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THE CONTINUOUS PROJECTION-GRADIENT METHOD OF THE FOURTH ORDER

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Abstract:* In this paper the continuous projection-gradient method of the fourth order for solving the convex minimization problem in Euclidean space is considered. The sufficient conditions for convergence are established.

Keywords: Projection-gradient method, convexity.

1. INTRODUCTION

Consider the following minimization problem

$$J(u) \to inf, \qquad u \in U, \tag{1}$$

where U is a closed, convex subset of a real Euclidean space $\mathbb{E}^n$, function $J(u)$ is continuously differentiable and convex on $\mathbb{E}^n$. The scalar product of two elements $u, v \in \mathbb{E}^n$ will be denoted by: $<u, v>$; $\| u \| = <u, u>^{1/2}$ is the norm of the element u. Suppose that

$$J_* = \inf_{u \in U} J(u) > -\infty, \qquad U_* = \{ u \in U : J(u) = J_* \} \neq \varnothing. \tag{2}$$

The continuous minimization methods of the projection-gradient type

$$u'(t) + u(t) = P_U[\, u(t) - \alpha(t) J'(u(t))\,], \qquad t \geq 0,$$

$$u(0) = u_0, \qquad u_0 \in \mathbb{E}^n,$$

have been proposed and investigated in [5, 6] for $U = \mathbb{E}^n$ and in [1] for $U \subseteq \mathbb{E}^n$, $\alpha(t) = \alpha > 0$, $t \geq 0$. The further investigation in this area, considering the continuous projection-gradient methods of the second and third order has been presented in [1, 2]. This paper presents the continuous projection-gradient method of the fourth order.

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2. THE CONDITIONS FOR CONVERGENCE

For solving the problem (1) we will use the continuous projection gradient method of the fourth order

$$\begin{aligned}&\beta_4(t)\,u^{iv} + \beta_3(t)\,u''' + \beta_2(t)\,u'' + u' + u = \\ &P_U[\,u - \alpha(t)\,J'(u)\,], \qquad t \ge 0, \qquad\qquad (3)\\ &u(0) = u_0, \qquad u'(0) = u_1, \qquad u''(0) = u_2, \qquad u'''(0) = u_3\end{aligned}$$

where $P_U(z)$ – denotes the projection of the point z on the set U; u_i, $i = 0, 1, 2, 3$ are given initial points from the Euclidean space $\mathbf{E}^n$; $\alpha(t)$, $\beta_i(t)$, $i = 2, 3, 4$ are the parameters of the method (3), $u = u(t)$, $u^{(i)}(t) = d^i u(t)/dt^i$, $i = 1, 2, 3, 4$, $J'(u)$ – gradient of the function $J(u)$.

Theorem 1. Suppose that

1) U is a convex closed set in Euclidean space $\mathbf{E}^n$; function $J(u)$ is convex and differentiable on $\mathbf{E}^n$; the gradient $J'(u)$ satisfies the Lipschitz condition

$$\| J'(u) - J'(v) \| \le L \| u - v \|, \qquad u, v \in \mathbf{E}^n, \qquad L = const < +\infty;$$

the conditions (2) are satisfied;

2) the parameters $\alpha(t), \beta_i(t), i = 2, 3, 4$ of the method (3) are such that

$$\alpha(t) \in \mathbf{C}[\,0, +\infty\,),\ 0 < \alpha_0 \le \alpha(t) \le \alpha_1, \qquad t \ge 0;$$

$$\beta_2(t) \in \mathbf{C}^2[\,0, +\infty\,), \qquad \beta_3(t) \in \mathbf{C}^3[\,0, +\infty\,), \qquad \beta_4(t) \in \mathbf{C}^4[\,0, +\infty\,),$$

$$\beta_i'(t) \le 0, \qquad \beta_i''(t) \ge 0, \quad i = 2, 3, 4, \qquad t \ge 0;$$

$$\beta_i'''(t) \le 0, \quad i = 3, 4; \qquad \beta_i^{iv}(t) \ge 0, \qquad t \ge 0;$$

$$\lim_{t\to\infty} \beta_i(t) = \beta_{i\infty} > 0, \quad i = 2, 3, 4;$$

$$1 - \alpha_1 L - \beta_{2\infty} > 0, \qquad \beta_{2\infty}^2 (1 - \alpha_1 L) + \beta_{4\infty} - 2\beta_{3\infty} > 0,$$

$$\beta_{3\infty}^2 (1 - \alpha_1 L) + 2\beta_{2\infty}\beta_{4\infty} > 0, \qquad \beta_{2\infty} - \frac{3}{2}\beta_{3\infty} - \beta_{3\infty}\beta_{4\infty} > 0,$$

$$\beta_{2\infty}\beta_{3\infty} - 3\beta_{4\infty} - \beta_{2\infty}\beta_{4\infty}^2 > 0.$$

Then for any initial values $u_i \in \mathbf{E}^n$, $i = 0, 1, 2, 3$ there is a point $u_\infty \in U_*$ such that

$$\lim_{t\to\infty} \{ \sum_{i=1}^{4} \| u^{(i)}(t) \| + \| u(t) - u_\infty \| \} = 0,$$

$$\int_0^{+\infty} \{ \sum_{i=1}^{4} \| u^{(i)}(s) \|^2 + f(s) \| u(s) - u_\infty \|^2 \}\, ds < +\infty$$

where $f(s) = \beta_2''(s) - \beta_3'''(s) + \beta_4^{iv}(s)$, for all $s \ge 0$.

