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DESCRIPTIVE SET THEORY AND INFINITARY LANGUAGES

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Kurepa trees, partitions, Jensen's principles, large cardinals, and other notions from combinatorial set theory play an enormous role in the model theory of generalized-quantifier languages. (See e.g. [29].) Borel and analytic sets, Polish group actions, and notions from descriptive set theory can play almost as large a role in the model theory of certain infinitary languages. (See [31] and [32].) The present paper is a study, by the methods of descriptive set theory, of the class of *strong first-order languages*. These, roughly, are the infinitary languages which are strong enough to express wellfoundedness, at least over countable structures, yet weak enough that the satisfaction relation is Δ_1-definable.

Examples, culled from the literature of exotic model theory, are present in § 1. The set-theoretic machinery for their study is set up in §§ 2-4. §§ 5 and 6 are devoted to an exposition of the properties shared by all strong first-order languages. Most notably: *There is a quasi-constructive complete proof procedure involving rules with* $\aleph_1$ *premisses for any strong first-order language*, and *even the weak version of Beth's Definability Theorem fails for every such language.*

Many of the results in this paper date from the author's days as a student in R.L. Vaught's seminar at Berkeley, 1972-73. At that time I had the benefit of correspondence with Profs. Barwise and Moschovakis, and especially of frequent discussions with Prof. Vaught and D. E. Miller. Most of this work was included in [6], and a few items have appeared in print ([5]; [8], § 2). More recent discussions with Miller led to the discovery of the proof procedure and the counter-example to Beth's Theorem alluded to above, and to the writing of this paper.

§ 1 Some Infinitary Languages

Throughout this paper *structure* means *infinite structure* and *vocabulary* (set of predicates, function symbols, and constants) means *countable vocabulary*. References for some possibly unfamiliar notions such as primitive recursive (*PR*) set functions or Δ_1^{ZFC} definability are recalled at the beginning of § 2.

1.1 *Borel-Game Logic* $L_{\infty B}$

We introduce *codes* for Borel subsets of the power set ω as follows: $\mathcal{E}(0)=\{(0, n): n\in\omega\}$; $\mathcal{E}(\alpha+1)=\mathcal{E}(\alpha)\cup\{(1, e): e\in\mathcal{E}(\alpha)\}$ for α even; $\mathcal{E}(\alpha+1)=$ $=\mathcal{E}(a)\cup\{(2, f): f:\omega\to\mathcal{E}(\alpha)\}$ for α odd; $\mathcal{E}(\lambda)=\cup\{\mathcal{E}(\alpha):\alpha<\lambda\}$ at limits; $\mathcal{E}=\mathcal{E}(\omega_1)$. The Borel set $\mathcal{B}(e)$ coded by $e\in\mathcal{E}$ is determined as follows: $\mathcal{B}((0, n))=\{u\subseteq\omega: n\in u\}$; $\mathcal{B}((1, e))=$ complement of $\mathcal{B}(e)$; $\mathcal{B}((2, f))=$ $=\cup\{\mathcal{B}(f(n)): n\in\omega\}$.

The class of formulas of $L_{\infty B}$ in a vocabulary **R** is the smallest class which (i) contains the atomic formulas of **R**; (ii) is closed under negation $\neg$; (iii) is closed under (single) quantification $\forall$, $\exists$; (iv) is closed under conjunction and disjunction $\bigwedge$, $\bigvee$, of arbitrary sets of formulas, so long as the result has only finitely many free variables; and (v) is closed under the following operation: Given $e \in \mathfrak{E}$ and $I \neq \varnothing$ and formulas $\varphi_{i_0 i_1 \ldots i_n}(u_1 \ldots u_k, v_0 \ldots v_n)$ indexed by $I^{<\omega}$ with free variables as shown, we may form the following formula $\varphi(u_1 \ldots u_k)$:

$$(*B) \qquad \bigwedge_{i_0 \in I} \forall_{v_0} \bigvee_{i_1 \in I} \exists_{v_1} \bigwedge_{i_2 \in I} \forall_{v_2} \bigvee_{i_3 \in I} \exists_{v_3} \ldots$$
$$\ldots \{n : \varphi_{i_0 i_1 \ldots i_n}(u_1 \ldots u_k, v_0 \ldots v_n)\} \in \mathscr{B}(e)$$

The class $L_{\infty B}(\mathbf{R})$ of sentences of $L_{\infty B}$ in vocabulary **R** consists of those formulas without free variables.

Satisfaction for $L_{\infty B}$ is defined as follows: Given an **R**-structure $\mathfrak{A}$ and $b_1 \ldots b_k \in |\mathfrak{A}|$, the formula $\varphi(u_1 \ldots u_k)$ of $(*B)$ suggests an infinite game for two players PRO and CON. CON opens by picking $i_0 \in I$, $a_0 \in |\mathfrak{A}|$. PRO responds with $i_1 \in I$, $a_1 \in |\mathfrak{A}|$. And so on until infinite sequences $\mathbf{i} = i_0, i_1, i_2 \ldots$ and $\underset{\sim}{a} = a_0, a_1, a_1 \ldots$ are generated. PRO wins if $\{n : \mathfrak{A} \models \varphi_{i_0 i_1 \ldots i_n}(b_1 \ldots b_k, a_0 \ldots a_n)\} \in \mathscr{B}(e)$. Since the set of pairs $\mathbf{i}$, $\mathbf{a}$ constituting wins for PRO is a Borel subset of $I^\omega \times |\mathfrak{A}|^\omega$, by Martin's Borel Determinacy Theorem [22], either PRO or else CON has a winning strategy for this game. We define $\mathfrak{A} \models \varphi(b_1 \ldots b_k)$ to hold if PRO has the winning strategy.

If we wish to identify formulas with set-theoretic objects, we can proceed much as is done in [17] for $L_{\omega_1 \omega}$. In particular we take nonlogical symbols to be just certain hereditarily countable sets. We can identify the formula of $(*B)$ with, say, $(e, (\varphi_\sigma : \sigma \in I^{<\omega}))$. It is then not hard to see that sentencehood for $L_{\infty B}$ is a PR notion.

1.2. PROPOSITION Satisfaction for $L_{\infty B}$ is Δ_1^{ZFC}.

PROOF. Any notion defined by a reasonable induction from Δ_1^{ZFC} notions is Δ_1^{ZFC}, so it suffices to show satisfaction for a formula of $L_{\infty B}$ can be defined in a Δ_1^{ZFC} fashion in terms of satisfaction for its subformulas. We consider the case of the formula φ introduced by $(*B)$. Fix $\mathfrak{A}$ and $b_1 \ldots b_k \in |\mathfrak{A}|$ as in the definition above of satisfaction for φ. Let $f : I^{<\omega} \times |\mathfrak{A}|^{<\omega} \to \{0, 1\}$ code satisfaction for subformulas of φ:

$$f(\sigma, (a_0 \ldots a_n)) = 0 \leftrightarrow \mathfrak{A} \models \varphi_\sigma(b \ldots b_k, a_0 \ldots a_n)$$

A strategy for PRO in the game associated with φ is essentially a pair of functions $\mathscr{S} : I^{<\omega} \times |\mathfrak{A}|^{<\omega} \to I$, $\mathcal{T} : I^{<\omega} \times |\mathfrak{A}|^{<\omega} \to |\mathfrak{A}|$. Applied to sequences $\mathbf{i} = i_0, i_1, i_2, \ldots$ and $\mathbf{a} = a_0, a_1, a_2, \ldots$ $\mathscr{S}$ and $\mathcal{T}$ produce the sequences:

$$(1) \qquad \mathbf{s} = i_0, \mathscr{S}(((i_0), (a_0))), i_1, \mathscr{S}(((i_0, i_1), (a_0, a_1))), i_2, \ldots$$
$$\mathbf{t} = a_0, \mathcal{T}(((i_0), (a_0))), a_1, \mathcal{T}(((i_0, i_1), (a_0, a_1))), a_2, \ldots$$

Let $\sigma_n = \sigma_n(\mathscr{S}, \mathcal{T}, \mathbf{i}, \mathbf{a})$ be the finite sequence of the 0^{th} through n^{th} terms of $\mathbf{s}$, and define $\mathcal{T}_n$ similarly from $\mathbf{t}$. In this notation, $\mathfrak{A} \models \varphi(b_1 \ldots b_k)$ iff:

$$(2) \qquad \exists \text{ strategies } \mathscr{S}, \mathcal{T}\ \forall\, \mathbf{i} \in I^\omega, \underset{\sim}{a} \in |\mathfrak{A}|^\omega\ \{n : f(\sigma_n, \tau_n) = 0\} \in \mathscr{B}(e)$$

Now it is well known that every Borel subset of the power set of ω can be obtained from clopen sets by the fusion operation (1). Indeed the usual proofs of this fact reveal that we can obtain an operation ($\mathcal{A}$) representation of $\mathcal{B}(e)$ in a PR fashion from the code e, i.e. there is a PR function $\mathcal{N}$ from $\mathcal{E}$ to the power set of $2^{<\omega}\times\omega^{<\omega}$ such that for all $x\in 2^{\omega}$:

$$\{n\colon x(n)=0\}\ G\ \mathcal{B}(e)\leftrightarrow \exists y\in\omega^{\omega}\ \forall n\,(x\,|\,n,\ y\,|\,n)\in\mathcal{N}(e)$$

Thus (2) is equivalent to:

(3) $$\exists\mathcal{S},\mathcal{T}\ \forall \mathbf{i},\ \mathbf{a}\ \forall x\in 2^{\omega}\ \forall y\in\omega^{\omega}\ \exists n$$
$$(x(n)\neq f(\sigma_n,\tau_n)\vee(x\,|\,n,\ y\,|\,n)\notin\mathcal{N}((1,e))$$

where, let us recall, $(1, e)$ codes the complement of $\mathcal{B}(e)$.

Now for given strategies $\mathcal{S}$, $\mathcal{T}$ let $\mathcal{Q}=\mathcal{Q}(\mathcal{S},\mathcal{T})$ be the set of all four-tuples $\mathbf{i}_n=(i_0,i_1\dots i_n)$, $\mathbf{a}_n=(a_0,a_1\dots a_n)$, $\xi=(x_0,x_1\dots x_{2n+1})$, $\eta=(y_0,y_1\dots y_{2n+1})$ such that for all $m\leq 2n+1$, $x_m=f(\sigma_n,\tau_n)$ (where σ_n, τ_n are the obvious initial segments of the sequences in (1)) and $(\xi\,|\,m+1,\ \eta\,|\,m+1)\in\mathcal{N}((1,e))$. Partially order $\mathcal{Q}$ by letting one four-tuple p be below another q if every component of p extends the corresponding component of q. Then (3) is equivalent to:

(4) (a) $\exists\mathcal{S},\mathcal{T}$ ($\mathcal{Q}$ is wellfounded)

Moreover, the existence of a winning strategy for PRO is equivalent to the nonexistence of a winning strategy for CON, so (a) is equivalent to:

(4) (b) $\neg\exists\mathcal{S},\mathcal{T}$ ($\mathcal{Q}'$ is wellfounded)

where $\mathcal{Q}'$ is defined dually to $\mathcal{Q}$. Examination of the construction shows $\mathcal{Q}$, $\mathcal{Q}'$ are obtained in a PR fashion from $\mathcal{S}$, $\mathcal{T}$ and the data e, f. Every PR function is Δ_1^{ZFC}, as is the notion of wellfoundedness. Further Martin's Borel Determinacy Theorem, which implies the equivalence of (4) (a) and (b) is provable in *ZFC*. It follows (4) provides a Δ_1^{ZFC} definition of satisfaction for φ in terms of satisfaction for its subformulas φ_σ, as required.

