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# A NOTE ON THE IMPROVEMENT OF THE MAXIMUM INDEPENDENT SET'S APPROXIMATION RATIO

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**Abstract:** We present an $O(n^{3.5})$ approximation algorithm for maximum independent set problem achieving, in unweighted case, a worst case approximation ratio strictly greater than $(2/\Delta)$, for a graph of order $n$ and maximum vertex degree $\Delta$. The best known ratio was, up to now, equal to $2/\Delta$ and its improvement is considered as an interesting open problem.

**Keywords:** Approximation algorithms, analysis of algorithms, combinatorial problems, independent set, computational complexity.

The aim of this note is to propose a new approximation algorithm for maximum independent set problem which, for the unweighted case, guarantees an approximation ratio better (greater) than $2/\Delta$ which, up to now, is the best known ratio [3, 5].

Let $G = (V, E)$ be a graph of order $n$; an independent set $V'$ is a subset of $V$ such that whenever $\{ v_i, v_j \} \subseteq V'$, $v_i v_j \notin E$, and the maximum independent set problem (IS) is to find an independent set of maximum size; in what follows, we denote by $\alpha(G)$ the stability number (cardinality of a maximum independent set) of $G$.

Nemhauser and Trotter prove, in [4], that the LP-relaxation[1] $\mathrm{IS}_r$ of IS has the semi-integral property, i.e., each basic feasible optimal solution $F$ of $\mathrm{IS}_r$ assigns to the variables values drawn from the set $\{ 0, 1/2, 1 \}$ (let us denote by $v(F)$ the value of $F$, and consequently, the optimal value of $\mathrm{IS}_r$); next, starting from this fact, they prove that if $S$ is the set of vertices corresponding to variables valued by 1 in $F$, then there exists a maximum independent set in $G$ that contains $S$. This elegant result, is at the

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[1] We replace the integrity constraints on the IS-solution by positivity ones.

origin of the very interesting Hochbaum's IS-algorithm [3].[2]

Here, we show how we can exploit further the result of [4] in such a way that, in combination with the natural greedy heuristic for IS, we obtain a better approximation bound for IS, i.e., an approximation ratio strictly greater than $2/\Delta$. The algorithm we propose uses as procedure an algorithm, proposed by Bourjolly et al. in [1], that decides if, in a graph $G$, $\upsilon(F) = \alpha(G)$, and if this is true, it finds a maximum independent set in $G$ in $O(n^3)$; in fact, it is proved there, that, in a graph $G$, the property "stability number equal size of a minimum edge covering", property which defines the class of König-Egervary graphs (KE-graphs), is equivalent to the property "$\upsilon(F) = \alpha(G)$".

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begin
        $S' \leftarrow \emptyset$; $V_C \leftarrow V$
        while the variables of $F$ for $G[V_C]$ are not all equal to 1/2 do
                solve $IS_r$ in order to define sets $S$, $\overline{S}$; $V_C \leftarrow V_C \setminus (S \cup \overline{S})$; $S' \leftarrow S' \cup S$
        od
        call the algorithm of [1] to decide if $G[V_C]$ is KE;
        if $G[V_C]$ is KE then
          use the algorithm of [1] to solve (optimally) IS on $G[V_C]$;
          let $S_C$ be the IS-solution for $G[V_C]$; $S' \leftarrow S' \cup S_C$
        exit
        fi
        repeat
                $v_j \leftarrow \mathrm{argmin}_{v_i \in V_C} \{\delta_i\}$; $S' \leftarrow S' \cup \{v_j\}$; $V_C \leftarrow V_C \setminus (\{v_j\} \cup \Gamma(v_j))$;
                delete all edges incident to $\{v_j\} \cup \Gamma(v_j)$;
                update the degrees of the vertices in $V_C$
        until $V_C = \emptyset$
end.
```

Algorithm 1. A modified greedy IS-algorithm; the solution constricted is denoted by S'.