Proof. Note that there exist functions, for example

$$\alpha(t) = \alpha_0 \frac{2+t}{1+t}, \qquad \beta_i(t) = \beta_{i\infty} + \frac{1}{1+t}, \qquad \alpha_0 > 0, \qquad \beta_{i\infty} > 0,$$

for $i = 2, 3, 4$; and $t \geq 0$, which satisfy the conditions of the theorem.

From the inequalities for the derivatives of $\beta_i(t)$ and $\beta_{i\infty} > 0$, $i = 2, 3, 4$; it can be proved that

$$\lim_{t\to\infty} \beta_i'(t) = 0, \quad i = 2,3,4 \qquad \lim_{t\to\infty} \beta_i''(t) = 0, \quad i = 2,3; \qquad \lim_{t\to\infty} \beta_4'''(t) = 0. \tag{4}$$

Besides that, from the conditions on the limits $\beta_{i\infty}$ and α_1 we can find

$$1 > \beta_{2\infty} > \frac{3}{2}\beta_{3\infty} > \frac{9}{2}\beta_{4\infty}, \qquad \beta_{3\infty} - 3\beta_{4\infty}^2 > 0,$$
$$1 - \beta_{2\infty}\beta_{4\infty} - \beta_{4\infty}^4 > 0, \qquad \beta_{3\infty} - 2\beta_{4\infty} - 2\beta_{4\infty}^2 > 0, \tag{5}$$
$$\beta_{2\infty} - 2\beta_{3\infty}\beta_{4\infty} > 0, \qquad \beta_{3\infty} - \beta_{2\infty}^2\beta_{4\infty} - \beta_{4\infty} > 0.$$

As we know [3], there is the unique solution $u = u(t)$, $t \geq 0$ to the differential equation (3) for any given initial values $u_i \in \mathbb{E}^n$, $i = 0, 1, 2, 3$. For every $u_* \in U_*$, it holds: (see [4], pp.165, Theorem 3)

$$< J'(u_*), w - u_* > \geq 0, \qquad w \in U. \tag{6}$$

From (3) and the property of the projection operator (see [4]) it can be derived

$$< \beta_4(t)\, u^{iv} + \beta_3(t)\, u''' + \beta_2(t)\, u'' + u' + \alpha(t)\, J'(u),$$
$$\beta_4(t)\, u^{iv} + \beta_3(t)\, u''' + \beta_2(t)\, u'' + u' + u - v > \leq 0, \tag{7}$$
$$v \in U, \qquad t \geq 0.$$

Multiplying the inequality (6) by $[-\alpha(t)]$ for $w = \beta_4(t)\, u^{iv} + \beta_3(t)\, u''' + \beta_2(t)\, u'' + u' + u$ and summing it with (7) for $v = u_*$, we have

$$< \beta_4(t)\, u^{iv} + \beta_3(t)\, u''' + \beta_2(t)\, u'' + u' + \alpha(t)\, [J'(u) - J'(u_*)],$$
$$\beta_4(t)\, u^{iv} + \beta_3(t)\, u''' + \beta_2(t)\, u'' + u' + u - u_* > \leq 0,$$
$$t \geq 0, \qquad u_* \in U_*.$$

The last inequality can be written in the following way

$$\begin{aligned}
&\beta_4^2(t)\, \| u^{iv} \|^2 + 2\beta_4(t)\beta_3(t) < u^{iv}, u''' > + 2\beta_4(t)\beta_2(t) < u^{iv}, u'' > + \\
&2\beta_4(t) < u^{iv}, u' > + \beta_4(t) < u^{iv}, u - u_* > + \beta_3^2(t)\, \| u''' \|^2 + \\
&2\beta_3(t)\beta_2(t) < u''', u'' > + 2\beta_3(t) < u''', u' > + \\
&\beta_3(t) < u''', u - u_* > + \beta_2^2(t)\, \| u'' \|^2 + 2\beta_2(t) < u'', u' > + \\
&\beta_2(t) < u'', u - u_* > + \| u' \|^2 + < u', u - u_* > \quad \leq \\
&\alpha(t) < J'(u) - J'(u_*), u_* - [\, u + u' + \beta_2(t) u'' + \beta_3(t) u''' + \beta_4(t) u^{iv}\,] >
\end{aligned} \tag{8}$$

$$t \geq 0, \qquad u_* \in U_*.$$

Since the function $J(u)$ is convex, differentiable and its gradient $J'(u)$ satisfies the Lipschitz' s condition we have (see [4], pp.175).

