist many cones over the same choice $\sigma^{s}_{\Lambda_s}$ as their basis. We could point this by writing ${}_{*\alpha}\sigma^{s}_{\Lambda_s}$, if it is the world about the creative concept $*$; here $\alpha \in \mathcal{A}$, where $\mathcal{A}$ means the number of creative concepts over one and the same choice. However, among the possible creative concepts we might decide on the ultimate ones, i.e., on f.c. and l.c.c. concepts: unique or not unique.

Now, if we have a $*$-completed functor ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$, then the following question arises: can we complete the functor ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ as a whole? Certainly, we can do this. Such a functor will be completed if there is a creative concept $\bullet$ which will constitute a cone or a cocone with it; clearly, $\bullet$ is also a many-valued functor having the shape of a creative concept. If the choice functor is $\bullet$-completed, then we shall denote it by ${}_\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$. Certainly, the functor ${}_\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$ has complete its elements and is completed as a whole.

There is an elementary proposition which establishes the link between constructive objects of ${}_\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$: that one of $\bullet$ and those of choices of $\mathfrak{S}^{\Omega}_{(\Lambda)}$.

PROPOSITION 1. *If the concept $\bullet$ in ${}_\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$ is its l.c.c. (or f.c.), and if moreover each creative concept ${}_*\sigma^{s}_{\Lambda_s}$ of ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ is l.c.c. (or f.c.), then the covertex (the vertex) of $\bullet$ is the presequent of presequents (the sequent of sequents) of choices of $\mathfrak{S}^{\Omega}_{(\Lambda)}$.*

PROOF. Denote by $\mathcal{H}$ the collection of all presequents (sequents) of choices $\sigma^{s}_{\Lambda_s}$, $s \in \mathcal{S}_{ob}$ and by $P\,(S)$ the covertex (the vertex) of the concept $\bullet$. Clearly, $P\,(S)$ is the covertex (the vertex) of a cocone (a cone) over $\mathcal{H}$. It is easy to see that such a cocone (cone) is l.c.c. (f.c.). ∎

Hence we have that the constructive object of the concept $*$ is a *construction of constructions* on choices of $\mathfrak{S}^{\Omega}_{(\Lambda)}$. In such a way we have a terminating procedure for the creation of objects within a fundamental world. Such a terminating procedure will be later utilized in the definition of proof.

A $\bullet$-completed functor $_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})$ will serve for the creation of a spatial whole from the fundamental worlds making its domain and codomain: J and W for instance. This functor assigns, to the world J, a collection $_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})\,(J)$ of fundamental worlds and functors chosen on W. We might claim that this functor is such that this collection is also a fundamental world which moreover possesses some properties: to have the first and the last object, to be directed, well-ordered, etc. We might add to it some further requirements which govern the formation of wanted spatial whole as, for instance, separation ones. Denote the collection of all such requirements on the functor by Θ and the functor itself by $_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$. By means of such a functor we shall define the concept of spatial whole on a fundamental world.

DEFINITION 6. By an *$\mathscr{S}J'$-spatial organization* on the fundamental world W of a level of $\mathscr{M}$ we mean a choice functor $_{*}\mathfrak{S}^{\Omega}_{(\Lambda)}: J \to W$, which assigns, to each object $i \in J$, a $*$-completed fundamental world $_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}(i)$ of W and, to each arrow $i \to i' \in J$, a relevant functor $_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}(i) \to {}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}(i')$, for which there is a functor $\bullet: J' \to W$, where $J' \subset J$, which assigns, to each object $i \in J'$, a creative concept $\bullet\,(i)$ of the same type as those of $_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}\,(i)$, $i \in J'$ and, to each J'-arrow $i \to i'$, a relevant arrow between these concepts, together with a natural transformation $\gamma: \bullet \to {}_{(*)}\mathfrak{S}^{\Omega}_{(\Omega)}$ or $\gamma': {}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)} \to \bullet$ such that the triples $(\bullet, \gamma, {}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})\,(i)$ and $({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}, \gamma', \bullet)\,(i)$, $i \in J'$ are a cocone and a cone respectively and such to satisfy certain conditions given in the collection Θ.

By an *$\mathscr{S}$ J'-spatial whole* we mean the triple $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W\rangle$ consisting of the worlds J and W and of an $\mathscr{S}J'$-spatial organization $\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$.

According to the definition, a spatial organization $_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ on a fundamental world gives a certain creative closeness and in such a way creative possibilities of the world. These possibilities and their peculiarities are, otherwise determined by the collections of conditions Λ, Ω and Θ in which, as we have already said, properties of the worlds $\mathscr{S}$ and J may be included. By means of these collections, we are able to *handle* choices and creations on the world in question. In such a way we enable that certain particularly chosen parts or the whole world allow creative activity and moreover to obtain wanted kind of created objects in it. In what follows we shall be concerned with certain properties and further specifications of spatial organizations on a fundamental world.

We shall get further peculiarities of spatial wholes if we suppose that the functor $\mathfrak{S}^{\Omega}_{(\Lambda)}$ is *transitive*, i.e., if we suppose that $\sigma^{s}_{\Lambda_s} \in \mathfrak{S}^{\Omega}_{(\Lambda)} \Rightarrow \sigma^{s}_{\Lambda_s} \subset \mathfrak{S}^{\Omega}_{(\Lambda)}$ for each $s \in \mathscr{S}_{\mathrm{ob}}$. Hence we would have that $\mathfrak{S}^{\Omega}_{(\Lambda)}$ is also a choice on the world in question and that $\sigma^{s}_{\Lambda_s}$, $s \in \mathscr{S}_{\mathrm{ob}}$ are its parts.