In what follows, we consider a connected graph[3] $G = (V, E)$ of order $n$; for a $V' \subseteq V$, we denote by $G[V']$ the subgraph of $G$ induced by $V'$; for every vertex $v_i$ of $G$, we denote by $\Gamma(v_i)$ the set of its neighbours, by $\delta_i$ the degree of $v_i$, and by $\Delta$ the quantity $\max_{v_i \in V} \delta_i$; also, let us denote by $S$ and $\overline{S}$ the subsets of $V$ to the corresponding variables of which $F$ assigns values 1 and 0, respectively.

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[2] We have to mention that Hochbaum is the first author who has shown how we can use Nemhauser–Trotter's method to obtain efficient bounds for combinatorial problems.

[3] If the graph is not connected, then the result of the theorem remains true by considering every connected component of the graph.

**THEOREM.** Algorithm 1 is an $O(n^{3.5})$ approximation IS-algorithm achieving a worst case approximation ratio greater than or equal to $(2/\Delta)[1 + 1/(n-2)]$.

Let us suppose that the **while** loop of Algorithm 1 is repeated $t$ times. Let us denote by $\alpha(G_j)$, $j = 1, \dots, t$, the stability number of the surviving graph at the beginning of the $i$th iteration ( $\alpha(G_j) = \alpha(G)$ ) ; of course, this graph is a graph $G[V_C^j]$ where, for purposes of facility, we denote by $V_C^j$ the set $V_C$ produced at the end of the $j$–1th iteration. Also, for an iteration $j$, let us denote by $S_j$ the set – and by $\sigma_j$ the cardinality – of vertices, the corresponding variables of which are assigned by value 1 in $F$; it is easy to see, from the constraint-set of the integer-linear program of IS, that the set $\cup_{1 \leq j \leq t} S_j$ is an independent set of $G$, so does the union of this set with an independent set of the graph surviving after the $t$ executions of the **while** loop.

Then, based on the result of [4], for $j = 1, \dots, t$, there exists a maximum independent set of $G[V_C^j]$ containing $S_j$; we so obtain a $t$-line system of equations

$$\alpha(G[V_C^j]) = \alpha(G[V_C^{j+1}]) + \sigma_j, \quad j = 1, \dots, t$$

where $G[V_C^j] = G$, and $G[V_C^{t+1}]$ is the final graph, all the vertices of which have their corresponding variables valued by 1/2 in $F$; let us denote $G[V_C^{t+1}]$ by $G_{t+1}$.

By summing the lefthand and righthand sides of the equations of the above system, we obtain

$$\alpha(G) = \sum_{j=1}^{t} \sigma_j + \alpha(G_{t+1}) \tag{1}$$

If, after the execution of the **while** loop of Algorithm 1, the **if** condition is executed, then the optimal solution (and, hence, an approximation ratio equal to 1) is found for $G$.

Plainly, since IS in KE-graphs is polynomial, the value of the set $S_C$ equals $\alpha(G_{t+1})$ (where $S_C$ is the set appearing in the third line of the **if** condition); so, the solution $S'$ produced by Algorithm 1 just before the **exit** instruction has value $\sum_{j=1}^{t} \sigma_j + \alpha(G_{t+1})$ and, by expression (1), this cardinality is equal to the optimal one.

Let us now estimate the approximation performance of Algorithm 1 (**repeat** loop) on $G_{t+1}$ (the execution of this loop survenes in the case where $G_{t+1}$ is not KE).

Let us denote by $n_{t+1}$ the order of $G_{t+1}$; we also denote by $S'_{t+1}$ the subset of vertices added in $S'$ during the execution of the second line of the **repeat** loop.

So, during the first iteration of the **repeat** loop, since the minimum-degree vertex is chosen, at most $\Delta + 1$ vertices of $V_C^{t+1}$ are deleted, one among them being included in $S'_{t+1}$ (or, equivalently, in $S'$). Next, during ulterior iterations of the **repeat** loop (and because of the facts that (i) the minimum degree vertex is always chosen and (ii) the graph $G_{t+1}$ is connected[3]), the degree of the chosen vertices will be at most $\Delta - 1$; so, at most $\Delta$ vertices will be deleted from $V_C^{t+1}$; this is due to the fact that, by the connexity of the graph $G_{t+1}$, there always exists at least one vertex, a neighbour of which has already been deleted; so, if the **repeat** loop is executed $i$ times, then $i$ vertices have been introduced in set $S'_{t+1}$ and, consequently, $n_{t+1} \leq \Delta + 1 + \sum_{j=1}^{i-1} \Delta$, or $i \geq (n_{t+1} - 1)/\Delta$.