$$< J'(u) - J'(v), v - w > \le \frac{L}{4} \| u - w \|^2, \qquad u, v, w \in \mathbb{E}^n \tag{9}$$

In order to write (8) in a more compact way we will use the denotations

$$\begin{aligned} r(t) &= \| u'''(t) \|^2, \\ q(t) &= \| u''(t) \|^2, \\ p(t) &= \| u'(t) \|^2, \\ x(t, u_*) &= \frac{1}{2} \| u(t) - u_* \|^2. \end{aligned} \tag{10}$$

Using (10), (9) and inequality $(a + b + c + d)^2 \le 4 (a^2 + b^2 + c^2 + d^2)$, which is true for all $a, b, c, d \in \mathbb{R}$, from (8) we will get

$$\begin{aligned} &\beta_4^2(t) [1 - \alpha(t) L] \| u^{iv} \|^2 + \beta_4(t) \beta_3(t) r'(t) + \\ &[\beta_3^2(t)(1 - \alpha(t) L) - 2 \beta_4(t) \beta_2(t)] r(t) + \beta_4(t) \beta_2(t) q''(t) + \\ &[\beta_3(t) \beta_2(t) - 3 \beta_4(t)] q'(t) + [\beta_2^2(t)(1 - \alpha(t) L) + \beta_4(t) - 2 \beta_3(t)] q(t) + \\ &\beta_4(t) p'''(t) + [\beta_3(t) - 2 \beta_4(t)] p''(t) + [\beta_2(t) - \frac{3}{2} \beta_3(t)] p'(t) + \\ &[1 - \alpha(t) L - \beta_2(t)] p(t) + \\ &\beta_4(t) x^{iv}(t, u_*) + \beta_3(t) x'''(t, u_*) + \beta_2(t) x''(t, u_*) + x'(t, u_*) \le 0, \\ &t \ge 0, \qquad u_* \in U_*. \end{aligned}$$

After the integration on the segment $[\xi, t]$, $t > \xi$, where $\xi \ge 0$ is arbitrary, taking into account the conditions 2) of the Theorem, we have

$$\begin{aligned} &\int_\xi^t \{ \beta_4^2(s) [1 - \alpha_1 L] \| u^{iv}(s) \|^2 + [\beta_3^2(s)(1 - \alpha_1 L) - 2 \beta_4(s) \beta_2(s)] r(s) + \\ &\quad [\beta_2^2(s)(1 - \alpha_1 L) + \beta_4(s) - 2 \beta_3(s) + 3 \beta_4'(s)] q(s) + \\ &\quad [1 - \alpha_1 L - \beta_2(s) + \frac{3}{2} \beta_3'(s) - 2 \beta_4''(s)] p(s) + \\ &\quad [\beta_2''(s) - \beta_3'''(s) + \beta_4^{iv}(s)] x(s, u_*) \} \, ds + \\ &\beta_4(t) \beta_3(t) r(t) + \beta_4(t) \beta_2(t) q'(t) + \\ &[\beta_3(t) \beta_2(t) - 3 \beta_4(t)] q(t) + \beta_4(t) p''(t) + \\ &[\beta_3(t) - 2 \beta_4(t) - \beta_4'(t)] p'(t) + [\beta_2(t) - \frac{3}{2} \beta_3(t) + 2 \beta_4'(t)] p(t) + \\ &\beta_4(t) x'''(t, u_*) + [\beta_3(t) - \beta_4'(t)] x''(t, u_*) + [\beta_2(t) - \beta_3'(t) + \beta_4''(t)] x'(t, u_*) + \\ &x(t, u_*) \le C_0(\xi, u_*), \\ &t > \xi \ge 0, \qquad u_* \in U_*, \end{aligned} \tag{11}$$

where

$$
\begin{aligned}
C_0(\xi,u_*) = {} & \beta_4(\xi)\,\beta_3(\xi)\,r(\xi)+\beta_4(\xi)\,\beta_2(\xi)\,q'(\xi)+\\
& [\,\beta_3(\xi)\,\beta_2(\xi)-3\,\beta_4(\xi)-(\beta_2(\xi)\,\beta_4(\xi))'\,]\,q(\xi)+\beta_4(\xi)\,p''(\xi)+\\
& [\,\beta_3(\xi)-2\,\beta_4(\xi)-\beta_4'(\xi)\,]\,p'(\xi)+\\
& [\,\beta_2(\xi)-\frac{3}{2}\beta_3(\xi)+2\,\beta_4'(\xi)-\beta_3'(\xi)+\beta_4''(\xi)\,]\,p(\xi)+\\
& \beta_4(\xi)\,x'''(\xi,u_*)+[\,\beta_3(\xi)-\beta_4'(\xi)\,]\,x''(\xi,u_*)+\\
& [\,\beta_2(\xi)-\beta_3'(\xi)+\beta_4''(\xi)\,]\,x'(\xi,u_*)+\\
& [\,1-\beta_2'(\xi)+\beta_3''(\xi)-\beta_4'''(\xi)\,]\,x(\xi,u_*)\,.
\end{aligned}
\tag{12}
$$