If we consider now a spatial organization such that the functor $_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ is transitive and such that together with $\sigma^{s}_{\Lambda_s}$ it contains the creative concept $_{*}\sigma^{s}_{\Lambda_s}$, then it can possess convenient properties. So, for instance, if we suppose that $\mathscr{S}$ is an ordinal, ω for example, then we can prove the following

PROPOSITION 2. *If the functor* ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ *is transitive and parametrized by the ordinal* ω, *then we can make it to be recursive.* ▮

PROOF. Take any choice $\sigma^{k}_{\Lambda k} \in \mathfrak{S}^{\Omega}_{(\Lambda)}$, $k \in \omega$ and the creative concept ${}_{*}\sigma^{k}_{\Lambda k}$ over it. If we specify the conditions of Ω in such a way that this concept is the choice for the next creation of the same type, i.e., if ${}_{*}\sigma^{k+1}_{\Lambda k+1} = {}_{*}({}_{*}\sigma^{k}_{\Lambda k})$ and in the same time specify the choice $\sigma^{0}_{\Lambda_0}$, then ${}_{*}\sigma^{\Omega}_{(\Lambda)}$ will obviously be as required. ▮

We can make it to have some other convenient properties: to be directed or filtered, to have a simplicial form [6]; or, in a special case of this, the form of a tree, etc. If it is the world about filters, then ${}_{\bullet}({}_{(*)}\sigma^{\Omega}_{(\Lambda)})$ will mean a $\bullet$-completed filter; with $\bullet$ as a single-valued functor. They are completed filters in the collection ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ (J) of filters on W. If we have convenient arrows in this collection, then, by means of such arrows, we can complete other filters relating them to the completed ones. This completion is the essence of topological spatial organization (see [7]).

All specifications which we carry into ${}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ determine the peculiarities of the structure of the whole in question. If this functor is completely specified, and will be if we specify mentioned collections of conditions and corresponding creative concepts, then we shall say that we have a specified spatial organization on the fundamental world under consideration and hence specified the *structural type* of the whole. Relevant arrows between spatial wholes with specified structural type, called *spatial whole-arrows,* are those ones preserving the type in question. Continuous arrows, in the case of topological spatial wholes, are such arrows [7].

Certainly, in the specifiication of the structural type of a spatial whole, we have to differ wholes with one kind of creations on chosen collections: a cone or a cocone creation and those with both kinds of them; which, of course, can be performed on the same or various collections. The former, we shall call spatial wholes with the *simple* type and latter, spatial wholes with the *mixed* type. If two spatial wholes have the same kind of creations — simple or mixed ones, we shall say that they have the same creative type.

Let us consider, o nce again, the collection **Ch**(W) of choice-functors $\mathfrak{S}^{\Omega_\alpha}_{(\Lambda_\beta)}$ $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, on a fundamental world W. As we know, any choice functor of **Ch**(W) gives a spatial organization on W. It means that the collection **Ch**(W) serves as a groundwork for the existence a new collection: a collection of spatial organizations on W. We shall denote it by **Sp**(W). We could now deal with this collection: select spatial organizations on W according to their structural type, define relevant arrows between them and accordingly involve a fundamental structure on selected collections, then define concepts of the *sequent* and *presequent spatial organization* over W, etc. In such a way we make **Sp**(W) itself to carry a certain spatial organization. However, this organization, with respect to those on W, is of the higher level. This fact will be incorporated in our general requirement concerning the existence of spatial wholes of various levels and their vertical connection in $\mathcal{M}$.

Now we shall deal with the question of involvement of new spatial organizations over living ones. If we have a spatial whole on a fundamental world and want to involve another one over it, we have to take into account that creations of the new organization are relevant with respect to the former one, i.e., to preserve it.

In such a way we ensure the compatibility of creations and together with choices the *compatibility* of spatial organizations over one and the same fundamental world; of course, at this, organizations may be of the same or various types.

Now we shall say a few words about the link of spatial organizations in a single organization. If we have two simple and opposite types of spatial organizations, those with cone and cocone creations, then we can combine them in an organization of mixed type assuming that one type of choices and creations is utilized for choice purposes of another type. We have such a situation, for instance, in the case of intuitionistic spatial organizations.

We shall point out one more moment. Namely, we can involve a spatial organization on a fundamental world from an already defined spatial organization on that world by means of certain *well-defined operators*. In this case we have to preserve a part of living spatial organization and to involve a new part; a part which we are going to involve. It means that operators have to be such to enable this. We have this, for instance, in the case of topological spatial wholes [7]: we have involved a topological organization on a fundamental world from an already defined spatial organization: an 1-semigroupoid by means of complementation and closure operators. We shall see later some other examples as for instance Post algebras [13], etc.

Now we shall be concerned with the concept of a subwhole of a spatial whole. This concept, we obtain in the following manner: Let $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W\rangle$ be a spatial whole. By a *spatial subwhole* of this whole we mean a spatial whole $\langle \tilde{J}, {}_{\bullet}({}_{(*)}\tilde{\mathfrak{S}}^{\Omega}_{(\Lambda)})_{\Theta}, \tilde{W}\rangle$, where $\tilde{J} \subset J$, $\tilde{W} \subset W$ and ${}_{\bullet}({}_{(*)}\tilde{\mathfrak{S}}^{\Omega}_{(\Lambda)})_{\Theta}$ is a subfunctor of ${}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ which imposes the same type structure on $\tilde{W}$ as ${}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ on W.

We can define one more kind of subwhole of a spatial whole $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W\rangle$ — a choice subwhole. Namely, if $\mathfrak{S}^{ch}_{\Theta} : J \to W$ is a many-valued functor consisting of single-valued functors I^s such that $I^s \in_{*} \sigma^{s}_{\Lambda_s}$ for each $s \in \mathcal{S}_{ob}$, which is moreover completed by a single-valued functor I, then the spatial whole $\langle J, {}_{I}\mathfrak{S}^{ch}_{\Theta}, W\rangle$ is a *choice subwhole* of the considered spatial whole. Certainly, it is fully embedded in the whole $\langle J, {}_{\bullet}({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W\rangle$. We could assume that functors I^s, $s \in \mathcal{S}_{ob}$ are constructive functors for choices σ^s. In that case the functor I for such a choice functor ${}_{I}\mathfrak{S}^{ch}_{\Theta}$ will be the construction of constructions on mentioned choices. Peculiarities of such a construction are determined by means of conditions of Ω.