By denoting by $\alpha'(G)$ the cardinality of the set $S'$ finally provided by the algorithm, we get

$$\alpha'(G) = \sum_{j=1}^{t} \sigma_j + i \geq \sum_{j=1}^{t} \sigma_j + \frac{n_{t+1} - 1}{\Delta} \tag{2}$$

Let us recall that the variables of $F$ for the vertices of $G_{t+1}$, are all equal to 1/2, hence $v(F) = n_{t+1}/2$; moreover, since IS is a maximization problem, $\alpha(G_{t+1}) \leq v(F) = n_{t+1}/2$. On the other hand, if $\alpha(G_{t+1}) = v(F)$, then $G_{t+1}$ is KE and the execution of the second **if** condition of Algorithm 1 would provide the optimal solution for $G_{t+1}$, the whole of the algorithm attaining an approximation ratio of 1 for IS; so, since an independent set in a graph is a set of discrete objects, $\alpha(G_{t+1}) \leq (n_{t+1}/2) - 1$ and, by using expression (1), we get

$$\alpha(G) \leq \sum_{j=1}^{t} \sigma_j + \frac{n_{t+1}}{2} - 1 \tag{3}$$

From expressions (1) and (3), we obtain

$$\frac{\alpha'(G)}{\alpha(G)} \geq \frac{\sum_{j=1}^{t} \sigma_j + \frac{n_{t+1} - 1}{\Delta}}{\sum_{j=1}^{t} \sigma_j + \frac{n_{t+1}}{2} - 1} \geq \frac{\frac{n_{t+1} - 1}{\Delta}}{\frac{n_{t+1}}{2} - 1} = \frac{2}{\Delta}\left(1 + \frac{1}{n_{t+1} - 2}\right) \geq \frac{2}{\Delta}\left(1 + \frac{1}{n - 2}\right) \tag{4}$$

Concerning the complexity of Algorithm 1, the call of the algorithm of [1] and the execution of the **if** condition take time $O(n^{2.5})$; the **while** loop will be executed at most $n$ times, each time requiring $O(n^{2.5})$ [4]; finally, if the vertices are, a priori, sorted in increasing degree–order ($O(n \log n)$), the reorganization of the sorted list, after the deletion of the selected vertex and the update of the degrees, can be performed in $O(\log n)$ by using a heap (and this reorganization will be repeated, at most $n$ times); on the other hand, the deletion of the edges takes, at most, $O(|E|)$; consequently, the whole of the complexity of the **repeat** loop takes $O(\max\{|E|, n \log n\})$. So, the total worst case complexity of Algorithm 1 is of $O(n^{3.5})$.

In fact, if the complexity of algorithm is $O(n^{3.5})$ (i.e., the **while** loop of Algorithm 1 is executed), the worst case approximation ratio of the algorithm is even better. Since the **while** loop is executed at least once, then at least a set $S$ (corresponding to variables of $F$ valued by 1) is formed; by supposing that $G$ is connected, and by the constraints of IS, a set $\overline{S}$ associated to $S$ is also defined and at least one vertex of $G[V_C \setminus (S \cup \overline{S})]$ has at least one neighbour in $\overline{S}$; consequently, the minimum degree of $G[V_C \setminus (S \cup \overline{S})]$ is at most $\Delta - 1$ (moreover, the degrees of the survived vertices, for ulterior executions of the **while** loop of Algorithm 1, decrease). Hence, $i \geq n_{t+1} / \Delta$ (recall that $i$ is the number of vertices added to $S'$ during the **repeat** loop of Algorithm 1); on the other hand $\sum_{j=1}^{t} \sigma_j \geq 1$ and, since the function $(x+a)/(x+b)$, $b > a$ is increasing in $x$, expression (4) gives now $(\alpha'(G)/\alpha(G)) \geq (2/\Delta)(1+\Delta/n_{t+1})$. By the remark made above on the creation of the sets $S$ and $\overline{S}$ (resulting from the execution of the **while** loop of Algorithm 1), we have $n_{t+1} \leq n-2$ (by supposing that there were only one execution of the **while** loop, this execution having produced sets $S$ and $\overline{S}$, both of size 1); consequently

$$\frac{\alpha'(G)}{\alpha(G)} \geq \frac{2}{\Delta}(1+\frac{\Delta}{n-2})$$

Let us remark here that even if the improvement obtained in this note is slight and the quantity $2/\Delta$ remains its lower bound, neither the fine analysis of [3], nor the analysis of [5], allow to obtain an improvement even as weak as the one obtained here.