From the condition 2) of the Theorem and (4), (5), it follows

$$
\begin{aligned}
& \beta_4^2(s)\,[\,1-\alpha_1 L\,]>0, \qquad s\ge 0,\\
& \lim_{s\to\infty}[\,\beta_3^2(s)\,(1-\alpha_1 L)-2\,\beta_4(s)\,\beta_2(s)\,]=\\
& \qquad \beta_{3\infty}^2\,(1-\alpha_1 L)-2\,\beta_{4\infty}\,\beta_{2\infty} \;>\; 0,\\
& \lim_{s\to\infty}[\,\beta_2^2(s)\,(1-\alpha_1 L)+\beta_4(s)-2\,\beta_3(s)+3\,\beta_4'(s)\,]=\\
& \qquad \beta_{2\infty}^2\,(1-\alpha_1 L)+\beta_{4\infty}-2\,\beta_{3\infty} \;>\; 0,\\
& \lim_{s\to\infty}[\,1-\alpha_1 L-\beta_2(s)+\frac{3}{2}\beta_3'(s)-2\,\beta_4''(s)\,]=\\
& \qquad 1-\alpha_1 L-\beta_{2\infty} \;>\; 0.
\end{aligned}
$$

Therefore there are ε, $0<\varepsilon<1/2$ and $t_0\ge 0$, such that:

$$
\begin{aligned}
& \beta_4^2(s)\,[\,1-\alpha_1 L\,]\ge\varepsilon,\\
& \beta_3^2(s)\,(1-\alpha_1 L)-2\,\beta_4(s)\,\beta_2(s)\ge\varepsilon,\\
& \beta_2^2(s)\,(1-\alpha_1 L)+\beta_4(s)-2\,\beta_3(s)+3\,\beta_4'(s)\ge\varepsilon,\\
& 1-\alpha_1 L-\beta_2(s)+\frac{3}{2}\beta_3'(s)-2\,\beta_4''(s)\ge\varepsilon,\\
& s\ge t_0\,.
\end{aligned}
$$

Then from (11) we have

$$
\begin{aligned}
& \varepsilon\int_\xi^t\{\sum_{i=1}^{4}\|u^{(i)}(s)\|^2+f(s)\,\|u(s)-u_\infty\|^2\}\,ds+\\
& \beta_4(t)\,\beta_3(t)\,r(t)+\beta_4(t)\,\beta_2(t)\,q'(t)+\\
& [\,\beta_3(t)\,\beta_2(t)-3\,\beta_4(t)\,]\,q(t)+\beta_4(t)\,p''(t)+\\
& [\,\beta_3(t)-2\,\beta_4(t)-\beta_4'(t)\,]\,p'(t)+[\,\beta_2(t)-\frac{3}{2}\beta_3(t)+2\beta_4'(t)\,]\,p(t)+\\
& \beta_4(t)\,x'''(t,u_*)+[\,\beta_3(t)-\beta_4'(t)\,]\,x''(t,u_*)+[\,\beta_2(t)-\beta_3'(t)+\beta_4''(t)\,]\,x'(t,u_*)+\\
& x(t,u_*) \;\le\; C_0(\xi,u_*),
\end{aligned}
\tag{13}
$$

for all $t > \xi \geq 0$, $u_* \in U_*$. Let $h(t) = \exp\{ \int_0^t \beta_4(s)\, ds \} > 0$, $t \geq 0$. In (13) the integral is positive and $\beta_4(t)\, \beta_3(t)\, r(t) \geq 0$, $t \geq 0$, so they can be omitted. After that we will multiply (13) by $h(t)$ and integrate it on the segment $[\xi, t]$, for $t > \xi \geq t_0$. In this way we will get