Now we shall be concerned with certain internal activities in the creation of a spatial whole on a fundamental world W. We shall enable this activity if, besides objects of the world W, we include collections of subobjects of its objects in the creative procedure. These new collections in W will allow certain new creations of spatial wholes within the whole which we create on W and, of course, their inclusion in the creation of the whole itself. In such a way we shall obtain more creative possibilities and hence convenient properties of the whole which we create on W. To do this we must claim that among the choices necessary for the creation of the whole there are also choices which will ensure the creativity of spatial wholes on collections of subobjects. Spatial wholes which allow such an activity we shall call spatial wholes with the local spatial organization. We define them as follows:

DEFINITION 7. We shall say that a spatial whole on the world W will admit a *local spatial organization* if there is a functor $\mathcal{T}: W \to W$ which assigns, to each object $a \in W$, a species $\mathcal{F}(a)$ such that there is an arrow $a \to \mathcal{F}(a)$ with respect to which $\mathcal{F}(a)$ strictly dominates a, and to each arrow $a \to a'$ of W a species-arrow $\mathcal{F}(a) \to \mathcal{F}(a')$.

Certainly, a power-object functor, i.e., a functor which assigns, to each object of W, the species of its subobjects is such a functor.

We now have to make the local spatial organization to be effective in the considered spatial whole $\langle J, \bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}, W \rangle$. We can ensure this by relating the functor $\mathcal{F}$ to a constructive functor being defined on this whole. For that purpose, however, we have to assume that among choice-functors with domains in J, there are also those ones with domains in the world W itself; it means that we assume that W is a subworld of J.

DEFINITION 8. We shall say that a local spatial organization on a spatial whole on the world W is *effective* if there is a choice-functor $\mathfrak{L} : W \to W$ such that the functor $\mathcal{F}$ is naturally equivalent to its sequent functor.

Topoi [10], for instance, are spatial wholes in which the local spatial organization is effective. This is realized by means of the existence of an object which is the representing object for the power-object functor.

Spatial wholes with the above property are very important because they have certain levels-organizations. Namely, on each $\mathcal{F}(a)$, where a is an arbitrary object of the regarded spatial whole, we can involve a spatial organization; these spatial organizations are internal ones. Hence we might say that the functor $\mathcal{F}$ is in fact a spatial whole-functor, i.e., a functor which bears the structure of a certain spatial whole. Spatial wholes on the object a and on $\mathcal{F}(a)$ are spatial organizations on two consecutive levels, within a living spatial whole, the structure on $\mathcal{F}(a)$ is the *hyperspatial structure* with respect to that on the object a.

Thus, spatial wholes with local spatial organizations admit different levels-organizations. With respect to levels of the collection $\mathcal{M}$ as a whole, these organizations are horizontal, i.e., along a fixed level of $\mathcal{M}$.

Since we have finished with considerations of horizontal organization of $\mathcal{M}$, i.e., with the organization of particular levels of $\mathcal{M}$, we shall do this with $\mathcal{M}$ as a whole. It means that we now have to organize $\mathcal{M}$ *vertically*, i.e., to find the link between symbols of various levels of $\mathcal{M}$, in order to obtain a coherent global organization of $\mathcal{M}$. As we have already seen, we organize symbols of a level of $\mathcal{M}$ in certain wholes. Now we assume that symbols of the first higher level with respect to a level of $\mathcal{M}$ which is under consideration represent wholes and arrows between these wholes of the latter level. In that case, we can talk about species of this new level. Its members are clearly symbols which stand for spatial wholes and connectives between these wholes of the first lower level. It means that, if we now want to realize a spatial organization on this species we have to take into account symbols which mean properties of symbols standing for its objects and arrows Namely, creative capabilities of this species are determined by means of structural and other characteristics of its objects and of course of specifications of arrows in it.

Before the creation of spatial wholes on species of the new level of $\mathcal{M}$, we have to make them to be fundamental worlds. It means that we have to specify them in that sense. Since arrows in species have to be relevant, then spatial wholes — objects in them have to be of the same structural type. Hence the notion — structural type is instrinsic for a species. This we could utilize to spefify them. Thus, as a supplement to the specification of species, we would have that *all objects in them are those having the same structural type*. In the same time we have the specification of arrows in $\mathcal{M}$: they are those which preserve structural types in question; we called them relevant arrows.

If we now assume that all we have said above is valid for any two consecutive levels of $\mathcal{M}$, then we could say that the world M is organized completely: horizontally and vertically.

Finally, if we now view the organization of the mathematical world $\mathcal{M}$, we shall notice that it is inductive. Namely, in its organizing we have first to make the organization of a level of $\mathcal{M}$ and after its ending have to pass over to organize the first higher level. What this means? This means that we have to find (all) mathematical entities on species of already created mathematical entities having the particular structure characterized by spatial organizations here given and to continue to create new mathematical entities on, in such a way, created entities. To see which species of mathematical entities will admit a spatial organization we have to know their choice and structural capabilities. Of course, this requires a separate study of spatial wholes and their properties.

In this approach, we assume that there exists a starting level with certain *starting objects* from which we begin the creation of the world; we could assume that these objects are undivisible. Hence we have that all symbols of $\mathcal{M}$, except the starting ones, are created by processes given in the paper: *objects have structural forms of a certain spatial whole and arrows are such to preserve these forms.* From the creative processes arise properties of symbols which stand instead of mathematical entities. Hence we could say that symbols adjoined to symbols of $\mathcal{M}$ to represent their characteristics are also creative and obtained in the process of creation of the world of mathematics.

3. Examples of spatial wholes

We shall deal in this section with certain concrete and typical examples of spatial wholes — wholes with specified structural types. They are topological and intuitionistic spatial whole. These wholes are detailly studied in [7] and [8]. Here we shall only deal with their mode of generation. Afterwards we shall compare these organizations to some standard mathematical conceptions as they are formalism and intuitionism and see what they mean from the standpoint of these organizations.