1.3 $L_{\varkappa B}$

For any uncountable cardinal $\varkappa$, the formulas of $L_{\varkappa B}$ are those formulas of $L_{\varkappa B}$ which, as set-theoretic objects, are of hereditary cardinality $<\varkappa$; briefly: $L_{\varkappa B}=L_{\infty B}\cap H(\varkappa)$. Up to a harmless relabelling, these are precisely the formulas with $<\varkappa$ subformulas; and for regular $\varkappa$ constitute the smallest class closed under $\neg$, $\forall$, $\exists$; under $\bigwedge$, $\bigvee$ for sets of $<\varkappa$ formulas; and under operation $(^*B)$ for index sets I of cardinality $<\varkappa$.

1.4 *Vaught's Closed-Game Logic* $L_{\infty G}$

Let $e\in\mathcal{E}$ be a code for $\{\omega\}$. For this e $(^*B)$ can be written more perspicuously:

$(_*G)$ $$\bigwedge\nolimits_{i_0\in I}\forall v_0\bigvee\nolimits_{i_1\in I}\exists v_1\dots\bigwedge\nolimits_n\varphi_{i_0\dots i_n}(u_1\dots u_k,\ v_0\dots v_n)$$

The sublanguage of $L_{\infty B}$ obtained by allowing *only* this special case of $(_*B)$ we call $L_{\infty G}$. We also set $L_{\varkappa G}=L_{\infty G}\cap H(\varkappa)$. Vaught [31], [32] has extensively investigated $L_{\omega_1 G}$, and formulas of form $(*G)$ with I countable and the φ_σ quantifier-free formulas of $L_{\omega\omega}$ are called *Vaught formulas*. The game associated with $(*G)$ is closed, and since the determinateness of such games can be proved in ZFC^-, satisfaction for $L_{\infty G}$ is Δ_1^{ZFC-}.

Other fragments of $L_{\infty B}$ can be obtained by restricting the matrix of $(*B)$ to other special forms, e.g. the G_δ-game logic of [6], ch.4C.

1.5 *On Keisler's $L(\omega)$ and Related Languages*

We form $L_{\infty QB}$ by restricting the game prefix in $(*B)$ to allow only quantifiers: Given $e \in \mathcal{E}$ and φ_n, $n \in \omega$, we form:

$$(*QB) \qquad \forall v_0 \exists v_1 \forall v_2 \exists v_3 \ldots \{n: \varphi_n(u_1 \ldots u_k, v_0 \ldots v_n)\} \in \mathcal{B}(e)$$

which can be regarded as a formula of $L_{\infty B}$ by inserting vacuous propositional operations.

$L_{\omega_1 QB} = L_{\infty QB} \cap HC$ coincides with the restriction to HC of the language Keisler [16] calls $L(\omega)$. This observation justifies our assertion in [5] that satisfaction for $L(\omega) \cap HC$ is Δ_1^{ZFC}.

$L_{\infty QG}$ is obtained by similarly restricting $L_{\infty G}$. Moschovakis and Barwise [2] have studied this language, which (unfortunately) is sometimes called $L_{\infty G}$.

Though obviously (considering propositional logic) $L_{\omega_1 QG} = L_{\infty QG} \cap HC$ is weaker than $L_{\omega_1 G}$, Vaught [32] remarks that *over countable models with some coding built-in* (e.g. models of arithmetic) the expressive power of the two languages coincides.

1.6. *Propositional Game Logic*

We form $L_{\infty PB}$ by restricting the game prefix in (*B) to allow only propositional operations. Thus given $e \in \mathcal{E}$ and $I \neq \varnothing$ and formulas $\varphi_\sigma(u_1 \ldots u_k)$ all in the same free variables, we form:

$$(*PB) \qquad \bigwedge_{i_0 \in I} \bigvee_{i_1 \in I} \bigwedge_{i_2 \in I} \bigvee_{i_3 \in I} \ldots \{n: \varphi_{i_0 i_1 \ldots i_n}(u_1 \ldots u_k)\} \in \mathcal{B}(e)$$

This is equivalent to a formula of $L_{\infty\omega}$, viz:

$$(1) \qquad \bigvee_{\mathcal{S}: I^{<\omega} \to I} \bigwedge_{i = i_0, i_1, i_2 \ldots} \bigvee_{y \in \omega^\omega} \bigwedge_{n \in \omega}$$

$$\bigvee_{\xi \in 2^{n+1}, (\xi, y | n+1) \in \mathcal{N}(e)}$$

$$\left(\bigwedge_{m \leq n, \xi(m)=0} \varphi_{\sigma_m} \wedge \bigwedge_{m \leq n, \xi(m)=1} - \varphi_{\sigma_m} \right)$$

where $\mathcal{N}(e)$ is as in § 1.2 and σ_n is the obvious initial segment of:

$$i_0, \mathcal{S}((i_0)), i_1, \mathcal{S}((i_0, i_1)), i_2, \ldots$$

In particular, wellfoundedness cannot be expressed in $L_{\infty PB}$. $L_{\omega_1 PB} = L_{\infty PB} \cap HC$, however, still vastly exceeds $L_{\omega_1\omega}$ in expressive power, since if the formula in $(*PB)$ is in $L_{\omega_1 PB}$, we can only say the equivalent formula (1) is in $L_{\lambda\omega}$ where $\lambda = (2^{\aleph_0})^+$.

$L_{\infty PG}$ and $L_{\omega_1 PG}$ (defined the obvious way) have been studied by Green [10], [11].

1.7 *Solitaire and Souslin-Quantifiers*

We form $L_{\infty SB}$ (resp. $L_{\infty SG}$) by restricting the game prefix (*B) (resp. (*G)) to allow only $\exists$ and $\bigvee$. Formulas of these languages correspond to games in which PRO makes all the moves and CON is a passive spectator. $L_{\infty SB}$ and $L_{\infty SG}$ coincide in expressive power. Indeed we can assign in a PR fashion to every formula of the former an equivalent formula of the latter.
For

$$(*SB) \quad \bigvee_{i_0\in I}\exists v_0 \bigvee_{i_1\in I}\exists v_1 \bigvee_{i_2\in I}\exists v_2 \ldots \{n\colon \varphi_{i_0 i_1\ldots i_n}(u_1\ldots u_k,\, v_0\ldots v_n)\}\in\mathscr{B}(e)$$

is equivalent to:

$$(1) \quad \bigvee_{i_0\in I}\exists v_0 \bigvee_{i_1\in I}\exists v_1 \ldots \bigvee_{y\in\omega^\omega} \bigwedge_{n\in\omega} \bigvee_{\xi\in 2^{n+1},\ (\xi,\, y|n+1)\in\mathscr{N}(e)}$$
$$(\bigvee_{m\leq n,\ \xi(m)=0} \varphi_{i_0\ldots i_m} \wedge \bigwedge_{m\leq n,\ \xi(m)=1} \neg\, \varphi_{i_0\ldots i_m})$$

and hence to:

$$(2) \quad \bigvee_{i_0\in I}\bigvee_{y_0\in\omega}\exists v_0 \bigvee_{i_1\in I}\bigvee_{y_1\in\omega}\exists v_1 \ldots \bigwedge_{n\in\omega}\bigvee_{\xi\in 2^{n+1},\ (\xi,(y_0\ldots y_n))\in\mathscr{N}(e)}$$

etc as in (1).

Distributing $\exists$ through $\bigvee$ and *vice versa*, we also see that any formula of $L_{\infty SG}$ is equivalent to a formula of $L_{\infty\omega_1}$. Malitz has shown that the class of wellorderings of type $\alpha+\alpha$ cannot be defined in $L_{\infty\infty}$, while Takeuti has observed that it is definable in $L_{\omega_1 QG}$.

Further restricting (*SG) to allow only $\exists$ produces $L_{\infty QSG}$. $L_{\omega_1 QSG} = L_{\infty QSG}\cap HC$ has been studied by Moschovakis and others under the name Souslin-Quantifier Logic. Note that the usual formula expressing wellfoundedness still belongs to this language.

1.8 *Souslin Logic*

Restricting (*SG) to allow only $\bigvee$ produces $L_{\infty PSG}$. Explicitly this language allows:

$$(*PSG) \qquad \bigvee_{i_0\in I}\bigvee_{i_1\in I}\bigvee_{i_2\in I}\cdots \bigwedge_{n\in\omega}\varphi_{i_0 i_1\ldots i_n}$$

$L_{\varkappa^+ PSG} = L_{\infty PSG}\cap H(\varkappa^+)$ has been called $\varkappa$-Souslin Logic, or just Souslin Logic for $\varkappa=\aleph_0$, and has been extensively investigated [9], [10], [11].

Of course (cf. § 1.6) $L_{\infty PSG}$ does not exceed $L_{\infty\omega}$ in expressive power; but Souslin logic vastly exceeds $L_{\omega_1\omega}$. For example, the class of countable wellfounded structures is a *PC* for Souslin logic, since a countable $\mathfrak{A}=(|\mathfrak{A}|, E^{\mathfrak{A}})$ is wellfounded iff it can be expanded to a model $(|\mathfrak{A}|, E^{\mathfrak{A}}, R^{\mathfrak{A}})$ of:

R linearly orders the universe in order type ω $\wedge$

$$\neg \bigvee_{i_0\in\omega}\bigvee_{i_1\in\omega}\bigvee_{i_2\in\omega}\cdots\bigwedge_n \varphi_n$$

where φ_n expresses that the ${i_{n+1}}^{st}$ element in the *R*-order stands in the relation *E* to the ${i_n}^{th}$ element. This means that the wellordering number of Souslin logic is $>\omega_1$, the wellordering number of $L_{\omega_1\omega}$. In fact, it may be as large as ω_2;

see [7]. It is perhaps worth noting (following Vaught) that Souslin logic and $L_{\omega_1 PG}$ coincide in expressive power. For

$$\bigwedge_{i_0\in\omega} \bigvee_{i_1\in\omega} \bigwedge_{i_2\in\omega} \bigvee_{i_3\in\omega} \cdots \bigwedge_n \varphi_{i_0 i_1 \dots i_n}$$

is equivalent to

$$\bigvee_{j_0\in\omega} \bigvee_{j_1\in\omega} \bigvee_{j_2\in\omega} \cdots \bigwedge_n \psi_{j_0 j_1 \dots j_n}$$

where the ψ_τ are determined as follows: For $\sigma\in\omega^{<\omega}$ let $\#(\sigma)$ be the natural code for σ, $2^{\sigma(0)}3^{\sigma(1)}5^{\sigma(2)}\dots$ For $\tau=(j_0, j_1, \dots j_n)$ let ψ_τ be the conjunction for all $\sigma=(i_0, i_1, \dots i_m)$ with $\#(\sigma)\leq n$ of $\varphi_{i_0}, \varphi_{i_0 k_0}, \varphi_{i_0 k_0 i_1}, \varphi_{i_0 k_0 i_1 k_1}, \dots$, where $k_0=j_{\#((i_0))}$, $k_1=j_{\#((i_0 i_1))}, \dots$.