$$
\begin{aligned}
&\beta_4(t)\,\beta_2(t)\,h(t)\,q(t) + \\
&\int_\xi^t [\,\beta_3(s)\,\beta_2(s) - 3\,\beta_4(s) - (\beta_2(s)\,\beta_4(s))' - \beta_2(s)\,\beta_4^2(s)\,]\,h(s)\,q(s)\,ds + \\
&\beta_4(t)\,h(t)\,p'(t) + [\,\beta_3(t) - 2\,\beta_4(t) - 2\,\beta_4'(t) - \beta_4^2(t)\,]\,h(t)\,p(t) + \\
&\int_\xi^t \{\,[\,\beta_4(s)\,h(s)\,]'' - [\,(\beta_3(s) - 2\,\beta_4(s) - \beta_4'(s))\,h(s)\,]' + \\
&\qquad [\,\beta_2(s) - \frac{3}{2}\,\beta_3(s) + 2\beta_4'(s)\,]\,h(s)\,\}\,p(s)\,ds + \\
&\beta_4(t)\,h(t)\,x'''(t,u_*) + [\,\beta_3(t) - 2\,\beta_4'(t) - \beta_4^2(t)\,]\,h(t)\,x'(t,u_*) + \\
&\{\,[\,\beta_2(t) - \beta_3'(t) + \beta_4''(t)\,]\,h(t) - [\,(\beta_3(t) - \beta_4'(t))\,h(t)\,]' + [\,\beta_4(t)\,h(t)\,]''\,\}\,x(t,u_*) + \\
&\int_\xi^t \{\,h(s) - [\,(\beta_2(s) - \beta_3'(s) + \beta_4''(s))\,h(s)\,]' + \\
&\qquad [\,(\beta_3(s) - \beta_4'(s))\,h(s)\,]'' - [\,\beta_4(s)\,h(s)\,]'''\,\}\,x(s,u_*)\,ds \quad \leq \\
&C_0(\xi,u_*)\int_\xi^t h(s)\,ds + C_1(\xi,u_*), \\
&t > \xi \geq t_0, \qquad u_* \in U_*.
\end{aligned}
$$

where

$$
\begin{aligned}
C_1(\xi,u_*) = {} & \beta_4(\xi)\,\beta_2(\xi)\,h(\xi)\,q(\xi) + \beta_4(\xi)\,h(\xi)\,p'(\xi) + \\
& [\,\beta_3(\xi) - 2\,\beta_4(\xi) - 2\,\beta_4'(\xi) - \beta_4^2(\xi)\,]\,h(\xi)\,p(\xi) + \\
& \beta_4(\xi)\,h(\xi)\,x''(\xi,u_*) + [\,\beta_3(\xi) - 2\,\beta_4'(\xi) - \beta_4^2(\xi)\,]\,h(\xi)\,x'(\xi,u_*) + \\
& \{\,[\,\beta_2(\xi) - \beta_3'(\xi) + \beta_4''(\xi)\,]\,h(\xi) - [\,(\beta_3(\xi) - \beta_4'(\xi))\,h(\xi)\,]' + \\
& [\,\beta_4(\xi)\,h(\xi)\,]''\,\}\,x(\xi,u_*).
\end{aligned}
$$

From the conditions 2) of the Theorem, (4) and (5) it can be shown that all integrals on the left hand side of the above inequality are non–negative for some $t_1 \geq t_0$ and every $t > \xi \geq t_1$. Besides that we have: $\beta_4'(t) \leq 0$, $\beta_3'(t) \leq 0$, $\beta_4''(t) \geq 0$, so the last inequality becomes

$$\begin{aligned}&\beta_4(t)\,\beta_2(t)\,h(t)\,q(t)+\beta_4(t)\,h(t)\,p'(t)+\\&[\,\beta_3(t)-2\,\beta_4(t)-\beta_4^2(t)\,]\,h(t)\,p(t)+\\&\beta_4(t)\,h(t)\,x''(t,u_*)+[\,\beta_3(t)-2\,\beta_4'(t)-\beta_4^2(t)\,]\,h(t)\,x'(t,u_*)+\\&[\,\beta_2(t)-\beta_3(t)\,\beta_4(t)+4\,\beta_4(t)\,\beta_4'(t)\,]\,h(t)\,x(t,u_*)\quad\le\\&C_0(\xi,u_*)\int_\xi^t h(s)\,ds+C_1(\xi,u_*),\end{aligned}\tag{14}$$

for every $t>\xi\ge t_1$, $u_*\in U_*$. Integrating (14) on $[\xi, t]$, we have

$$\begin{aligned}&\int_\xi^t \beta_4(s)\,\beta_2(s)\,h(s)\,q(s)\,ds+\int_\xi^t[\,\beta_3(s)-2\,\beta_4(s)-2\,\beta_4^2(s)-\beta_4'(s)\,]\,h(s)\,p(s)\,ds+\\&\beta_4(t)\,h(t)\,p(t)+\beta_4(t)\,h(t)\,x'(t,u_*)+[\,\beta_3(t)-3\,\beta_4'(t)-2\,\beta_4^2(t)\,]\,h(t)\,x(t,u_*)+\\&\int_\xi^t\{\,[\,\beta_2(s)-\beta_3(s)\,\beta_4(s)+4\,\beta_4(s)\,\beta_4'(s)\,]\,h(s)-\\&\quad[\,(\beta_3(s)-2\,\beta_4'(s)-\beta_4^2(s))\,h(s)\,]'+[\,\beta_4(s)\,h(s)\,]''\,\}\,x(s,u_*)\,ds\quad\le\\&C_0(\xi,u_*)\int_\xi^t\int_\xi^s h(\theta)\,d\theta\,ds+C_1(\xi,u_*)\,(t-\xi)+C_2(\xi,u_*),\end{aligned}$$

for $t>\xi\ge t_1$, $u_*\in U_*$ where

$$\begin{aligned}C_2(\xi,u_*)=&\ \beta_4(\xi)\,h(\xi)\,p(\xi)+\beta_4(\xi)\,h(\xi)\,x'(\xi,u_*)+\\&[\,\beta_3(\xi)-3\,\beta_4'(\xi)-2\,\beta_4^2(\xi)\,]\,h(\xi)\,x(\xi,u_*)\,.\end{aligned}$$