We close this note by proving that for graphs admitting a perfect matching (of cardinality $\lfloor n/2 \rfloor$) the greedy algorithm (the **repeat** loop of Algorithm 1) attains approximation ratio strictly greater than $2/\Delta$.

Let us consider a minimum vertex cover $C^*$ and the corresponding maximum independent set $S^*$ in a graph $G$. Let us also suppose that, given a matching $M$, there are $f$ matching edges such that both their extremities belong to $C^*$, for the remaining ones, one of their extremities belonging to $C^*$ and the other one to $S^*$. Let us call these edges "dissymmetric" and denote by $F$ the set of theses "dissymmetric" edges ($|F| = f$). For $M$ (in the case that is not perfect), let us denote by $X$ the set of the exposed (non-saturated) vertices of $G$ with respect to $M$, and by $X_C$ ($|X_C| = g$) and $X_S$ the subsets of $X$ belonging to $C^*$ and $S^*$, respectively (of course, $X = X_C \cup X_S$). The numbers $f$ and $g$, consequently the sets $F$ and $X_C$, depend not only on $M$ but also, for a fixed matching, on the sets $C^*$ (and $S^*$) considered. However, the sum $f+g$ is a quantity depending only on $G$ ($f+g$ can be considered as the "discrete duality gap").

In fact, for every graph $|C^*| = \tau(G) = m + (f+g)$, $|S^*| = \alpha(G) = n - m - (f+g)$.

We have seen (see expression (2)) that the size of the solution delivered by the greedy IS-algorithm on a graph of order $n$ is $\alpha'(G) \geq (n-1)/\Delta$. From the expression for $|S^*|$, we get $\alpha(G) = n - m - (f+g) = m + e - (f+g)$, where $e$ is the number of the exposed vertices (of $G$) with respect to a maximum matching of $G$.

On the other hand, let us suppose that $f = n/x$; so, given that $G$ admits a perfect matching $M$, we have the following for the terms of the third part of the last expression for $\alpha(G)$: $m \leq n/2$, $e-g \leq 1$ and $f = n/x$. From these last expressions, the last one for $\alpha(G)$, as well as the one for $\alpha'(G)$, we obtain $\alpha'(G)/\alpha(G) \geq [[n-1/\Delta]/[(n/2)+1-(n/x)]]$.

The function on the righthand side is decreasing in $x$ ; on the other hand, we can, without loss of generality, suppose[4] that $f = n / x \geq k$, for any fixed positive integer $k$, or $x \leq n / k$ ; so, after some easy algebra, we get

$$\frac{\alpha'(G)}{\alpha(G)} \geq \frac{2}{\Delta}(1 + \frac{2k+1}{n-2(k-1)})$$

The second term in the parentheses can be quite large, always remaining, however, of $o(1)$. Moreover, this improvement for the approximation ratio of the greedy algorithm is obtained (if the information on the existence of a perfect matching is given) in time linear to the number of edges of $G$.

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[4] In the opposite case, as we show in [2], IS is polynomial.

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# THE CONTINUOUS PROJECTION-GRADIENT METHOD OF THE FOURTH ORDER

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**Abstract:**[*] In this paper the continuous projection-gradient method of the fourth order for solving the convex minimization problem in Euclidean space is considered. The sufficient conditions for convergence are established.