Green [11] shows that for all $\varkappa$ the wellordering numbers of $\varkappa$-Souslin logic and $L_{\varkappa^+ PG}$ coincide and equal the least ordinal not $H(\varkappa^*)$ recursive in the sense of [4]. Moreover she shows for cf $\varkappa>\omega$, $\varkappa$-Souslin logic and $L_{\varkappa^+\omega}$ coincide in expressive power.

1.9 *Kolmogorov R-Operation Logic*

The formation rules of $L_{\infty R}$ allow us, given formulas indexed by $(I^{<\omega})^{<\omega}$ to form the following horror:

(*R)
$$\begin{aligned}
&\bigwedge_{i_{00}} \forall v_{00} \bigvee_{i_{01}} \exists v_{01} \bigwedge_{i_{02}} \forall v_{02} \bigvee_{i_{03}} \exists v_{03} \cdots\\
&\bigvee_{n_0\in\omega} \bigvee_{i_{10}} \exists v_{10} \bigwedge_{i_{11}} \forall v_{11} \bigvee_{i_{12}} \exists v_{12} \bigwedge_{i_{13}} \forall v_{13} \cdots\\
&\bigwedge_{n_1\in\omega} \bigwedge_{i_{20}} \forall v_{20} \bigvee_{i_{21}} \exists v_{21} \bigwedge_{i_{22}} \forall v_{22} \bigvee_{i_{23}} \exists v_{23} \cdots\\
&\cdots \bigwedge_r \varphi_{(i_{00}\dots i_{0n_0})\dots(i_{r0}\dots i_{rn_r})}(u_1 \dots u_k,\ v_{00} \dots v_{rn_r})
\end{aligned}$$

For fixed $\mathfrak{A}$ and $b_1 \dots b_k \in |\mathfrak{A}|$ the obvious game of length ω^2 associated with (*R) is equivalent to the following game of length ω, in the sense that the same player has a winning strategy: CON picks elements of I and $|\mathfrak{A}|$ which we call i_{00} and a_{00}. PRO then has three options: to *challenge* immediately, to pick elements we call i_{01}, a_{01} and then challenge, or to pick such elements without challenging. If PRO does not challenge, CON then picks elements we call i_{02}, a_{02}, and PRO then again has the same three options. If PRO eventually does challenge just after i_{0n_0}, a_{0n_0} have been picked, PRO then also picks elements we call i_{10}, a_{10}. CON then has three options analogous to those PRO had earlier. If CON does not challenge, PRO picks another pair of elements, and CON has the same three options, and so on. If CON eventually challenges after i_{1n_1}, a_{1n_1} have been picked, he also picks elements we call i_{20}, a_{20}, and PRO has three options again, and so on. In the end, PRO wins if either each player challenges infinitely often and the matrix of (*R) comes out true with the a's replacing the v's and the b's the u's, or if at some point it is PRO's option to challenge and he lets infinitely many moves go by without doing so. We leave it to the reader to see that this game really is equivalent to that suggested by (*R). Note that the set of sequences $i\in I^\omega$, $a\in|\mathfrak{A}|^\omega$, which constitute a win for PRO is a Borel (in fact, G_δ) set. This means we can associate to each formula of $L_{\infty R}$, in a PR fashion, an equivalent formula of $L_{\infty B}$, and former language can be regarded as a sublanguage of the latter in a generalized sense.

$L_{\omega_1 R}=L_{\infty R}\cap HC$ was mentioned under the name L^2 in [8], § 2. The languages L^ν, $\nu>\omega_1$, mentioned there are all sublanguages of $L_{\infty B}$ in the same sense that $L_{\infty R}$ is.

§ 2. Some Definability Theory

For any vocabulary **R**, let $\mathfrak{X}(\mathbf{R})$ be the set of all **R**-structures with universe ω. $K \subseteq \mathfrak{X}(\mathbf{R})$ is *invariant* if for all $\mathfrak{A} \in \mathfrak{X}\mathbf{R})$, $\mathfrak{A} \cong \mathfrak{B} \in K$ implies $\mathfrak{A} \in K$. We will be concerned with four classifications of invariant subsets of $\mathfrak{X}(\mathbf{R})$.

2.1 *Recursion Theory*

Let $X(\mathbf{R})$ be the product of one copy of 2^{ω^n} for each n-ary predicate in **R**, one copy of ω^{ω^n} for each n-ary function symbol, and one copy of for each constant. Any $x \in X(\mathbf{R})$ corresponds in an obvious way to an $\mathfrak{A}_x \in \mathfrak{X}(\mathbf{R})$. E.g. if **R** has just one binary predicate, $x \in 2^{\omega \times \omega}$ corresponds to the structure consisting of universe ω equipped with the binary relation whose characteristic function x is. $K \subseteq X(\mathbf{R})$ is called *invariant* if the corresponding subset of $\mathfrak{X}(\mathbf{R})$ is. This amounts to invariance under a natural action of the group $\omega!$ of permutations of ω on $X(\mathbf{R})$; see [32].

At least for finite **R**, we can classify subsets of $X(\mathbf{R})$ as Σ^0_n, Π^0_n, Δ^0_n, ari-arithmetical, HYP, Σ^1_n, Π^1_n, Δ^1_n, analytical, etc. according to their definability by various types of formulas of second-order arithmetic. For the elements of this theory see [27], ch. 14—16. If we allow parameters to appear in the definitions we obtain the boldface notions $\underset{\sim}{\Sigma}^0_n$, etc. By tedious but routine coding, these boldface notions can be applied even to infinite **R**. We call a subset of $\mathfrak{X}(\mathbf{R})$ $\underset{\sim}{\Sigma}^0_n$, etc., if the corresponding subset of $X(\mathbf{R})$ is.

2.2 *Topology*

Give $2=\{0,1\}$ and ω the discrete topologies. Give each 2^I, ω^I the product topology (making them homeomorphs of the Cantor and of the irrationals, respectively). Give each $X(\mathbf{R})$ the product topology. Finally give $\mathfrak{X}(\mathbf{R})$ the topology that makes $x \to \mathfrak{A}_x$ a homeomorphism. Then each of these spaces is Polish (separable, admitting a complete metric). We may classify subsets as open, closed, F_σ, G_δ, Borel, analytic, co-analytic (CA), PCA, projective, etc. For the elements of this theory see [19].

2.3 *Set Theory*

We assume familiarity with the Levy hierarchy of formulas of the language of set theory. The appendix to [2] contains a useful summary of the needed material. A class K is $\Sigma_n(V)$ (resp. $\underset{\sim}{\Sigma}_n(V)$) if it is definable over the universe V of set theory by a Σ_n formula without parameters (resp. with parameters). $\Pi_n(V)$ is defined similarly; and K is $\Delta_n(V)$ if both $\Sigma_n(V)$ and $\Pi_n(V)$. The boldface notions are defined similarly. K is $\Delta_n{}^T$, where T is a fragment of ZFC, if it is $\Delta_n(V)$ by Σ_n and Π_n definitions whose equivalence is provable in T. K is $\underset{\sim}{\Delta}_n{}^T$, if of form $\{x: (t, x) \in K'\}$ where K' is $\Delta_n{}^T$, and t is a parameter. We are most interested in the cases $T=$KP (Kripke-Platek admissible set theory, with Infinity), ZFC$^-$ (Zermelo-Frankel set theory with Choice and without Power Set), and ZFC.

$HC = H(\aleph_1)$ is the set of hereditarily countable sets. $K \subseteq HC$ is $\Sigma_n(HC)$ (resp. $\underset{\sim}{\Sigma}_n(HC)$) if K is definable over HC by a Σ_n formula without parameters (resp. with parameters from HC). The Π and Δ notions are similarly defined.

Familiarity with the primitive recursive (PR) set functions of [14] is also assumed. These functions include all functions with reasonably simple inductive definitions. They are all Δ_1^{KP}. A class K is PR if its characteristic function is, and is **PR** if of form $\{x : (t, x) \in K'\}$ for some PR K' and some parameter t.

2.4 *Model Theory*

Let L^* be a language. A class K of **R**-structures is an *elementary class* for L^*, in symbols $EC(L^*)$, if K is of form $\mathrm{Mod}(\varphi) = \{\mathfrak{A} : \mathfrak{A} \models \varphi\}$ for some $\varphi \in L^*(\mathbf{R})$. K is a *pseudo-elementary* or *projective class* for L^*, in symbols $PC(L^*)$, if for some vocabulary **S** disjoint from **R** and some sentence $\varphi' \in L^*(\mathbf{R} \cup \mathbf{S})$ such that K is the class of all **R**-reducts of models of φ'. Equivalently, K is $PC(L^*)$ if it is of form $\mathrm{Mod}(\exists \mathbf{S}\, \varphi')$ for some existential second-order sentence $\exists \mathbf{S}\, \varphi'$, $\varphi' \in L^*(\mathbf{R} \cup \mathbf{S})$. By abuse of language, we call $K \subseteq \mathcal{X}(\mathbf{R})$ $EC(L^*)$ or $PC(L^*)$ if it is the restriction of such a class to structures with universe ω.

For the definition of *language* in the abstract see [2] or [3] (where languages are respectively called systems of logics and logics). We call a language L^* *first-order* if:

(1) Sentencehood for L^* is a notion PR, or **PR** in parameters from HC; or the restriction of such a notion to some $H(\varkappa)$.

(2) Satisfaction for L^* is a notion $\Delta_1(V)$, or $\underset{\sim}{\Delta}_1(V)$ in parameters from HC; or the restriction of such a notion to $\varphi \in$ some $H(\varkappa)$.