Taking into account the conditions 2) of the Theorem and relations (4), (5), we can find that there exists $t_2\ge t_1$, such that the integrals on the left hand side of the last inequality are non–negative for all $t>\xi\ge t_2$. Hence

$$\begin{aligned}&\beta_4(t)\,h(t)\,p(t)+\beta_4(t)\,h(t)\,x'(t,u_*)+\\&[\,\beta_3(t)-3\,\beta_4'(t)-2\,\beta_4^2(t)\,]\,h(t)\,x(t,u_*)\quad\le\\&C_0(\xi,u_*)\int_\xi^t\int_\xi^s h(\theta)\,d\theta\,ds+C_1(\xi,u_*)+(t-\xi)+C_2(\xi,u_*)\,,\end{aligned}\tag{15}$$

$t>\xi\ge t_2$, $\qquad u_*\in U_*$.

After the integration in (15) on $[\xi, t]$, we get

$$\begin{aligned}&\int_\xi^t\{\,\beta_4(s)\,p(s)+[\,\beta_3(s)-4\,\beta_4'(s)-3\,\beta_4^2(s)\,]\,x(t,u_*)\,\}\,h(s)\,ds+\beta_4(t)\,h(t)\,x(t,u_*)\quad\le\\&C_0(\xi,u_*)\int_\xi^t\int_\xi^s\int_\xi^v h(\theta)\,d\theta\,dv\,ds+\frac{1}{2}\,C_1(\xi,u_*)\,(t-\xi)^2+C_2(\xi,u_*)\,(t-\xi)+C_3(\xi,u_*)\,,\end{aligned}$$

for $t>\xi\ge t_2$, $u_*\in U_*$, where

$$C_3=\beta_4(\xi)\,h(\xi)\,x(\xi,u_*)\,.$$

From $\beta_4' \leq 0$ and (5) it follows that there is t_3, $t_3 \geq t_2$, such that the integrals on the left hand side of the above inequality are non–negative for all $t > \xi \geq t_3$, so that

$$\beta_4(t)\, h(t)\, x(t,u_*) \leq C_0(\xi,u_*) \int_\xi^t \int_\xi^s \int_\xi^v h(\theta)\, d\theta\, dv\, ds + \frac{1}{2} C_1(\xi,u_*)(t-\xi)^2 + C_2(\xi,u_*)(t-\xi) + C_3(\xi,u_*),$$

for $t > \xi \geq t_3$, $u_* \in U_*$. Consequently

$$\overline{\lim_{t\to\infty}} \| u(t) - u_* \|^2 \leq 2\, \overline{\lim_{t\to\infty}} [\beta_4(t)\, h(t)]^{-1} \{ C_3(\xi,u_*) + C_2(\xi,u_*)(t-\xi) + \frac{1}{2} C_1(\xi,u_*)(t-\xi)^2 + C_0(\xi,u_*) \int_\xi^t \int_\xi^s \int_\xi^v h(\theta)\, d\theta\, dv\, ds \} =$$

$$2\beta_{4\infty}^{-1} \lim_{t\to\infty} h^{-1}(t) \{ C_3(\xi,u_*) + C_2(\xi,u_*)(t-\xi) + \frac{1}{2} C_1(\xi,u_*)(t-\xi)^2 + C_0(\xi,u_*) \int_\xi^t \int_\xi^s \int_\xi^v h(\theta)\, d\theta\, dv\, ds \}.$$

It is clear that $\lim_{t\to\infty} h(t) = +\infty$. In order to calculate the limit of the right hand side we will use the L'Hospital rule three times. After that we will get

$$\overline{\lim_{t\to\infty}} \| u(t) - u_* \|^2 \leq b_0\, C_0(\xi,u_*), \tag{16}$$

$\xi \geq t_3$, $u_* \in U_*$.

where $b_0 = 2\beta_{4\infty}^{-4}$. Using (5) and $\beta_4' \leq 0$, it is not difficult to see that the third term on the left hand side of (15) is non–negative for some $t_4 \geq t_3$ and all $t \geq t_4$. From here and denotations (10) we have

$$\| u'(t) \|^2 + \langle u'(t), u(t) - u_* \rangle \leq 2 \{ C_0(\xi,u_*) \int_\xi^t \int_\xi^s h(\theta)\, d\theta\, ds + C_1(\xi,u_*)(t-\xi) + C_2(\xi,u_*) \} [\beta_4(t)\, h(t)]^{-1},$$

$t > \xi \geq t_4$, $u_* \in U_*$.