We obtain a *topological spatial whole* $\langle J, \mathfrak{S}^{top}, W\rangle$ if we assume that J is a discrete fundamental world, i.e., a world consisting of objects and identity arrows such that there is an injection functor I of it to W and that $\mathfrak{S}^{top}$ is a transitive functor which assigns, to each $i \in J$, a filter $\mathfrak{S}^{top}(i)$ and that each such filter allows a cocone creation in W; it means that it is completed in such a way to make a cocone. The collection of all filters on W is endowed with relevant arrows called opposite inclusions. With respect to these arrows the functor $\mathfrak{S}^{top}$ is supposed to obey certain conditions (see [7]). We can see that such a spatial organization has two types of choices and two types of constructions: it allows i) arbitrary f.c. creations and ii) l..c.c. creations on collections with restricted size; it moreover contains the objects o and 1. Otherwise, a topological spatial organization one can involve by means of certain operators as they are the complementation and closure operator (see [7]).

We could also here define the concept of the *pseudotopological spatial whole.* It is enough to take, for this purpose, that the range of choice-functions is in the collection of filters of a fundamental world. It means then that objects of these functors are filters with filter-arrows as connectives. Certainly, we now can impose the spatial structure of topological type on this new fundamental world consisting of filters and filter-arrows.

An *intuitionistic spatial whole* or an intuitionistic topological space $\langle J, \mathfrak{S}^{\text{int}}, W\rangle$ one can obtain if one assumes that each $\sigma^a \in \mathfrak{S}^{\text{int}}$, $a \in W_{\text{ob}}$, is obtained by means of presequent constructions, which every finite collection of W is assumed to admit, in the following manner: each $\sigma^a(i)$, $i \in J_{ob}$ consists of all those objects a' of W for which there is a W-arrow $F(i) \wedge a' \to a$, where $\wedge$ means the presequent construction and $F: J \to W$ is a single-valued functor. We assume that for each $a \in W_{ob}$, there is a single-valued functor $S^a: J \to W$ and a natural transformation $\eta^a: \sigma^a \to S^a$ such that (σ^a, η^a, S^a) (i), $i \in J_{ob}$ is an l.c.c. in W: it will allow that creation in itself if moreover $S^a(i) \in \sigma^a(i)$. The functor S^a is the creative functor for the functor σ^a, i.e., its *sequent functor.* We still claim that the existence of a connection — an arrow between objects $a, a' \in W$ implies the existence of a natural transformation between sequent functors S^a and $S^{a'}$.

Hence we could say that an intuitionistic topological space has constructively closed parts. However, it has not this property as a whole. To ensure this we shall assume that J and W have strict first objects [6] and that F is such to preserve such an object. If this is fulfilled, then the space as a whole will posses the sequent of all its objects. We shall denote it by 1. This object is, otherwise, equal to $S^o(o')$, where o and o' are strict first objects of W and J, respectively, Such a space has the following properties:

a) it contains the objects o and 1,

b) it is closed with respect to finite p esequents, and

c) it is closed with respect to particular sequents, i.e., sequents of particularly chosen subcollections.

We gave in [8] certain characterizations of intuitionistic topological spaces. Moreover we gave the link between these and topological spaces. We proved the following

PROPOSITION 3. *An $\aleph_0$-topological space is an intuitionistic topological space.* ❙

Now we shall select certain operators on an intuitionistic topological space having the object o. Let $\langle J, \mathfrak{S}^{int}, W\rangle$ be such a space determined by the functor F and parametrized by the world W itself. The object $S^o(i)$, $i \in J$, in it is an W-object satisfying the following condition: the presequent $P(F(i), S^o(i)) = o$. If this object is unique then we might call it a *pseudocomplement* of the object $F(i)$. Furthermore, if $J = W$, then the composition $\mathbf{C}^o = S^o \cdot S^o$ of the sequent functor S^o with itself gives us an operator called the *closure operator.* This operator has the following properties: there is a unique arrow $a \to \mathbf{C}^o(a)$, $a \in W$; then $\mathbf{C}^o \cdot \mathbf{C}^o \cong \mathbf{C}^o$, $\mathbf{C}^o(o) = 1$, etc. (see [8]).

A particular kind of intuitionistic topological spaces are those which are realized and parametrized by the world W itself, i.e., choices of which are those for which $J={}_{C}\mathcal{S}=W$. These spaces, we can involve by means of certain operators: functors possessing certain properties.

Let A be a (quasi) category with defined presequent creations $P(a, a')$, $a,\ a'\in A_{\mathrm{ob}}$. Denote by δ^a, $a\in A_{ob}$, a relative functor of A to A which assigns, to each object $b\in A$, an object $\delta^a(b)$ and, to each arrow $b\to c\in A_{\mathrm{ar}}$, an arrow $\delta_a(c)\to\delta_a(b)$. If the functor δ_a, where δ_a is a functor such that $\delta_a(b)=\delta^b(a)$, is right adjoint to the functor $P(\ ,a)\colon A\to A$, i.e., if there is a natural isomorphism

$$(P(a', a), b)\cong(a',\ \delta_a(b)),$$

then we have the following

PROPOSITION 4. *The pair $\langle A;\delta_a\rangle$ consisting of a (quasi)category A having finite presequents and of a functor $\delta_a\colon A\to A$, which is right adjoint of the presequent functor $P(\ ,a)$ is an intuitionistic topological space.*

PROOF. Certainly, the object $\delta_a(b)$ is the unique sequent of all objects a' of A satisfying the above relation. Hence we can define a collection $\mathfrak{S}$ of choice functors, varying objects a and b of A, such that each has the sequent functor and which moreover obeys the connection condition: if there is an arrow $a\to c$, then there is a natural transformation $\delta^a\to\delta^c$. ∎

There is a characterization of the space $\langle A;\delta_a\rangle$, which is specified in the above proposition, given by the following

PROPOSITION 5. *The intuitionistic topological space $\langle A;\delta_a\rangle$ is a distributive $1^{\aleph_0}$-semigroupoid.*

PROOF. It is an $1^{\aleph_0}$-semigroupoid by the definition: this follows from its bicompleteness. Next we have to show that the distributive law

$$\bigvee_{a'\in A'}(a'\wedge b)\cong\bigvee_{a'\in A'}a'\wedge b,$$

where A' is a subcollection of objects of A and $\vee$ and $\wedge$ are the marks for sequents and presequents, respectively, holds in the space $\langle A;\delta_a\rangle$.