These conditions correspond roughly to *absoluteness* as in [2] (where the terminology *first-order* is given some intuitive justification). All the languages of § 1 are first-order, as is each $L_{\varkappa\omega}$. We call a first-order language *strong* if:

(3) L^* is closed under $\neg, \forall, \exists$; under countable $\bigwedge, \bigvee$; under substitution of formulas of $L_{\omega\omega}$ for predicates; and the functions corresponding to these closure conditions, e.g. the function $\varphi \mapsto \neg\varphi$, are PR, or **PR** in parameters from HC, or the restriction of such functions to some $H(\varkappa)$.

(4) The class of countable wellfounded structures is $PC(L^* \cap HC)$.

Much of (3) is included in the definition of language in [3] (though not in [2]). These closure conditions guarantee that any $PC(L^*)$ class of **R**-structures is of form $\mathrm{Mod}(\exists \mathbf{S}\, \varphi')$ where **S** contains just a single binary predicate not in **R**. (4) corresponds roughly to the notion *not bounded below* ω_1 of [2]. The languages of § 1 are, but $L_{\infty\omega}$ is not, strong.

2.5 *Connections Among the Classifications*

Addison [1] observed that for any of the spaces we have been considering, the class of open sets and the class of $\underset{\sim}{\Sigma}_1^0$ sets coincide, and similarly: $\underset{\sim}{\Pi}_1^0 =$ closed, $\underset{\sim}{\Sigma}_2^0 = F_\sigma$, $\underset{\sim}{\Pi}_2^0 = G_\delta$, $\underset{\sim}{\Delta}_1^1 =$ Borel, $\underset{\sim}{\Sigma}_1^1 =$ analytic, $\underset{\sim}{\Pi}_1^1 = CA$, $\underset{\sim}{\Sigma}_2^1 = PCA$.

Ryll-Nardzewski, using Lopez-Escobar's Interpolation Theorem for $L_{\omega_1\omega}$, showed that for *invariant* subsets of $\mathcal{X}(\mathbf{R})$, Borel $= EC(L_{\omega_1\omega})$. Also analytic $=$ $PC(L_{\omega_1\omega})$. See [20].

Kleene [18] in effect showed that for subsets of any of the spaces we have been considering $\underset{\sim}{\Sigma}^1_{n+1} = \underset{\sim}{\Sigma}_n(HC)$. (Note that these spaces $\mathcal{X}(\mathbf{R}), X(\mathbf{R})$ are **PR** in parameter **R**, and are subsets of HC.)

Lévy's Teorem (cf. Appendix to [2]) tells us that each $H(\varkappa)$ is an elementary substructure of the universe V with respect to Σ_1 formulas. It follows that for subsets of HC, $\underset{\sim}{\Sigma}_1(HC) = \underset{\sim}{\Sigma}_1(V)$ in parameters from HC.

Barwise [2] in effect shows that for cardinals $\varkappa > \omega$ and for *invariant* classes of structures, **PR** in parameters from $H(\varkappa) = \underset{\sim}{\Delta}_1^{KP}$ in parameters from $H(\varkappa) = EC(L_{\varkappa\omega})$.

Jensen and Karp apparently knew that for subsets of the spaces we have been considering, $\underset{\sim}{\Delta}^1_1 = \mathbf{PR}$ in parameters from HC.

Vaught's work [32] discloses the following: For a fixed Polish space, let $\mathcal{U}(0) =$ Borel sets; $\mathcal{U}(\alpha+1) = \mathcal{U}(\alpha)$ plus complements of sets in $\mathcal{U}(\alpha)$ for α odd; $\mathcal{U}(\alpha+1) =$ sets obtainable from sets in $\mathcal{U}(\alpha)$ by $(\mathcal{A})$ for α even; $\mathcal{U}(\lambda) = \bigcup\{\mathcal{U}(\alpha)\colon \alpha < \lambda\}$ at limits; $\mathcal{U} = \mathcal{U}(\omega_1)$. Where the fusion operation $(\mathcal{A})$ given sets A_σ, $\sigma \in \omega^{>\omega}$, produces $\bigcup_{f\in\omega^\omega}\bigcap_{n\in\omega} A_{f|n}$. Classically the sets in $\mathcal{U}$ are known as C-sets, and it is known $\mathcal{U}(1) =$ analytic sets. Then for invariant subsets of $\mathcal{X}(\mathbf{R})$, C-sets $= EC(L_{\omega_1 G})$, and moreover there is a level-by-level correspondence between the $\mathcal{U}$-hierarchy and the complexity of sentences of $L_{\omega_1 G}$, with analytic $= EC$ (Vaught sentences), where the Vaught sentences are, as in § 1.4, the simplest sentences of $L_{\omega_1 G} - L_{\omega_1\omega}$. Moreover Ryll-Nardzewski's equation Borel $= EC(L_{\omega_1\omega})$ can be improved to establish a level-by-level correspondence between the Borel hierarchy and the complexity of sentences of $L_{\omega_1\omega}$.

We extended this work of Vaught's to some other hierarchies in [8], § 2 and [6], ch. 4. The following has been noted with varying degrees of explicitness by several people:

2.6 PROPOSITION. For any strong first-order language L^*, for invariant subsets of $\mathcal{X}(\mathbf{R})$, $\underset{\sim}{\Sigma}_1(V)$ in parameters from $HC = PC(L^* \cap HC)$.

PROOF. That every $PC(L^*\cap HC)$ class is $\Sigma_1(V)$ in parameters from HC is immediate from the fact that satisfaction for $\underset{\sim}{L}^*$ is. To prove the converse fix a Σ_1 formula ψ and a parameter $t \in HC$ defining an invariant $K \subseteq \mathcal{X}(\mathbf{R})$.

Let $\in$ be the binary predicate of the language of set theory. The class of countable wellfounded $\in$-structures is $PC(L^*\cap HC)$. Say it is $\mathrm{Mod}(\exists\, \mathrm{S}\, \vartheta)$ where $\vartheta \in L^*(\{\in\}\cup \mathrm{S}) \cap HC$. Define inductively for $x \in HC$ a characterizing formula χ_x of $L_{\omega_1\omega}$ by letting $\chi_x(v)$ be:

$$\bigwedge\nolimits_{y\in x} \exists u \in v\, \chi_y(u) \wedge \forall u \in v \bigvee\nolimits_{y\in x} \chi_y(u).$$

Let F be a singulary function symbol, and let $\bar{r}, \bar{a}, \bar{t}$ be constants. We assume these symbols and $\in$ and the symbols in $\mathbf{S}$ are all distinct from the symbols of $\mathbf{R}$. Let $\mathbf{T} = \mathbf{R} \cup \mathbf{S} \cup \{F, \bar{r}, \bar{a}, \bar{t}\}$, and let $\varphi \in L(\mathbf{T}) \cap HC$ be the conjunction of:

(1) A large enough finite fragment of ZFC.

(2) ϑ

(3) $\chi_{\underset{\sim}{R}}(\bar{r}) \wedge \chi_t(\bar{t})$

(4) $\bar{a}$ is an $\bar{r}$-structure with universe ω

(5) $\psi(\bar{t}, \bar{a})$

(6) F is an injection $\wedge$ range $F =$ universe of $\bar{a}$.

Plus for each n-ary predicate $R \in \mathbf{R}$:

$(7)_R$ $\quad \forall v (\chi_R(v) \to \forall v_1 \ldots v_n (R(v_1 \ldots v_n) \leftrightarrow (F(v_1) \ldots F(v_n)) \in$ the $\bar{a}$-interpretation of the symbol v))

and similarly for function symbols and constants. Here in (4), (6), (7), the definitions of structure, universe, and interpretation are to be written out in terms of $\in$ using the usual set-theoretic definitions.

If $\mathfrak{A} \in K$, then by Lévy's Reflection Principle there is a countable transitive model M of enough of ZFC with $t, \mathfrak{A} \in M$ and $M \models \psi(t, \mathfrak{A})$. Using such an M it is easy to construct a $\mathfrak{L} \in \mathscr{X}(\mathbf{T})$ with $\mathfrak{L} \models \varphi$ and $\mathfrak{L} | \mathbf{R} = \mathfrak{A}$.

Conversely given $\mathfrak{L} \models \varphi$ with $\mathfrak{L} | \mathbf{R} = \mathfrak{A}$, (1) and (2) guarantee that $\mathfrak{L}$ is up to isomorphism a transitive set. Then (3), (4), (5) guarantee that the interpretation $\bar{a}^{\mathfrak{L}}$ of $\bar{a}$ in $\mathfrak{L}$ is an $\mathbf{R}$-structure satisfying the definition of K. (We use here the fact that a Σ_1 statement true inside some transitive set is actually true in the universe V.) Finally (5), (6) guarantee that $\mathfrak{A} \cong \bar{a}^{\mathfrak{L}}$, so by invariance of K, $\mathfrak{A} \in K$.

2.7 *Summary*

For any strong first-order language L^*, and for invariant subsets of $\mathscr{X}(\underset{\sim}{R})$, we have:

(a) $\underset{\sim}{\Delta}_1^1 = \text{Borel} = \mathbf{PR}$ in parameters from $HC = EC(L_{\omega_1\omega})$,

(b) $\underset{\sim}{\Sigma}_1^1 = \text{analytic} = PC(L_{\omega_1\omega}) = EC$ (Vaught sentences),

(c) $\underset{\sim}{\Sigma}_2^1 = PCA = \underset{\sim}{\Sigma}_1(HC) = \underset{\sim}{\Sigma}_1(V)$ in parameters from $HC = PC(L^* \cap HC)$.

§ 3 A Question of Vaught

3.1 PROPOSITION. For any first-order language L^*, and for invariant subsets of $\mathscr{X}(\mathbf{R})$, we have:

$$EC(L^* \cap HC) \subseteq \underset{\sim}{\Delta}_2^1 = \underset{\sim}{\Delta}_1(HC)$$

PROOF. We only give a sketch since our proof has appeared in [21]. The inclusion and the identity are immediate from 2.7 (*c*). We tacitly assume **R** is nontrivial, i.e. contains at least one binary predicate E. We say $\mathfrak{A}\in\mathcal{X}(\{E\})$ *codes* $x\in HC$ if $\mathfrak{A}\cong(TC(y),\in)$ where $TC(y)=\{y\}\cup y\cup\cup y\cup\cup\cup y\cup\ldots$ is the transitive closure of y. An example to show the inclusion is proper is provided by $\{\mathfrak{A}\in\mathcal{X}(\mathbf{R})\colon\ \exists\varphi\in L^*(\mathbf{R})\cap HC\ ((|\mathfrak{A}|,E^{\mathfrak{A}})\text{ codes }\varphi\wedge\mathfrak{A}\models\neg\varphi)\}$.