Therefore from (16) and the inequality

$$2 | \langle a, b \rangle | \leq \| a \|^2 + \| b \|^2, \qquad a, b \in \mathbf{E}^n, \tag{17}$$

it follows

$$\overline{\lim_{t\to\infty}} \|u'(t)\|^2 \le b_0\, C_0(\xi,u_*) +$$

$$2\,\overline{\lim_{t\to\infty}}\,[\,\beta_4(t)\,h(t)\,]^{-1}\,\{\,C_2(\xi,u_*)+C_1(\xi,u_*)(t-\xi)+C_0(\xi,u_*)\int_\xi^t\int_\xi^s h(\theta)\,d\theta\,ds\,\} =$$

$$b_0\, C_0(\xi,u_*) +$$

$$\frac{2}{\beta_{4\infty}}\lim_{t\to\infty}\{\,C_0(\xi,u_*)\int_\xi^t\int_\xi^s h(\theta)\,d\theta\,ds + C_1(\xi,u_*)(t-\xi)+C_2(\xi,u_*)\,\}\,h^{-1}(t)\,.$$

Using the L'Hospital rule two times, it can be obtained

$$\overline{\lim_{t\to\infty}} \|u'(t)\|^2 \le b_1\, C_0(\xi,u_*) \qquad \xi\ge t_4,\quad u_*\in U_*, \tag{18}$$

where $b_1 = b_0 + 2/\beta_{4\infty}^3$. From (4), (5) it is clear that there is $t_5 \ge t_4$, such that the third and the sixth terms on the left hand side of (14) are non–negative. Taking into account (10) and (17) in a similar way as before, we can get

$$\overline{\lim_{t\to\infty}}\,\beta_4(t)\,[\,\beta_2(t)-\frac{3}{2}\beta_4(t)\,]\,\|u''(t)\|^2 \le$$

$$\overline{\lim_{t\to\infty}}\,\{\,\frac{1}{2}[\,1+|f_1(t)|\,]\,\|u-u_*\|^2+[\,1+\frac{|f_1(t)|}{2}\,]\,\|u'(t)\|^2 +$$

$$[\,h(t)\,]^{-1}\,[\,C_0(\xi,u_*)\int_\xi^t h(s)\,ds + C_1(\xi,u_*)\,]\,\},$$

where $f_1(t)=\beta_3(t)-2\beta_4'(t)-\beta_4^2(t)$, $t\ge 0$. Let $M_1>0$, such that $|f_1(t)|\le M_1$, $t\ge 0$. Then

$$\beta_{4\infty}[\,\beta_{2\infty}-\frac{3}{2}\beta_{4\infty}\,]\,\overline{\lim_{t\to\infty}}\,\|u''(t)\|^2 \le$$

$$\lim_{t\to\infty}\,\{\,\frac{1+M_1}{2}\|u-u_*\|^2+[\,1+\frac{M_1}{2}\,]\,\|u'(t)\|^2 +$$

$$[\,h(t)\,]^{-1}\,[\,C_0(\xi,u_*)\int_\xi^t h(s)\,ds + C_1(\xi,u_*)\,]\,\}\,.$$

From (5) we have $\beta_{4\infty}[\,\beta_{2\infty}-(3/2)\,\beta_{4\infty}\,]>0$. Using L'Hospital's rule on the right hand side of the above inequality and the estimates (16), (18) we will get

$$\overline{\lim_{t\to\infty}} \|u''(t)\|^2 \le b_2\, C_0(\xi,u_*), \qquad \xi\ge t_5,\quad u_*\in U_*, \tag{19}$$

where b_2 does not depend on ξ and u_*. Taking into account (4), (5), (17) and denotations (10), from (13) we can obtain

$$\varepsilon\int_{\xi}^{t} \{ \sum_{i=1}^{4} ||u^{(i)}(s)||^2 + f(s)||u(s)-u_\infty||^2 \} ds +$$
$$[\beta_4(t)\beta_3(t) - \beta_4^2(t)\beta_2^2(t) - \beta_4^2(t)]||u'''||^2 \le$$
$$|f_2(t)|||u''||^2 + |f_3(t)|||u'||^2 + |f_4(t)|||u-u_*||^2 + C_0(\xi, u_*),$$
$$t > \xi \ge t_5, \qquad u_* \in U_*,$$

where the functions f_i, $i = 2, 3, 4$; are bounded and do not depend on ξ and u_*. Let $M_2 > 0$, such that $|f_i(t)| \le M_2$, $i = 2, 3, 4$; $t \ge 0$. Since