**Keywords:** Projection-gradient method, convexity.

## 1. INTRODUCTION

Consider the following minimization problem

$$J(u) \to inf, \qquad u \in U, \tag{1}$$

where $U$ is a closed, convex subset of a real Euclidean space $\mathbb{E}^n$, function $J(u)$ is continuously differentiable and convex on $\mathbb{E}^n$. The scalar product of two elements $u, v \in \mathbb{E}^n$ will be denoted by: $<u, v>$; $\| u \| = <u, u>^{1/2}$ is the norm of the element $u$. Suppose that

$$J_* = \inf_{u \in U} J(u) > -\infty, \qquad U_* = \{ u \in U : J(u) = J_* \} \neq \varnothing. \tag{2}$$

The continuous minimization methods of the projection-gradient type

$$u'(t) + u(t) = P_U[ u(t) - \alpha(t) J'(u(t)) ], \qquad t \geq 0,$$

$$u(0) = u_0, \qquad u_0 \in \mathbb{E}^n,$$

have been proposed and investigated in [5, 6] for $U = \mathbb{E}^n$ and in [1] for $U \subseteq \mathbb{E}^n$, $\alpha(t) = \alpha > 0$, $t \geq 0$. The further investigation in this area, considering the continuous projection-gradient methods of the second and third order has been presented in [1, 2]. This paper presents the continuous projection-gradient method of the fourth order.

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## 2. THE CONDITIONS FOR CONVERGENCE

For solving the problem (1) we will use the continuous projection gradient method of the fourth order

$$\begin{aligned} &\beta_4(t)\,u^{\mathrm{iv}} + \beta_3(t)\,u''' + \beta_2(t)\,u'' + u' + u = \\ &P_U[\,u - \alpha(t)\,J'(u)\,], \qquad t \ge 0, \\ &u(0) = u_0, \qquad u'(0) = u_1, \qquad u''(0) = u_2, \qquad u'''(0) = u_3 \end{aligned} \tag{3}$$

where $P_U(z)$ – denotes the projection of the point $z$ on the set $U$; $u_i$, $i = 0, 1, 2, 3$ are given initial points from the Euclidean space $\mathbb{E}^n$; $\alpha(t)$, $\beta_i(t)$, $i = 2, 3, 4$ are the parameters of the method (3), $u = u(t)$, $u^{(i)}(t) = d^i u(t)/d t^i$, $i = 1, 2, 3, 4$, $J'(u)$ – gradient of the function $J(u)$.

**Theorem 1.** Suppose that

1) $U$ is a convex closed set in Euclidean space $\mathbb{E}^n$; function $J(u)$ is convex and differentiable on $\mathbb{E}^n$; the gradient $J'(u)$ satisfies the Lipschitz condition

$$\| J'(u) - J'(v) \| \le L \, \| u - v \|, \qquad u, v \in \mathbb{E}^n, \qquad L = const < +\infty;$$

the conditions (2) are satisfied;

2) the parameters $\alpha(t)$, $\beta_i(t)$, $i = 2, 3, 4$ of the method (3) are such that

$$\alpha(t) \in \mathbb{C}[\,0, +\infty\,),\ 0 < \alpha_0 \le \alpha(t) \le \alpha_1, \qquad t \ge 0;$$