Vaught has asked whether for any invariant Δ^1_2 $K\subseteq\mathcal{X}(\mathbf{R})$ there is some first-order language L^* for which K is $EC(L^*\cap\tilde{H}C)$. We will show this question cannot be answered in *ZFC*.

3.2 *A Positive Answer*

For any partially ordered set of forcing conditions (*PO* set) $\mathcal{P}$, let $V^{\mathcal{P}}$ be the corresponding extension of the universe of set theory. (If you will, the Boolean-valued model associated with the complete Boolean algebra of regular open subsets of $\mathcal{P}$) For simplicity let us assume **R** finite. Then we may define $K\subseteq X(\mathbf{R})$ to be *absolutely* Δ^1_2 if there exist Σ^1_2 and Π^1_2 formulas φ, φ in a parameter t from, say, ω^ω, defining K, such that for any *PO* set $\mathcal{P}$:

$$V^{\mathcal{P}}\models\forall x(\varphi(t,x)\leftrightarrow\psi(t,x)) \tag{1}$$

Here we are using elements t of V autonymously (writing t rather than t^v). We extend this notion in the obvious way to $\mathcal{X}(\mathbf{R})$. Note that if K is invariant, then so is the set defined by φ and t in any $V^{\mathcal{P}}$, since

$$\neg\exists x,y\,(\mathfrak{A}_x\cong\mathfrak{A}_y\wedge\varphi(t,x)\wedge\neg\psi(t,y)) \tag{2}$$

is a true Π^1_2 statement, and Π^1_2 statements are absolute by Shoenfield's Theorem. We show now how, given an invariant absolutely Δ^1_2 $K\subseteq\mathcal{X}(\mathbf{R})$ to constrult a first-order language $L^*\subseteq HC$ for which K is an $\tilde{EC}$. We begin by fixing definitions of the corresponding subset of $X(\mathbf{R})$ satisfying (1) above. To further simplify matters we suppose **R** contains just one binary predicate E.

For an arbitrary **R**-structure $\mathfrak{A}$, let $\mathcal{P}(\mathfrak{A})$ be the *PO* set of the injective elements of $|\mathfrak{A}|^{<\omega}$, partially ordered by reverse inclusion, i.e. the usual conditions for adjoining a generic bijection between ω and $|\mathfrak{A}|$. Let $\bar{x}(\mathfrak{A})$ be the following term of the forcing language for $\mathcal{P}(\mathfrak{A})$:

$$\{(p,((m,n),i))\colon\ p\in\mathcal{P}(\mathfrak{A})\wedge m,n\in\operatorname{dom}p\wedge$$
$$(((m,n)\in E^{\mathfrak{A}}\wedge i=0)\vee((m,n)\notin E^{\mathfrak{A}}\wedge i=1))\}$$

i.e. the canonical term for an element x of $X(\mathbf{R})$ with $\mathfrak{A}_x\cong\mathfrak{A}$. By (1) and (2) we have:

$$V^{\mathcal{P}(\mathfrak{A})}\models\varphi(t,\bar{x}(\mathfrak{A}))\leftrightarrow\psi(t,\bar{x}(\mathfrak{A})) \tag{3}$$

(4) $$V^{\mathcal{P}(\mathfrak{A})} \models \varphi(t, \bar{x}(\mathfrak{A})) \leftrightarrow \exists y \in X(\mathbf{R})(\mathfrak{A}_x \cong \mathfrak{A} \wedge \varphi(t, y))$$
$$\leftrightarrow \forall y \in X(\mathbf{R})(\mathfrak{A}_y \cong \mathfrak{A} \rightarrow \varphi(t, y))$$

Any permutation h of ω induces an automorphism H_h of $\mathcal{P}(\mathfrak{A})$ and a permutation $\bar{u} \rightarrow \bar{u}^h$ of the terms of the forcing language. For any $p, q \in \mathcal{P}(\mathfrak{A})$ there is an h such that p, $H_h(q)$ are compatible (weak homogeneity). For any h, $\bar{x}(\mathfrak{A})^h$ is still a term for an isomorph of $\mathfrak{A}$. It follows by (4) there cannot exist $p, q \in \mathcal{P}(\mathfrak{A})$ one of which forces $\varphi(t, \bar{x}(\mathfrak{A}))$ and the other of which forces its negation. Thus:

(5) Either $V^{\mathcal{P}(\mathfrak{A})} = \varphi(t, \bar{x}(\mathfrak{A}))$ or else $V^{\mathcal{P}(\mathfrak{A})} \models \neg\varphi(t, \bar{x}(\mathfrak{A}))$

Let $K^+ = \{\mathfrak{A} : V^{\mathcal{P}(\mathfrak{A})} \models \varphi(t, \bar{x}(\mathfrak{A}))\}$.

K^+ is invariant. For if $\mathfrak{B} \cong \mathfrak{A}$, there is an isomorphism $\mathcal{P}(\mathfrak{B}) \cong \mathcal{P}(\mathfrak{A})$ such that the induced map on terms carries $\bar{x}(\mathfrak{B})$ to $\bar{x}(\mathfrak{A})$.

$K^+ \cap \mathfrak{X}(\mathbf{R}) = K$. For if $x \in X(\mathbf{R})$ and $\mathfrak{A}_x \in K$, then $\varphi(t, x)$ is true and remains true in $V^{\mathcal{P}(\mathfrak{A}_x)}$ by Shoenfield's Theorem whence by (4) $V^{\mathcal{P}(\mathfrak{A}_x)} \models$ $\models \varphi(t, \bar{x}(\mathfrak{A}_x))$, i.e. $\mathfrak{A}_x \in K^+$. Conversely, if $\mathfrak{A}_x \notin K$, by (3) and (4) $\mathfrak{A}_x \notin K^+$.

K^+ is $\Delta_1(V)$ in parameter t. For φ is equivalent over all models to some Σ_1 condition $\tilde{\theta}$; and by the general theory of forcing there is a Σ_1 θ' such that for all PO sets $\mathcal{P}$, all $p \in \mathcal{P}$, and all terms $\bar{u}$, $V^{\mathcal{P}} \models \theta(t, \bar{u})$ iff $\theta'(\mathcal{P}, p, t, u)$ holds. Since, $\mathcal{P}(\mathfrak{A})$ and $\bar{x}(\mathfrak{A})$ are PR functions of $\mathfrak{A}$, this implies K^+ is $\underset{\sim}{\Sigma}_1(V)$ in parameter t. Using ψ in place of φ we get Π_1 in place of Σ_1.

Now let L^* be a language with but a single sentence $\rho \in HC$, and $\mathfrak{A} \models \rho$ iff $\mathfrak{A} \in K^+$. L^* is certainly first-order, and we can without difficulty fatten L^* up to a strong language without losing the first-order property. (Cf. [2].) Finally, K is $EC(L^*)$.

The Solovay Absoluteness Theorem, [23], p. 152, implies that if $\forall\varkappa\,\exists\lambda\,\lambda \rightarrow$ $\rightarrow (\varkappa)_2^{<\omega}$, then *every* Δ^1_2 set is absolutely $\underset{\sim}{\Delta}^1_2$. Thus *if enough large cardinals exist, Vaught's question has a positive answer.*

3.3 *A Negative Answer*

It is wellknown that any class K which is $\underset{\sim}{\Sigma}_1(V)$ in parameters from HC having $\omega_1 \in K$ contains a closed unbounded (CUB) subset of ω_1. It is also wellknown that if F assigns to each countable ordinal α a wellordering of ω in type α, and for $i = 2^m(2n+1) \in \omega$, $D_1 = \{\alpha\colon m$ precedes n in $F(\alpha)\}$, then for some i, neither D_i nor $\omega_1 - D_i$ contains a CUB set. Finally it is wellknown that if $\omega_1^L = \omega_1$ then the function F may be taken to be $\Sigma_1(V)$ and hence (since its domain is $OR \cap HC$) $\Delta_1(HC)$. On this assumption, for suitable i, $K = \{\mathfrak{A} \in \mathfrak{X}(\{E\})\colon \mathfrak{A}$ is a wellordering with order type $\in D_i\}$ is a subset of $\mathfrak{X}(\{E\})$ which is invariant (in $\mathfrak{X}(\{E\})$) and $\Delta_1(HC)$ hence Δ^1_2, but which cannot be the restriction to $\mathfrak{X}(\{E\})$ of any (fully) invariant class which is $\underset{\sim}{\Delta}_1(V)$ in parameters from HC. Thus *if* $\omega_1^L = \omega_1$, *Vaught's question has a negative answer.*

§ 4 Approximation Theory

Let L^*, L^0 be languages. By an *approximation function* for L^*, L^0 we mean a function $\mathcal{C}$: $\mathrm{OR}\times L^* \to L^0$ which preserves vocabulary; is *PR*, or **PR** in parameters from *HC*, or is the restriction of such a function to some $H(\varkappa)$; and which has the property that for any sentence φ of L^* the following is valid: $\varphi \leftrightarrow \bigwedge_{\alpha\in OR} \mathcal{C}(\alpha, \varphi)$.

4.1 LEMMA. There exists an approximation function for $L_{\infty G}, L_{\infty\omega}$.

PROOF. The basic idea goes back to Moschovakis [25]; see also [31].

We define by induction of subformulas two preliminary functions $\mathcal{A}, \mathcal{S}$: $\mathrm{OR}\times L_{\infty G} \geqq L_{\infty\omega}$. The easy clauses of the induction are:

$$\mathcal{A}(\alpha, \neg\varphi) = \neg\mathcal{A}(\alpha, \varphi) \qquad \mathcal{S}(\alpha, \neg\varphi) = \mathcal{S}(\alpha, \varphi)$$

$$\mathcal{A}(\alpha, \bigwedge\Phi) = \bigvee\{\mathcal{A}(\alpha, \varphi) : \varphi\in\Phi\}$$

$$\mathcal{A}(\alpha, \bigvee\Phi) = \bigvee\{\mathcal{A}(\alpha, \varphi) : \varphi\in\Phi\}$$

$$\mathcal{S}(\alpha, \bigwedge\Phi) = \mathcal{S}(\alpha, \bigvee\Phi) = \bigwedge\{\mathcal{S}(\alpha, \varphi) : \varphi\in\Phi\}$$

$$\mathcal{A}(\alpha, \forall v\varphi) = \forall v\,\mathcal{A}(\alpha, \varphi) \qquad \mathcal{A}(\alpha, \exists v\varphi) = \exists v\,\mathcal{A}(\alpha, \varphi)$$

$$\mathcal{S}(\alpha, \forall v\varphi) = \mathcal{S}(\alpha, \exists v\varphi) = \forall v\,\mathcal{S}(\alpha, \varphi).$$