$$\lim_{t\to\infty} [\beta_3(t) - \beta_4(t)\beta_2^2(t) - \beta_4(t)] = \beta_{3\infty} - \beta_{4\infty}\beta_{2\infty}^2 - \beta_{4\infty} > 0,$$

then there exist a moment $t_7 \ge t_6$ and a number δ, $0 < \delta < \varepsilon$, such that for all $t \ge t_7$: $\beta_4(t)[\beta_3(t) - \beta_4(t)\beta_2^2(t) - \beta_4(t)] > \delta$,. Hence

$$\delta \{ \int_{\xi}^{t} [\sum_{i=1}^{4} ||u^{(i)}(s)||^2 + f(s)||u(s)-u_\infty||^2] ds + ||u'''||^2 \} \le$$
$$M_2 [||u''||^2 + ||u'||^2 + ||u-u_*||^2] + C_0(\xi, u_*),$$

for all $t > \xi \ge t_7$, $u_* \in U_*$. Consequently

$$\delta \{ \int_{\xi}^{t} [\sum_{i=1}^{4} ||u^{(i)}(s)||^2 + f(s)||u(s)-u_\infty||^2] ds \le b_3 C_0(\xi, u_*), \tag{20}$$

$$\overline{\lim_{t\to\infty}} ||u'''(t)||^2 \le b_3 C_0(\xi, u_*), \tag{21}$$
$$\xi \ge t_7, \quad u_* \in U_*.$$

where $b_3 = \delta^{-1}[M_2(b_2 + b_1 + b_0) + 1]$. From (20) it follows that there is a sequence $\{s_j\} \subseteq [0, +\infty)$ such that

$$\underline{\lim_{s\to\infty}} [\sum_{i=1}^{4} ||u^{(i)}(s)||^2] = \lim_{j\to\infty} [\sum_{i=1}^{4} ||u^{(i)}(s_j)||^2] = 0,$$

i.e.

$$\lim_{j\to\infty} ||u^{(i)}(s_j)|| = 0, \qquad i = 1, 2, 3, 4. \tag{22}$$

Then from (16) and the conditions 2) of the Theorem, it is obvious that there exist a subsequence $\{\hat{s}_j\}$, a point $u_\infty \in \mathbb{E}^n$ and a real number $\alpha_\infty > 0$, such that

$$\lim_{j\to\infty} ||u(\hat{s}_j) - u_\infty|| = 0, \qquad \lim_{j\to\infty} \alpha(\hat{s}_j) = \alpha_\infty. \tag{23}$$

Setting $t = \hat{s}_j$ in the differential equation (3), from (22) and (23), we can get

$$\lim_{j\to\infty} ||\beta_4(\hat{s}_j)u^{iv}(\hat{s}_j) + \beta_3(\hat{s}_j)u'''(\hat{s}_j) + \beta_2(\hat{s}_j)u''(\hat{s}_j) + u'(\hat{s}_j) + u(\hat{s}_j) -$$
$$P_U[u(\hat{s}_j) - \alpha(\hat{s}_j)J'(u(\hat{s}_j))]|| =$$
$$||u_\infty - P_U[u_\infty - \alpha_\infty J'(u_\infty)]|| = 0.$$

Consequently (see[4], pp.171) $u_\infty \in U_*$. From (12) where $\xi = \hat{s}_j$, $u_* = u_\infty \in U_*$, along with (22), (23) we have

$$\lim_{j\to\infty} C_0(\hat{s}_j, u_\infty) = 0.$$

Let $j_0 \geq 1$ be such a number that $\hat{s}_j \geq t_7$, for every $j \geq j_0$. Then the relations (16), (18)-(21) hold for $\xi = \hat{s}_j$, $j \geq j_0$ and $u_* = u_\infty$. Therefore

$$\overline{\lim_{t\to\infty}} \| u(t) - u_\infty \|^2 \leq b_0 \lim_{j\to\infty} C_0(\hat{s}_j, u_\infty) = 0.$$

$$\overline{\lim_{t\to\infty}} \| u^{(i)}(t) \|^2 \leq b_i \lim_{j\to\infty} C_0(\hat{s}_j, u_\infty) = 0, \qquad i = 1, 2, 3.$$

Hence

$$\lim_{t\to\infty} \| u(t) - u_\infty \| = 0, \qquad \lim_{t\to\infty} \| u^{(i)}(t) \| = 0, \qquad i = 1, 2, 3. \tag{24}$$

From the differential equation (3) and the relations, when $t \to \infty$, it can be obtained

$$\lim_{t\to\infty} \| u^{\text{iv}}(t) \| = 0. \tag{25}$$

The inequality (20) and the relations (24), (25) give the statement of the theorem.

3. CONCLUSION

The projection–gradient methods of the higher order are important because of their higher rate of convergence in comparison with the methods of that type of the first order [1]. Besides that, the continuous methods give us a large choice of the numerical integration methods for solving the corresponding differential equations. These facts justify the investigation of the continuous methods of the higher order.

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