Let A' be the collection of all those a' of A such that $(a',\ \delta_b(a))=(a'\wedge b,\ a)$. Denote by $\mathbf{P}$ the collection of all presequents $a'\wedge b,\ a'\in A$, and by r the last object of $\mathbf{P}$ [6]; it is the unique sequent of all $\mathbf{P}$, i.e., $r=\bigvee_{a'\in A'}(a'\wedge b)$. Hence we have that for every $a\in A$ there is a morphism $a'\wedge a\to r$ and then also a morphism $a'\to\delta_b(r)$. Thus $\delta_b(r)$ is a vertex of a cone over A'. Since $\delta_b(a)$ is the unique sequent of all A', then there is a unique morphism $\delta_b(a)\to\delta_b(r)$ and hence a unique morphism $\delta_b(a)\wedge b\to r$. On the other hand, since $\delta_b(a)\wedge b$ is a vertex of a cone over all $\mathbf{P}$, then there is also a unique morphism $r\to\delta_b(a)$. Hence we have $r\cong\delta_b(a)\wedge b$. Since $\delta_b(a)=\bigvee_{a'\in A'}a'$, then the above relation holds. ∎

If we now claim that the space $\langle A ; \delta_a \rangle$ has an effective local spatial organization ensured by the existence of an object which is the representing object for the power-object functor, then we shall obtain the concept of a topos.

We could give many more specifications of the functor $\bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta}$ in $\langle J, \bullet({}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)})_{\Theta'}, W \rangle$ in order to obtain various kinds of spatial wholes and relationships between them. We could obtain various algebras, lattices, topological algebras, numbers: natural and real, equational classes, etc. For instance, pseudo-Boolean algebras, called also Heyting algebras, one can obtain as a special case of an intuitionistic topological space $\langle A ; \delta_a \rangle$: it is enough to take that there are unique arrows between objects in it. If we provide this algebra by two collections of operators which have some preserving properties concerning the structure of the algrebra and the collections themselves have some structure and connecting properties, then we could obtain Post algebras. It means that these algebras are certain special cases of spatial wholes involved by means of certain operators. We shall not concern further cases, but shall proceed to consider two standard mathematical views: formalism and intuitionism. We shall see what these views mean from the standpoint of spatial wholes.

A *formal system* or *formalism* can be regarded as a systematic scheme according to which we organize a collection of symbols in a whole with precisely established internal relations: relations between its concepts and rules for the creation of these. We shall show that it creates a kind of spatial whole from such a collection. In what follows we shall sketch such a system given in [9].

Let S be a collection of symbols. As it is well-known, a formal system distinguishes two collections of expressions made from elements of S: the collection of terms $T(S)$ and the collection of formulas $F(S)$. It also gives modes of generation of these collections. The collection $T(S)$ is generated from elements of S by means of certain operations and the collection $F(S)$ from $T(S)$, which is provided with certain relations, by means of logical operations. It is moreover endowed with effective rules for the derivation of formulas from some collections of these, as premises. These rules are known as the rules of inference. According to them, we may take a certain collection of fundamentally valid formulas of axioms and extend it up to a collection of valid formulas or theorems.

Now we shall see what this story means from the standpoint of spatial wholes. Certainly, the collection of symbols S, we can consider as a discrete fundamental world. We are going to specify the kind of spatial whole which a formal system involves on S. According to the above description, the collection $T(S)$ is generated in such a way to contain the collection S and to be closed with respect to finite sequents and presequents. Hence, it is certainly a spatial whole on S.

The next collection of expressions is $F(S)$. Let us see what kind of structure involves the formal system on this collection. To show this we shall first deal with a topological spatial organization on it. Suppose first that $F(S)$ is endowed with certain arrows by means of which it will become a fundamental world; it is enough to take arrows called implications. A topological spatial organization is defined on such a world by means of a many-valued functor ${}_{(*)}\mathfrak{S}$ of S to $F(S)$ which assigns, to each symbol $s \in S$, a filter ${}_{(*)}\mathfrak{S}(s)$ in such a way that there is a single-valued functor $I : S \rightarrow F(S)$ and a natural transformation $\eta : I \rightarrow {}_{(*)}\mathfrak{S}$. Hence we have that $(I, \eta, {}_{(*)}\mathfrak{S})(s)$, $s \in S$, is a cocone on $F(S)$. With respect to this structure, $F(S)$ becomes closed with respect to arbitrary f.c. and restricted l.c.c. creations. We know [7] that such a structure we can involve by means of certain operators. Hence, we can represent it as a system $\langle F(S); \wedge, \vee, \mathscr{C}, \mathbf{C}\rangle$, where $\wedge$ and $\vee$ are the signs for the pre-sequent and sequent operation respectively, $\mathscr{C}$ is the functor of complementation and $\mathbf{C}$ is the closure functor on $F(S)$. All these functors are defined in [7]. If we now look at the structure which the formal system involves on $F(S)$, we shall notice that it is just such a structure and hence a spatial structure.