For φ given by (*G*) of §1.4 the definition is more complex. Fixing α and φ for the moment we define auxiliary functions $\mathcal{A}^*$, $\mathcal{S}^*$ with domains $\mathrm{OR}\times I^{<\omega}$, OR respectively, by a subinduction:

$$\mathcal{A}^*(0, \sigma) = \bigwedge_{n\leq \mathrm{length}\,\sigma} \mathcal{A}(\alpha, \varphi_\sigma)$$

$$\mathcal{A}^*(\beta+1, \sigma) = \bigvee_{i\in I} \mathcal{A}^*(\beta, \sigma^\frown i)$$

$$\mathcal{A}^*(\lambda, \sigma) = \bigwedge_{\beta<\lambda} \mathcal{A}^*(\beta, \sigma) \text{ at limits}$$

$$\mathcal{S}^*(\beta) = \bigwedge_{n\in\omega} \bigwedge_{\sigma\in I^n} \forall_{v_0}\ldots\forall_{v_n}(\mathcal{A}^*(\beta, \sigma) \to \mathcal{A}^*(\beta+1, \sigma)).$$

We then set:

$$\mathcal{A}(\alpha, \varphi) = \mathcal{A}^*(\alpha, 0)$$

$$\mathcal{S}(\alpha, \varphi) = \mathcal{S}^*(\alpha) \wedge \bigwedge_{n\in\omega} \bigwedge_{\sigma\in I^n} \forall_{v_0}\ldots\forall_{v_n}\mathcal{S}(\alpha, \varphi_\sigma).$$

Readers of [31] should then have no difficulty in verifying that the following are valid:

(1) $\mathcal{S}(\alpha, \varphi) \to \mathcal{S}(\beta, \varphi)$ for $\alpha<\beta$

(2) $\bigvee_{\alpha\in OR} \mathcal{S}(\alpha, \varphi)$

(3) $\mathcal{S}(\alpha, \varphi) \to (\varphi \leftrightarrow \mathcal{A}(\alpha, \varphi))$ for all α

(4) $\varphi \leftrightarrow \bigvee_{\alpha\in OR}(\mathcal{S}(\alpha, \varphi) \wedge \mathcal{A}(\alpha, \varphi))$

(5) $\varphi \leftrightarrow \bigwedge_{\alpha\in OR}(\mathcal{S}(\alpha, \varphi) \to \mathcal{A}(\alpha, \varphi))$.

So it suffices to set $\mathcal{C}(\alpha, \varphi) = (\mathcal{S}(\alpha, \varphi) \to \mathcal{A}(\alpha, \varphi))$.

4.2 APPROXIMATION THEOREM. Let L^* be any first-order language. Then there exists an approximation function for $L^*, L_{\infty\omega}$.

PROOF. By the Lemma it suffices to obtain an approximation function for $L^*, L_{\infty G}$. For simplicity we will consider only the vocabulary $\mathbf{R} = \{E\}$, E a binary predicate, and we will assume satisfaction for L^* is $\Sigma_1(V)$ (no parameters). On these assumptions the approximation function will be PR.

From the Σ_1 definition of satisfaction we obtain a Σ^1_2 formula θ defining $S = \{(x, y) \in X(\mathbf{R})^2\colon \exists \varphi \in L^*(\mathbf{R}) \cap HC$ (y codes $\varphi \wedge \mathfrak{A}_x \vdash \varphi$)$\}$ and a Σ^1_2 formula θ^- defining the set S^- obtained by replacing φ by $\neg\varphi$ in the definition of S. (Cf. proof of Prop. 3.1.) The statement:

(1) $\neg \exists x, y\, (\theta(x, y) \wedge \theta^-(x, y) \wedge \mathfrak{A}_x \cong \mathfrak{A}_y)$

is Π^1_2, hence absolute.

From θ we can obtain the index of a recursive functional F such that $(x, y) \in S$ iff:

(2) $\exists z\ F(x, y, z)$ is wellfounded.

By Shoenfield's Theorem, the required z can be found in $J(x, y)$, the class of sets constructible from x, y. Hence (2) is equivalent to the existence of $\alpha < \omega_1$ such that:

(3) $\exists_z \in J_\alpha(x, y)\ F(x, y, z)$ wellorders ω in order type $< \alpha$ where J_α is the α^{th} level of the constructible hierarchy. From (3) we can readily obtain a Σ^1_1 formula ψ such that the following holds:

(4) $\forall x, y, z, z'$ ($\mathfrak{A}_z$ is embeddable in $\mathfrak{A}_{z'} \rightarrow (\psi(x, y, z) \rightarrow \psi(x, y, z'))$) and for any x, y and for fixed α and z wellordering ω in order type α, (3) is equivalent to $\psi(x, y, z)$. Note that (4) is Π^1_2, hence absolute.

From this ψ we can compute the index of an RE set W such that $\psi(x, y, z)$ is equivalent to:

(5) $\exists w \in \omega^\omega\, \forall n \in \omega\, (x \| n, y \| n, z \| n, w \mid n) \in W$

where $x \| n$ denotes the restriction of x to $(n+1) \times (n+1)$ for $x \in X(\mathbf{R})$ $(= 2^{\omega \times \omega})$.

Now let φ be a sentence of $L^*(\mathbf{R})$, $\mathfrak{A}$ an arbitrary $\mathbf{R}$-structure. Let $\mathcal{P} = \mathcal{P}(\mathfrak{A})$, $\bar{x} = \bar{x}(\mathfrak{A})$ be as in § 3.2. Let $\mathcal{Q} = \mathcal{Q}(\varphi)$ be the PO set of forcing conditions for making $TC(\varphi)$ countable (i.e. for making $\varphi \in HC$), and let $\bar{y} = \bar{y}(\varphi)$ be the canonical term for an element of $X(\mathbf{R})$ coding φ. Now if $\mathfrak{A} \models \varphi$, then $\mathfrak{A}$, φ satisfy the Σ_1 definition of satisfaction for L^* in V, and will continue to do so in $V^{\mathcal{P} \times \mathcal{Q}}$. Hence in that extension $\bar{x}$ and $\bar{y}$ will satisfy the Σ^1_2 definition θ of S. Conversely, if $\mathfrak{A} \models \neg\varphi$, $\bar{x}$, $\bar{y}$ satisfy θ^- in $V^{\mathcal{P} \times \mathcal{Q}}$ and so by (1) do not satisfy θ. So $\mathfrak{A} \models \varphi$ iff $V^{\mathcal{P} \times \mathcal{Q}} \models \theta(\bar{x}, \bar{y})$. By our detailed analysis of θ above, this condition is equivalent to:

(6) $V^{\mathcal{P} \times \mathcal{Q}} \models \exists \alpha < \omega_1 \exists z \in X(\mathbf{R})\ (\mathfrak{A}_z \cong (\alpha, \in) \wedge \psi(\bar{x}, \bar{y}, \bar{z}))$.

For fixed $\alpha \in \mathrm{OR}$, let $\mathcal{R}(\alpha)$ be the *PO* set of forcing conditions for collapsing α, and let $\bar{z}(\alpha)$ be the canonical term for an element of $X(\mathbf{R})$ with $\mathfrak{A}_z \cong (\alpha, \in)$. We claim (6) is equivalent to the existence of α such that:

$$(7) \qquad V^{\mathcal{P}\times\mathcal{Q}\times\mathcal{R}(\alpha)} \models \psi(\bar{x}, \bar{y}, \bar{z}(\alpha)).$$

For if (7) holds for some α, then the Σ_2^1 statement $\exists z(\mathfrak{A}_z$ is a wellordering $\wedge\, \psi(\bar{x}, \bar{y}, z))$ holds in $V^{\mathcal{P}\times\mathcal{Q}\times\mathcal{R}(\alpha)}$, and hence by Shoenfield's Theorem in $V^{\mathcal{P}\times\mathcal{Q}}$, so (6) holds. Conversely, suppose (6) holds and let $\beta = \mathrm{card}(\mathcal{P}\times\mathcal{Q})^+$, so β is still uncountable in $V^{\mathcal{P}\times\mathcal{Q}}$. For any $p \in \mathcal{P}$, $q \in \mathcal{Q}$, there will exist $p' \leq p$, $q' \leq q$ and $\alpha < \beta$ such that (p', q') forces $\exists z(\mathfrak{A}_z \cong (\alpha, \in) \wedge \psi(\bar{x}, \bar{y}, z))$. It follows (p', q', l_β) forces the same thing, where l_β is the trivial element of $\mathcal{R}(\beta)$. By (4), (p', q', l_β) forces $\exists z(\mathfrak{A}_z \cong (\beta, \in) \wedge \psi(\bar{x}, \bar{y}, z))$, and since p, q were arbitrary, (7) follows.

Now fixing α and $\mathcal{R} = \mathcal{R}(\alpha)$, $\bar{z} = \bar{z}(\alpha)$, (7) is equivalent to:

$$(8) \qquad V^{\mathcal{P}\times\mathcal{Q}\times\mathcal{R}} = \exists w \in \omega^\omega \,\forall n \in \omega\, (x\|n, y\|n, z\|n, w\|n) \in W.$$

For $p \in \mathcal{P}$ with $\mathrm{dom}\, p \geqq n$, define $\xi(n, p)$ to be what p forces $x\|n$ to be. Thus for $i, j < n$, $(\xi(n,p))(i,j)$ is 0 if $(p(i), p(j)) \in E^{\mathfrak{A}}$, and 1 if not. Let η, ζ be similarly defined. Then we claim (8) is equivalent to:

$$(9) \qquad \forall p_0 \in \mathcal{P}, q_0 \in \mathcal{Q}, r_0 \in \mathcal{R}\, \exists p_1 < p_0, q_1 < q_0, r_1 < r_0\, \exists w_0, w_1 \in \omega$$
$$\forall p_2 < p_1, q_2 < q_1, r_2 < r_1\, \exists p_3 < p_2, q_3 < q_2, r_3 < r_2\, \exists w_2, w_3 \in \omega \ldots$$
$$\ldots \forall n (\xi(n, p_n), \eta(n, q_n), \zeta(n, r_n), (w_0 \ldots w_n)) \in W.$$

We will omit the proof of this equivalence, since it is a special case of more general theorems of [15]. Now (9) is equivalent to the following sentence holding in $\mathfrak{A}$:

$$(10) \qquad \bigwedge_{k_0 \in \omega} \forall v_0 \ldots v_{k_0-1} \text{ distinct } \bigwedge_{k_0 \in \mathcal{Q}} \bigwedge_{r_0 \in \mathcal{R}}$$
$$\bigvee_{k_1 \in \omega} \exists v_{k_0} \ldots v_{k_0+k_1-1} \text{ distinct } \bigvee_{q_1 < q_0} \bigvee_{r_1 < r_0} \bigvee_{w_0, w_1 \in \omega} \ldots$$
$$\ldots \bigwedge_n \bigvee_\xi \text{ with } (\xi, \eta(n, q_n), \zeta(n, r_n), (w_0 \ldots w_n)) \in W$$
$$\left(\bigwedge_{i,j \leq n, \xi(i,j)=0} v_i E v_j \wedge \bigwedge_{i,j \leq n, \xi'(i,j)=1} \neg v_i E v_j\right)$$

where here *distinct* means not merely that $v_{k_0} \ldots$ are distinct from each other, but also that they are distinct from $v_0 \ldots v_{k_0-1}$. Tedious but routine coding (cf. Vaught's remarks [32], § 3, on the closure of $L_{\omega_1 G}$ on passage to weak second-order logic) produces a sentence $\mathcal{Q}(\alpha, \varphi)$ equivalent to (10) which belongs to $L_{\infty G}$, and is independent of $\mathfrak{A}$. It suffices to set $\mathcal{C}(\alpha, \varphi) = \neg \mathcal{Q}(\alpha, \neg\varphi)$.