$$\beta_2(t) \in \mathbb{C}^2[\,0, +\infty\,), \qquad \beta_3(t) \in \mathbb{C}^3[\,0, +\infty\,), \qquad \beta_4(t) \in \mathbb{C}^4[\,0, +\infty\,),$$

$$\beta_i'(t) \le 0, \qquad \beta_i''(t) \ge 0, \quad i = 2, 3, 4, \qquad t \ge 0;$$

$$\beta_i'''(t) \le 0, \quad i = 3, 4; \qquad \beta_i^{\mathrm{iv}}(t) \ge 0, \qquad t \ge 0;$$

$$\lim_{t\to\infty} \beta_i(t) = \beta_{i\infty} > 0, \quad i = 2, 3, 4;$$

$$1 - \alpha_1 L - \beta_{2\infty} > 0, \qquad \beta_{2\infty}^2 (1 - \alpha_1 L) + \beta_{4\infty} - 2\beta_{3\infty} > 0,$$

$$\beta_{3\infty}^2 (1 - \alpha_1 L) + 2\beta_{2\infty}\beta_{4\infty} > 0, \qquad \beta_{2\infty} - \frac{3}{2}\beta_{3\infty} - \beta_{3\infty}\beta_{4\infty} > 0,$$

$$\beta_{2\infty}\beta_{3\infty} - 3\beta_{4\infty} - \beta_{2\infty}\beta_{4\infty}^2 > 0.$$

Then for any initial values $u_i \in \mathbb{E}^n$, $i = 0, 1, 2, 3$ there is a point $u_\infty \in U_*$ such that

$$\lim_{t\to\infty} \{ \sum_{i=1}^{4} \| u^{(i)}(t) \| + \| u(t) - u_\infty \| \} = 0,$$

$$\int_0^{+\infty} \{ \sum_{i=1}^{4} \| u^{(i)}(s) \|^2 + f(s) \, \| u(s) - u_\infty \|^2 \} \, ds < +\infty$$

where $f(s) = \beta_2''(s) - \beta_3'''(s) + \beta_4^{\mathrm{iv}}(s)$, for all $s \ge 0$.

**Proof.** Note that there exist functions, for example

$$\alpha(t)=\alpha_0\frac{2+t}{1+t},\qquad \beta_i(t)=\beta_{i\infty}+\frac{1}{1+t},\qquad \alpha_0>0,\qquad \beta_{i\infty}>0,$$

for $i=2,3,4$; and $t\geq 0$, which satisfy the conditions of the theorem.

From the inequalities for the derivatives of $\beta_i(t)$ and $\beta_{i\infty}>0$, $i=2,3,4$; it can be proved that

$$\begin{aligned}&\lim_{t\to\infty}\beta_i'(t)=0,\quad i=2,3,4 &&\lim_{t\to\infty}\beta_i''(t)=0,\quad i=2,3;\\ &\lim_{t\to\infty}\beta_4'''(t)=0. &&\end{aligned}\tag{4}$$

Besides that, from the conditions on the limits $\beta_{i\infty}$ and $\alpha_1$ we can find

$$\begin{aligned}&1>\beta_{2\infty}>\frac{3}{2}\beta_{3\infty}>\frac{9}{2}\beta_{4\infty}, &&\beta_{3\infty}-3\beta_{4\infty}^2>0,\\ &1-\beta_{2\infty}\beta_{4\infty}-\beta_{4\infty}^4>0, &&\beta_{3\infty}-2\beta_{4\infty}-2\beta_{4\infty}^2>0,\\ &\beta_{2\infty}-2\beta_{3\infty}\beta_{4\infty}>0, &&\beta_{3\infty}-\beta_{2\infty}^2\beta_{4\infty}-\beta_{4\infty}>0.\end{aligned}\tag{5}$$

As we know [3], there is the unique solution $u=u(t)$, $t\geq 0$ to the differential equation (3) for any given initial values $u_i\in\mathbb{E}^n$, $i=0,1,2,3$. For every $u_*\in U_*$, it holds: (see [4], pp.165, Theorem 3)

$$<J'(u_*),w-u_*>\geq 0,\qquad w\in U.\tag{6}$$

From (3) and the property of the projection operator (see [4]) it can be derived