It is clear that the structure on $F(S)$ is of the first higher level with respect to that on $T(S)$; objects of $T(S)$ are otherwise included in $F(S)$ through atomic formulas: collections of $T(S)$ selected by certain relations. Thus, the following proposition holds:

PROPOSITION 6. *A formal system involves a two-levels spatial organization on a collection of symbols S.* ▮

Since the structure on $F(S)$ is of topological type, then we could involve certain topological concepts in it, as they are open and closed formulas, separation and compactness conditions, etc. All these concepts one can derive from those for topological spatial wholes. So, for instance, we can see quantifiers as closure and interior operators which we defined in [7]. We shall here mention the definition of the closure operator. A *closure operator* on a fundamental world W is a covariant functor $\mathbf{C}: W \rightarrow W$ which fulfils the following conditions:

$C1$: $\mathbf{C}$ is a *successor functor,* i.e., a functor which assigns, to each object $a \in W$, an object $\mathbf{C}(a)$ which is the successor of a with respect to a W-arrow;

$C2$: $\mathbf{C}$ is an *idempotent functor,* i.e., such that $\mathbf{C}\cdot\mathbf{C} \cong \mathbf{C}$ holds;

$C3$: $\mathbf{C}$ is an f.c.$^{<c_\beta}$*-functor,* i.e., a functor which preserves f.c.'s over any $<c_\beta$-subcollection of W, where c_β means its size;

$C4$: $\mathbf{C}$ leaves *fixed* the first object of W.

The interior operator is defined in a similar manner. The complementation operator is defined as a contravariant functor with certain properties. All these functors are not defined in general to be necessarily unique ones.

If objects of the fundamental world are formulas with many variables, then the quantification by variables we can realize by the iteration of these operators along variables, i.e., as a system $I \rightarrow \mathbf{C}_{x_1} \rightarrow C_{x_1} \cdot C_{x_2} \rightarrow \ldots$ of functors and natural transformations, where I is the identity functor and $x_1, x_2, \ldots$ stand for variables in question. By the application of the complementation operator to this system we could obtain the case with the interior operator.

However, beside these concepts, there are other syntactic and semantic concepts which are relevant to various types of formal systems such as proof, consistency, model, etc. Therefore we have to put a general question: in which manner we can find the place of these concepts within those of a spatial whole,

In what follows we shall deal with this question. We shall be concerned with it only in general. We shall first consider the concept of proof.

It is well-known that a *proof* in a system is a procedure by means of which we can deduce (produce) a formula from a collection of formulas using rules which are established in the system which we are concerned with. Since our concept of spatial whole contains in itself various creative procedures, then we can say generally that an object, a formula for instance, is deducible — creative from a collection of objects and arrows if there is a convergent — terminating procedure of applications of creative concepts which starts in this collection and terminates in the desired object; otherwise, a production (derivation) in logical sense can be represented by our creative concept $*$; or, in a more broader case, by the concepts of cylinder and cocylinder. Namely, a production or a derivation is a figure of the form $t_1 t_2 \dots t_n \to t$, where $t_1, t_2, \dots, t_n$ are mathematical objects of a certain kind, terms and formulas for instance, called premises and the object t, the conclusion of the production. We can represent such a figure by our creative concept $*$ in which the vertex will be the conclusion; $t_1 t_2 \dots t_n$ is its basis. Certainly, in such a case, we can consider $\mathfrak{S}^{\Omega}_{(\Lambda)}$ as a collection of mutually linked produticons. In such a way we could obtain a Post system [12].

Now we have concepts like consistency and model. These concepts are concerned with the characterization of a system, in our case, of a spatial whole. What the *consistency* means. By means of this concept we ensure that the creative procedure of the spatial whole in question cannot produce in it an object which is in a certain sense contestable. Let us see in which manner we determine contestability of an object. The standard way is by selecting certain valuation-fibers. We do this by a relevant arrow — a morphism from the whole in question to the spatial whole consisting of two distinguished and different objects denoted by 0 and 1; in topoi, the representing object for the power-object functor serves for these purposes. Let W be a spatial whole and f a morphism of W to $\{0, 1\}$. By f we select on W two disjoint subcollections called fibers and take them as frames for our purposes: they contain contestable and incontestable objects, respectively and are otherwise bridged over by means of the complementation type functor. Having these frames, we say that an object a is a consequence of a subcollection C of W if $a \in f^{-1}(1)$ for any $f: W \to \{0, 1\}$ such that $C \subset f^{-1}(1)$, i.e., if a belongs to the same fiber as C does. This fact is known as the *semantic implication* $\models$. This implication we can represent as a certain natural transformation between a functor $I: W \to W$ having its values in the collection $C \subset W$ and a constant functor $c_a: W \to W$ having as its values the object a. We can represent this situation as a many-valued functor $\mathbf{S}^C_a = (I, \models, c_a)$ of W to itself.

If there is no f such that $f(C)=1$, then one says that C is *semantic inconsistent*, otherwise it is semantic consistent. If $f(C)=1$, then it is customary to say that f is a *model* for C. Hence we have that a collection C is semantic consistent if it possesses a model. Certainly, models in this approach are certain subcollections which are closed with respect to certain objects; by such a process we can establish if a created object belongs to the fiber or not. Having now models, we could further deal with the concept of spatial structures on collections of them. One could notice that such a situation belongs to our case of spatial wholes with a local spatial organization.

Now we shall deal with the syntactic implication and its connection with the semantic one. A *syntactic implication* $C \vdash a$, from a subcollection $C \subset W$ to an object $a \in W$, as we have already seen, is a proof of a from C. We can represent it as a many-valued functor $\mathbf{P}^C_a$ consisting of inductively connected creative concepts which starts in C and terminates in a. If such a production gives us an object which is contestable, then we shall say that C is *deductively consistent*.

We could now connect these two implications and hence many-valued functors: the semantic $\mathbf{S}_a^C$ and the syntactic one $\mathbf{P}_a^C$ of W to itself. Clearly, we might say that $\vdash$ is a specified form of the implication $\models$; namely, if there is an incontestable and terminating procedure from a collection C, then there is also the implication $\models$. Conversely, it is not always the case. Namely, in a general case of spatial organizations, we do not know always if there is a production which realize this implication.