§ 5 The Anti-Beth Theorem

Beth's Definabillity Theorem for a language L^* asserts that for any vocabulary $\mathbf{R}$ and any binary predicate S and constants c, d not in $\mathbf{R}$, that if $\varphi \in L^*(\mathbf{R} \cup \{S\})$ is such that any $\mathbf{R}$-structure $\mathfrak{A}$ has at most one expansion to a model of φ, then there exists $\theta \in L^*(\mathbf{R} \cup \{c,d\})$ such that for any $\mathbf{R}$-structure $\mathfrak{A}$, if $\mathfrak{A}$ has an expansion to a model of φ, then $(\mathfrak{A}, \{(a,b): (\mathfrak{A}, a, b) \models \theta\})$ is that expansion. Replacing "at most one" by "exactly one" produces the weak version of Beth's Theorem.

5.1. ANTI-BETH THEOREM. Let L^* be any strong first-order language. Then even the weak version of Beth's Theorem fails for $L^* \cap HC$.

PROOF. It may help to isolate first the descriptive-set-theoretic content of the construction. Let $X = 2^{\omega \times \omega}$. We think of subsets of X^n, as n-ary relations on X, writing $Z(x_1 \dots x_n)$ for $(x_1 \dots x_n) \in Z$. For $x \in X$, $i \in \omega$, define $(x)_1 \in X$ by $(x)_1 (j,k) = x(1, 2^j (2k+1))$.

Suppose we are given a family Γ of subsets of and relations on X containing a $T \subseteq X^2$ such that for all x:

(1) $\mathfrak{A}_x$ is wellfounded $\leftrightarrow \exists y\, T(x,y)$

and satisfying:

(2) $\Gamma \subseteq \underset{\sim}{\Delta}^1_2$

(3) All closed sets belong to Γ

(4) Γ is closed under countable —

(5) Γ is closed under taking inverse images under continuous functions

We show how, given an arbitrary $\underset{\sim}{\Delta}^1_2$ set K, to construct a $\underset{\sim}{\Pi}^1_1 H \in \Gamma$ such that:

(6) $\forall x \exists ! y\, H(x,y)$

(7) $\forall x, y\, (H(x,y) \rightarrow (K(x) \leftrightarrow y(0,1) = 0))$.

To begin with, fix $\underset{\sim}{\Pi}^1_1$ sets P, Q such that:

(8) $\forall x\, (K(x) \leftrightarrow \exists y P(x,y) \leftrightarrow \neg \exists y Q(x,y))$.

Define a $\underset{\sim}{\Pi}^1_1$ set A by:

(9) $A(x,y) \leftrightarrow ((y(0,0) = 0 \wedge P(x,(y)_0)') \vee (y(0,0) = 1 \wedge Q(x,(y)_0))$.

Note:

(10) $\forall x \exists y\, A(x,y)$.

Let B_0 be a $\underset{\sim}{\Pi}^1_1$ set uniformizing A, i. e. $B_0 \subseteq A$ and

(11) $\forall x \exists ! y\, B_0(x,y)$.

By the standard analysis of $\underset{\sim}{\Pi}^1_1$ sets there is a continuous $F_0 : X^2 \rightarrow X$ such that:

(12) $\forall x, y\, (B_0(x,y) \leftrightarrow \mathfrak{A}_{F_0(x,y)}$ is wellfounded).

Define:

(13) $C_0(x,y,z,u) \leftrightarrow z = F_0(x,y) \wedge T(z,u)$.

Note the graph of F_0 is closed so by (3), (4), $C_0 \in \Gamma$. Moreover by (1):

(14) $\forall x, y\,(B_0(x, y) \leftrightarrow \exists z, u\, C_0(x, y, z, u))$.

By (2) C_0 is $\underset{\sim}{\Delta}^1_2$, so there exists a $\underset{\sim}{\Pi}^1_1$ set $D_0 \subseteq X^5$ such that:

(15) $\forall x, y, z, u\,(C_0(x, y, z, u) \leftrightarrow \exists v\, D_0(x, y, z, v))$.

Let B_1 be a $\underset{\sim}{\Pi}^1_1$ set uniformizing D_0, so:

(16) $\forall x \exists !\, y, z, u, v\, B_1(x, y, z, u, v)$.

Reviewing the construction, it is clear the same y is involved in (11) and (16).

Now iterate the above steps, picking $F_1 : X^5 \to X$, $C_1 \subseteq X^7$, $D_1 \subseteq X^8$, etc. In the end we define:

(17) $E_n(x, y) \leftrightarrow B_n(x, (y)_0 \ldots (y)_{3n+3})$,

(18) $G_n(x, y) \leftrightarrow C_n(x, (y)_0 \ldots (y)_{3n+5})$.

Since the maps $y \to ((y)_0 \ldots (y)_i)$ are continuous, the E_n will be $\underset{\sim}{\Pi}^1_1$ and, by (5), the G_n will belong to Γ. Finally, set:

(19) $H(x, y) \leftrightarrow \forall n\, E_n(x, y)$.

Reviewing the construction, and noting that $(y)_0(0, 0) = y(0, 1)$, we get (6), (7). Moreover:

(20) $\forall x, y\,(H(x, y) \leftrightarrow \forall n\, G_n(x, y))$,

which, with (4), implies $H \in \Gamma$.

Now to apply this construction to model theory. For $n \in \omega$ let $\mathbf{R}^n = \{R_1 \ldots R_n\}$ where the R_i are binary predicates, and let $\mathbf{S}^n = \mathbf{R}^n \cup \{\oplus, \otimes\}$, where $\oplus$, $\otimes$ are binary function symbols. Let L^* be a strong first-order language. By the definition of strong, cf. § 2.4, there is a sentence $\tau \in L^*(\mathbf{R}^2) \cap HC$ such that the class of countable wellfounded $\mathbf{R}^1$-structures is $\mathrm{Mod}(\exists \mathbf{R}_2 \tau)$. Define $T \subseteq X^2$ by:

(21) $T(x, y) \leftrightarrow \mathfrak{A}_{(x,y)} \models \tau$,

and let Γ be the smallest class containing T and closed under (3)—(5) above. It is wellknown that for any Borel $Z \subseteq X^n$ there is a sentence $\zeta \in L_{\omega_1\omega}(\mathbf{S}^n)$ such that for all $x_1 \ldots x_n$:

(22) $Z(x_1 \ldots x_n) \leftrightarrow (\mathfrak{A}_{(x_1 \ldots x_n)}, +, \times) \models \zeta$,

where $+$, $\times$ are the usual arithmetical operations on ω. Now the closure conditions required of Γ correspond to the closure conditions satisfied by strong languages: (3) corresponds to $L_{\omega_1\omega} \subseteq L^*$, (4) to closure of L^* under countable $\bigwedge$, and (5) to closure under substetution of formulas for predicates. Exploiting this correspondence, for every $Z \in \Gamma$ we can find a ζ in $L^* \cap HC$ satisfying (22). This, with 2.7 (c), implies (2).

Let now a $\Delta_2^1 K \subseteq X$ be given, and suppose K is invariant. Let H be as constructed above from K, and let $\eta \in L^*(\mathbf{S}^2) \cap HC$ correspond to H. Let $\varphi_0 \in L_{\omega_1\omega}(\mathbf{S}^0)$ express that $\oplus$, $\otimes$ are up to isomorphism the usual arithmetical operations on ω. Let $\varphi = (\varphi_0 \wedge \eta) \vee (\neg\varphi_0 \wedge \forall u, v \neg R_2(u, v))$. Then by (6) every $\mathbf{S}^1$-structure $\mathfrak{A}$ has a unique expansion to a model of φ. Suppose $\theta \in L^*(\mathbf{S}^1 \cup \cup \{c, d\})$ is as required by Beth's Theorem. Using the closure properties of L^* we can obtain from θ a $\psi \in L^*(\mathbf{S}^1) \cap HC$ expressing that θ holds of the identity element of $\oplus$ and the identity element of $\otimes$. Then by (7):

$$\forall x\, (K(x) \leftrightarrow (\mathfrak{A}_x, +, \times) \models \psi). \tag{23}$$

It is not hard to see no ψ satisfying (23) can exist if K is the counterexample constructed in the proof of Prop. 3.1. This contraditiction shows Beth's Theorem fails.

§ 6 Some Model Theory

We collect here what is known about first-order languages from §§ 1-5, from Barwise' work [2], and elsewhere.

6.1 DOWNWARD LÖWENHEIM-SKOLEM THEOREM. Let L^* be a first--order language, $\varkappa$ an infinite cardinal, $\varphi \in L^* \cap H(\varkappa^+)$, $\mathfrak{A}$ a model of φ, Z a subset of $|\mathfrak{A}|$ with card $Z \leq \varkappa$. Then there is a substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with $Z \subseteq |\mathfrak{B}|$, card $|\mathfrak{B}| = \varkappa$, and $\mathfrak{B} \models \varphi$.

PROOF. This is Prop. 2.1 of [2]. For the languages of § 1, a direct proof using Skolem functions is possible.

If L^* is a language and $\mathfrak{A}$, $\mathfrak{B}$ are structures of the same vocabulary, we say $\mathfrak{A}$ and $\mathfrak{B}$ are L^*-elementarily equivalent, in symbols $\mathfrak{A} \equiv^* \mathfrak{B}$, if they are models of exactly the same sentences of L^*. We say $\mathfrak{A} \approx \mathfrak{B}$ if there exists a family $\mathscr{F}$ of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ with the back-and-forth property ($\forall f \in \mathscr{F} \forall a \in |\mathfrak{A}| \exists b \in |\mathfrak{B}|\ f \cup \{a, b)\} \in \mathscr{F}$ and *vice versa*).

6.2 KARP PROPERTY. Let L^* be a first-order language. Then for all structures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A} \equiv^* \mathfrak{B}$ iff $\mathfrak{A} \approx \mathfrak{B}$.

PROOF. For $\equiv_{\infty\omega}$ this is due to Karp. For the general case it is Prop. 2.5 of [2]. The equivalence of $\equiv^*$ and $\equiv_{\infty\omega}$ is greatly strengthened by the Approximation Theorem 4.2.

We say a sentence φ in vocabulary $\mathbf{R}$ is *compact* if for any vocabulary $\mathbf{S}$ disjoint from $\mathbf{R}$, where we here allow, contrary to our convention everywhere else in this paper, uncountable $\mathbf{S}$, and for any theory $T \subseteq L_{\omega\omega}(\mathbf{R} \cup \mathbf{S})$, if φ is consistent with every finite subtheory of T, then φ is consistent with T.

6.3 GOLD PROPERTY. Let L^* be a first-order language, and φ a sentence of L^* such that both φ and $\neg\varphi$ are compact. Then φ is equivalent to a sentence of $L_{\omega\omega}$ in the same vocabulary.

PROOF. Gold [12] proves this for $L_{\infty\omega}$ but examining her proof one sees it only uses the Karp Property. —

6.4 UPWARD LÖWENHEIM-SKOLEM THEOREMS

(a) Let L^* be a strong first-order language such that the class of *all* wellfounded structures is $PC(L^* \subset HC)$. Then for invariant classes of structures $\underset{\sim}{\Sigma}_1(V)$ in parameters from $HC = PC(L^* \cap HC)$.

(b) Let $L^*, L^\#$ be languages satisfying the hypothesis of part (a). Then $L^* \cap HC$, $L^\# \cap HC$ have the same Hanf number.

(c) Let η be the common value of the Hanf numbers in part (b), then:

$$\mu\varkappa[\varkappa \to (\omega)_2^{<\omega}] < \eta < \mu\varkappa[\varkappa \to (\omega_1)_2^{<\omega}]$$

provide these large cardinals exist.

(d) Let L^* be any first-order language. Then the Hanf number of $L^* \cap HC$ is less than $\mu\varkappa[\varkappa \to (\omega_1)_2^{<\omega}]$ if it exists.

PROOF. (a) By *invariant* we here mean fully invariant (not just invariant in $\mathfrak{X}(\mathbf{R})$). (a) is then proved just like Prop. 2.6, but we need the stronger hypothesis. Of the languages in § 1, $L_{\omega_1 G}$, for example, satisfies this hypothesis, while Souslin logic does not.

(b) is immediate since the Hanf number depends only on the *PCs*.

(c) These bound were computed by Silver for the language of purely universal sentences of $L_{\omega_1\omega_1}$. Technically this language is not strong, but it is close enough for the arguments for parts (a) and (b) to go through. These bounds apply, for example, to $L_{\omega_1 G_1}$ but not to Souslin logic. For the Hanf number of the latter, see [9], [7], [11].

(d) is now immediate since any first-order language can be fattend up to a strong one. (d) is Prop. 2.4 of [2], and our 2.6 and 3.4 are more explicit formulations of things implicit in Barwise' proof. —

Craig's Interpolation Theorem for a language L^* states that disjoint $PC(L^*)$ classes (in a given fixed vocabulary) can be separated by an $EC(L^*)$. This is equivalent to the conjuction of the *Δ-Interpolation Theorem*, which states that disjoint $PC(L^*)$ classes can be separated by a class which is simultaneously $PC(L^*)$ and co-$PC(L^*)$, with the *Souslin-Kleene Theorem*, which states that any class both $PC(L^*)$ and co-$PC(L^*)$ is $EC(L^*)$. Craig's Theorem implies Beth's, and the Souslin-Kleene Theorem implies the weak version of Beth's Theorem.

6.5 ANTI-CRAIG THOREM. Let L^* be a first-order language containing $L_{\omega_2\omega}$. Then Craig's Theorem fails for L^*.

PROOF. In Prop. 2.11 of [2] Barwise derives this from Malitz' counterexample to Craig's Theorem for $L_{\omega_2\omega}$, which depends on the facts that $(\omega, \in)$ and $(\omega_1, \in)$ can be characterized up to isomorphism in $L_{\omega_2\omega}$, and that any two structures for the empty vocabulary (vocabulary with no nonlogical symbols, just the logical predicate $=$) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy $\mathfrak{A} \approx \mathfrak{B}$.

$A \subseteq HC$ is complete $\underset{\sim}{\Pi}_1(HC)$ if A is $\Pi_1(HC)$ and for any $\Pi_1(HC)\, B$ there exist a PR function F and a parameter $\tilde{r} \in HC$ such that $B = \{x : F(r,x) \in A\}$. No such set can be $\underset{\sim}{\Sigma}_1(HC)$ or $\underset{\sim}{\Sigma}_1(V)$ in parameters from HC.

6.6 INCOMPLETENESS THEOREM. Let L^* be a strong first-order language. Then the set of logically valid sentences of $L^* \cap HC$ is complete $\underset{\sim}{\Pi}_1(HC)$.

PROOF. Barwise, Prop. 2.15 of [2], shows this set is not $\Sigma_1(HC)$. An obvious simplification of his proof shows it is indeed complete $\Pi_1(HC)$. Given any complete proof procedure for $L^* \cap HC$, the set of valid sentences is $\{\varphi : \exists P (P$ is a proof of $\varphi)\}$. Thus 6.6 says there can be no such proof procedure in which proofs are countable objects and being a proof is a property Δ_1 in parameters from HC.

Let a first-order language L^* be given. We introduce a proof procedure for $L^* \cap HC$ by adjoining then the proof procedure for $L_{\omega_1\omega}$ given in [1] the following rule of inference with $\aleph_1$ premisses:

If $\vdash \mathscr{C}(\alpha, \varphi)$ for all $\alpha < \omega_1$, then $\vdash \varphi$,

where $\mathscr{C}$ is as in the Approximation Theorem 4.2.

6.7 COMPLETENESS THEOREM. Let L^* be a first-order language. The above proof procedure for $L^* \cap HC$ is sound and complete.

PROOF. $\varphi \in L^*$ is *not* valid if $\exists \mathfrak{A} \exists \alpha \neg \mathfrak{A} \vdash \mathscr{C}(\alpha, \varphi)$. This is a Σ_1 statement, and if $\varphi \in HC$, it is true iff it is true in HC, i.e. the ordinal α may be taken $< \omega_1$ and the model $\mathfrak{A}$ may be taken countable. Soundness and completeness are now immediate from the soundness and completeness of the proof procedure in [17].

6.7 shows validity for $L^* \cap HC$ is Σ_1 in parameters from HC plus the parameter ω_1. For particular languages from § 1 similar proof procedures have been obtained by Moschovakis (unpublished) and Green [10]. —

In the next four results **R**, **S**, **T** are disjoint nontrivial vocabularies.

6.8 DECOMPOSITION THEOREM. Let L^* be a first-order language, $\varphi \in L^*(\mathbf{R} \cup \mathbf{S}) \cap HC$. Then there exist $\varphi_\alpha \in L_{\omega_1\omega}(\mathbf{R})$, $\alpha < \omega_1$, such that the followingis valid over countable structures:

$$\exists S \varphi \leftrightarrow \bigvee_{\alpha < \omega_1} \varphi_\alpha .$$

6.9 NUMBER OF MODELS. Let L^* be a first-order language. $\varphi \in L^*(\mathbf{R} \cup \mathbf{S}) \cap HC$. Then up to isomorphism the number of countable models of $\exists \mathbf{S} \varphi$ is either $\leqslant \aleph_1$ or else exactly $2^{\aleph_0}$.

6.10 REDUCTION THEOREM. Let L^* be a strong firstorder language. Then for every $\varphi, \psi \in L^*(\mathbf{R} \cup \mathbf{S}) \cap HC$ there exist $\varphi_0, \psi_0 \in L^*(\mathbf{R} \cup \mathbf{S}) \cap HC$ such that the following are valid over countable models:

$$(\exists \mathbf{S} \varphi_0 \to \exists \mathbf{S} \varphi) \wedge (\exists \mathbf{S} \psi_0 \to \exists \mathbf{S} \psi)$$

$$\exists \mathbf{S} (\varphi \vee \psi) \to \exists \mathbf{S} (\varphi_0 \vee \psi_0)$$

$$\neg(\exists \mathbf{S} \varphi_0 \wedge \exists \mathbf{S} \psi_0)$$

6.11 UNIFORMIZATION THEOREM. Assume every real is constructible. Let L^* be a strong first-order language. Then for every $\varphi \in L^*(\mathbf{R} \cup \mathbf{S} \cup \mathbf{T}) \cap HC$ there exists $\psi \in L^*(\mathbf{R} \cup \mathbf{S} \cup \mathbf{T}) \cap HC$ such that the following are valid over conuntable structures:

$$\exists \mathbf{T} \psi \to \exists \mathbf{T} \varphi$$

$$\exists \mathbf{S} \exists \mathbf{T} \varphi \to \exists ! \mathbf{S} \exists \mathbf{T} \psi .$$

PROOF. 6.8—6.10 are the model-theoretic translations of results about invariant $\underset{\sim}{\Sigma}_2^1$ sets in [31]. 6.9 is of course immediate from 6.8 and a theorem of Morley on the number of countable models of a sentence of $L_{\omega_1\omega}$. 6.11 is similarly the model-theoretic translation of an invariant uniformization theorem (see [26], or [8] §1). An (unpublished) example of Silver shows the restriction to countable models cannot be lifted in 6.10. Myers [26] shows 6.11 cannot be proved in ZFC alone. Cf. also [30] for related observations.

6.12 THEOREM. Let L^* be a strong first-order language. Then the following fail for $L^* \cap HC$:

(a) Craig's Interpolation Theorem

(b) The Souslin-Kleene Theorem

(c) The Δ-Interpolation Theorem

(d) Beth's Definability Theorem

(e) Weak Beth's Theorem

PROOF. For (c) this is the model-theoretic translation of the fact that there exist disjoint invariant $\underset{\sim}{\Sigma}_2^1$ sets which cannot be separated by a $\underset{\sim}{\Delta}_2^1$ set. See [2], Prop. 2.13. (e) is Thm. 5.1, and this implies the rest.

One large problem in the model theory of strong first-order languages remains open, which does not lend itself to abstract, descriptive-set-theoretic statement: Can we prove for, say, $L_{\omega_1 G}$, that any sentence preserved under substructure (resp. homomorphic image) is equivalent to a universal (resp. positive) sentence? Harnik [13] has proved preservation theorems for $L_{\omega_1 G}$ for some symmetric relations ($L_{\omega\omega}$-elementary equivalence, the ρ-isomorphism of Scott, isomorphism of direct squares, etc.); his results (by the proofs of [13] or by alternative proofs due to Miller) extend to some of the other languages of § 1.

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