Now we shall be concerned with *intuitionism*. First we shall deal with the formal part of intuitionistic mathematics. We shall be concerned with the structural type of the intuitionistic propositional logic. We shall show that the system of axioms for this logic involves an intuitionistic topology on the collection of its formulas.

Let us consider the system of axioms for the intuitionistic propositional logic given for instance in [13]. This system we shall write in a form which is more convenient for us at this moment. Namely, we shall write $\delta^b(a)$, or $\delta_a(b)$, instead of $a \to b$, $S(a, b)$ instead of $a \vee b$ and $P(a, b)$ instead of $a \wedge b$, for two formulas a and b. Taking this into account, we shall write the system of axioms in the following form:

A1. $a \to \delta^a(b)$,

A2. $\delta^{b\to c}(a) \to (\delta^b(a) \to \delta^c(a))$,

A3. $a \to S(a, b)$, $b \to S(a, b)$,

A4. $\delta^b(a) \to (\delta^b(c) \to \delta^b(S(a, c))$,

A5. $P(a, b) \to a$, $P(a, b) \to b$,

A6. $\delta^b(a) \to (\delta^C(a) \to \delta^{P(b, c)}(a)$,

A7. $(P(a, b) \to c) \leftrightarrow (a \to \delta^c(b))$,

A8. $P(a, \mathcal{N}(a)) \to b$, $\delta^{\underline{P}(a, \mathcal{N}(a))}(a) \to \mathcal{N}(a)$.

In what follows we shall analyse this system of axioms. We shall see what these axioms mean from the standpoint of spatial whole.

Denote the class of formulas of this logic by $\mathcal{F}$. Elements of this class we shall call objects. This class is certainly provided with a class of unique connectives; there is just one connective between two objects of $\mathcal{F}$. Endowed with such connectives, $\mathcal{F}$ becomes a category. If we have a connective $a \to b$ between objects a and b of $\mathcal{F}$, then the object a is the hypothesis and b, the conclusion of the connective.

Now we shall see what the above axioms specify on the category $\mathcal{F}$. Before all the axioms A4. — A6. specify the category $\mathcal{F}$ to be closed with respect to finite sequent and presequent operations denoted by S and P, respectively; it means that the category $\mathcal{F}$ possesses finite sequents and presequents and hence that it is an $l\aleph_0$-semigroupoid ([6].

Let us consider now a functor $\delta^b : \mathcal{F} \to \mathcal{F}$ which assigns, to each object $a \in \mathcal{F}$, with respect to the chosen object $b \in \mathcal{F}$, an object $\delta^b(a)$. Assume further the object $\delta^b(a)$ to be the unique connective $a \to b$ between objects $a, b, \in \mathcal{F}$. In such a way connectives between objects of $\mathcal{F}$ also become objects of $\mathcal{F}$.

The axioms A1. and A2. specify the functor δ^b, $b \in \mathcal{F}$. According to the axiom A1., there is a connective between object a and the object $\delta^a(b)$ being a connective with respect to the object a. We can express this as the existence of a natural transformation $\eta : I \to \delta_b$, where I is the identity functor of $\mathcal{F}$ to itself and δ_b the functor such that $\delta_b(a) = \delta^a(b)$. According to the axiom A2. we have the existence of connectives between functors. Namely, let $b \to c$ be an object, then, for the functor $b^{b \to c}$, with respect to the connective object $b \to c$, we have the existence of a natural transformation $\eta^{b,c} : \delta^b \to \delta^c$.

The axiom A7. means the adjointness relation of the functor $\delta_b^{\cdot}$ and the presequent functor $P(\ ,b)$. This relation enables us to construct the connectives of $\mathcal{F}$ in an effective manner.

Finally, the axiom A8. specifies a functor $\mathcal{N} : \mathcal{F} \to \mathcal{F}$ which assigns, to each object $a \in \mathcal{F}$, an object $\mathcal{N}(a)$ such that the presequent of a and $\mathcal{N}(a)$ precedes all objects of $\mathcal{F}$; it is certainly the strict first object.

From the above analysis of the axioms for the intuitionistic propositional logic we have that the collection of formulas of this logic has the structure of an intuitionistic space of the form $\langle A; \delta_a \rangle$ which possesses the strict first object; here, A is an $l\aleph_0$-semigroupoid and δ_a a functor on A having mentioned properties; this is the covariant form of the above functor. Hence the following proposition holds:

PROPOSITION 7. *The collection of formulas of the intuitionistic propositional logic has the structure of an intuitionistic topological space.* ∎

Hence we have that the system of axioms of the intuitionistic propositional logic involves a certain kind of spatial structure of intuitionistic type on the collection of its formulas; a spatial structure of another type is contained in building up the collection of terms; this structure is of the first lower level with respect to that on the collection of formulas.

Now we shall deal with certain concepts of nonformalized intuitionistic mathematics. The concepts which we shall concern here are those given in [2], [11] and [16]. First we have the concept of a *species*. This concept is already studied in the paper and therefore we shall not be further concerned with it. Next concept is that of a *spread*. We can obtain this concept by specifying the choice functor $\mathfrak{S}^{\Omega}_{(\Lambda)}$; it means by specifying the collections of conditions Λ and Ω. Let us see in which way.

If we assume that the functor ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ is specified, of course, by specifying Λ and Ω, in such a way to consists of many-valued functors σ^{α}_{k}, $k \in \mathcal{K}$ and $\alpha \in \omega$, such that any $(\alpha+1)$th functor is in fact a cone over a αth one, then we can represent ${}_{(*)}\mathfrak{S}^{\Omega}_{(\Lambda)}$ as a collection $\{{}_*\sigma^{\alpha}_{k} \mid k \in \mathcal{K} \wedge \alpha \in \omega\}$, where ${}_*\sigma^{\alpha}_{k}$ are many-valued functors of the form ${}_*\sigma^{\alpha+1}_{k} = ({}_*\sigma^{\alpha}_{k'}, \rho^{\alpha}_{k'k}, G^{\alpha}_{k})$ for k', $k \in \mathcal{K}$; here, $\rho^{\alpha}_{k'k}$ are natural transformations, ${}_*\sigma^{o}_{k'} = F_{k'}$ is a single-valued functor and G^{α} are constant functors. The determination of the successive concepts ${}_*\sigma^{\alpha+1}_{k}$ may be pictured as a process of progressive ramification with simplicial branches: each branch gives a simplicial concept [6]. A spread is a *fan* if the collection $\mathcal{K}$ is finite.

We could deal with other concepts of this logic, as they are choice and lawlike sequences, apartness relation, etc. We shall see for instance what the *apartness relation* means. This relation, usually denoted by $\#$, differs objects in a fundamental world and can serve for choice purposes. Namely, by means of it we can select certain subcollections of the world, objects of which are either identical or in this relation; this relation is otherwise defined to be symmetrical, i.e., such that $\# (a, b) \Leftrightarrow \# (b, a)$, for any two objects a and b. We could deal with this relation as a special arrow, or to express it by means of arrows of the world in question.

According to this, we could say that we could find the position of (all) concepts of (non)formalized intuitionistic mathematics within those of spatial whole. And since we have already said this for the case of formalism, then we might say in general that the creation of spatial wholes contains in itself main parts of mathematical activity.

4. Fundamental acts in creation of the world of mathematics

In this section, we shall formulate, but only in general, fundamental acts which occur in the creation of the world which is intended to contain (all) objects of mathematics and in the creation of which (all) mathematical activity is to be exhausted. Such a world, we have called the world of mathematics. The acts in question are extracted from preceding investigations. All preceding story, we can summarize in five general acts. The first among the acts is the following one:

A1. *Specification of the frame of the world*

We have seen that it is enough to take as a symbolic frame for the creation of the world of mathematics a collection $\mathcal{M}$ consisting of two-sort symbols of various levels. If we adjoin to these symbols some new symbols characterizing these, then we shall arrive at the new act:

A2. *Selection of basic collections*

According to the adjoined symbols, representing properties of symbols of $\mathcal{M}$, all symbols of any level of $\mathcal{M}$ one can select in particular collections called species. Namely, we first select objects on the considered level. which we call species, and afterwards make the distribution of arrows over them. Then we provide such collections of objects and arrows with certain fundamental structure. Hence, the next act is

A3. *Formation of fundamental worlds*

We assume that each species of any level of $\mathcal{M}$, provided with a collection of arrows, bears a fundamental structure — the structure of a (quasi)category. It will possess such a structure if arrows in it are relevant, i.e., if they preserve intrinsic properties of its objects. This structure serves as a groundwork for further purposes contained in the following act:

A4. *Organization of spatial wholes*

Each fundamental world of any level of $\mathcal{M}$, one can organize in a spatial whole. This act is the central one. It contains in itself two activities: choice and creative activity. These activities are comprised in creations of spatial wholes of various levels. We might say that spatial wholes are the main products of mathematical activity and hence objects of an edifice which we have called the world of mathematics. Hence we have that the world of mathematics consists of spatial wholes of various sorts and levels; of course, together with arrows between them. These arrows horizontally connect spatial wholes. Meanwhile, their vertical connection, i.e., the connection between levels is established by the following act:

A5. *Vertical connection of spatial wholes*

Each spatial whole of an arbitrary level of $\mathcal{M}$ is an object of a species of the first higher level with respect to this level. Hence,species of each level of $\mathcal{M}$ consist of spatial wholes, with specified structural type, of the first lower level with respect to their level; they are also endowed with relevant arrows.

The above five acts give us a general procedure for the creation of the world of mathematics. By following them and specifying structural types of spatial wholes we specify the mathematical world. For the complete specification of the world, it is necessary to know all structural types of spatial whole which we can involve on a species of spatial wholes. This problem, however, requires a separate study of various types of spatial wholes and their relationships. Therefore we shall not deal with it.

If we forget the structure of spatial wholes, then the world of mathematics will consist of (quasi)categories of various levels; each (quasi)category of any level of $\mathcal{M}$ has as objects (quasi) categories of the first lower level with respect to its level and functors between them as arrows. If we assume that (quasi) categories are discrete and accept a necessary part of spatial structure we could obtain the frame of the world $\mathcal{U}$ of [5].

Furthermore, as an idealization of the world of mathematics, we could obtain the world of ordinal numbers and also of their cardinal capacities. Going along levels we would have ordinals of various number classes. We could realize this by assuming that choice-functors $\mathfrak{S}^{\Omega}_{(\Lambda)}$ are completed transitive functors having the tree structure.

It would be of an interest to find the link between our approach and some other approaches to the foundations of mathematics given for instance in [1]. Moreover one could deal with the connection of some other mathematical worlds, as they are for instance worlds of set theory ([9], [14]), then the world of [3] and others with our world. We shall deal with some of these questions in a separate paper. Moreover we shall apply these investigations to develop some other mathematical theories.

5. Conclusion

We have been concerned in this paper with general aspects of mathematical activity. We have seen that this activity has as its primary goal the creation of certain mathematical entities which we have called spatial wholes and of the world which contain all these entities. This world, we have called the world of mathematics. We have specified certain features of it and its constituents. We have also given fundamental acts for its creation. Certainly, there still remains much work concerning further characterizations of spatial wholes, heredity of their properties along levels, etc.

In the next part of this paper, we shall try to formalize these investigations in a system. We shall give main features of that system and then compare it to some known system. Afterwards we shall return, once again, to the discussion of goals of mathematical activity.

Since we consider that the investigations given in this paper reflect certain features of the real world: its horizontal and vertical evolution and structure, then we shall try to apply them to natural science. Thus, beside the problem of further characterization of the concepts given in the paper and the formalization of this program, we have one more task.

Finally we should say a few words about all what we have done here. The basic idea which has been leading us in this work has been to see mathematical conceptions as various kind procedures by means of which one can create mathematical entities called spatial wholes which have to comprise in themselves mathematical and logical concepts We do not know if we have yet succeded in this.

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