$$\begin{aligned}&<\beta_4(t)\,u^{\text{iv}}+\beta_3(t)\,u'''+\beta_2(t)\,u''+u'+\alpha(t)\,J'(u),\\ &\quad\beta_4(t)\,u^{\text{iv}}+\beta_3(t)\,u'''+\beta_2(t)\,u''+u'+u-v>\leq 0,\\ &v\in U,\qquad t\geq 0.\end{aligned}\tag{7}$$

Multiplying the inequality (6) by $[-\alpha(t)]$ for $w=\beta_4(t)\,u^{\text{iv}}+\beta_3(t)\,u'''+\beta_2(t)\,u''+u'+u$ and summing it with (7) for $v=u_*$, we have

$$\begin{aligned}&<\beta_4(t)\,u^{\text{iv}}+\beta_3(t)\,u'''+\beta_2(t)\,u''+u'+\alpha(t)\,[\,J'(u)-J'(u_*)\,],\\ &\quad\beta_4(t)\,u^{\text{iv}}+\beta_3(t)\,u'''+\beta_2(t)\,u''+u'+u-u_*>\leq 0,\\ &t\geq 0,\qquad u_*\in U_*.\end{aligned}$$

The last inequality can be written in the following way

$$\begin{aligned}&\beta_4^2(t)\,||u^{\text{iv}}||^2+2\beta_4(t)\,\beta_3(t)<u^{\text{iv}},u'''>+2\beta_4(t)\,\beta_2(t)<u^{\text{iv}},u''>+\\ &2\beta_4(t)<u^{\text{iv}},u'>+\beta_4(t)<u^{\text{iv}},u-u_*>+\beta_3^2(t)\,||u'''||^2+\\ &2\beta_3(t)\,\beta_2(t)<u''',u''>+2\beta_3(t)<u''',u'>+\\ &\beta_3(t)<u''',u-u_*>+\beta_2^2(t)\,||u''||^2+2\beta_2(t)<u'',u'>+\\ &\beta_2(t)<u'',u-u_*>+||u'||^2+<u',u-u_*>\ \leq\\ &\alpha(t)<J'(u)-J'(u_*),u_*-[\,u+u'+\beta_2(t)u''+\beta_3(t)u'''+\beta_4(t)u^{\text{iv}}\,]>\\ &t\geq 0,\qquad u_*\in U_*.\end{aligned}\tag{8}$$

Since the function $J(u)$ is convex, differentiable and its gradient $J'(u)$ satisfies the Lipschitz's condition we have (see [4], pp.175).

$$< J'(u) - J'(v), v - w > \;\le\; \frac{L}{4}\|u - w\|^2, \qquad u, v, w \in \mathbb{E}^n \qquad (9)$$

In order to write (8) in a more compact way we will use the denotations

$$\begin{aligned}
&r(t) = \| u'''(t) \|^2,\\
&q(t) = \| u''(t) \|^2,\\
&p(t) = \| u'(t) \|^2,\\
&x(t, u^*) = \frac{1}{2}\| u(t) - u^* \|^2.
\end{aligned} \qquad (10)$$

Using (10), (9) and inequality $(a + b + c + d)^2 \le 4\,(a^2 + b^2 + c^2 + d^2)$, which is true for all $a, b, c, d \in \mathbb{R}$, from (8) we will get

$$\begin{aligned}
&\beta_4^2(t)\,[\,1 - \alpha(t)\,L\,]\,\|u^{\mathrm{iv}}\|^2 + \beta_4(t)\,\beta_3(t)\,r'(t) +\\
&[\,\beta_3^2(t)\,(1 - \alpha(t)\,L) - 2\,\beta_4(t)\,\beta_2(t)\,]\,r(t) + \beta_4(t)\,\beta_2(t)\,q''(t) +\\
&[\,\beta_3(t)\,\beta_2(t) - 3\,\beta_4(t)\,]\,q'(t) + [\,\beta_2^2(t)\,(1 - \alpha(t)\,L) + \beta_4(t) - 2\,\beta_3(t)\,]\,q(t) +\\
&\beta_4(t)\,p'''(t) + [\,\beta_3(t) - 2\,\beta_4(t)\,]\,p''(t) + [\,\beta_2(t) - \frac{3}{2}\beta_3(t)\,]\,p'(t) +\\
&[\,1 - \alpha(t)\,L - \beta_2(t)\,]\,p(t) +\\
&\beta_4(t)\,x^{\mathrm{iv}}(t, u_*) + \beta_3(t)\,x'''(t, u_*) + \beta_2(t)\,x''(t, u_*) + x'(t, u_*) \;\;\le\;\; 0,\\
&t \ge 0, \qquad u_* \in U_*.
\end{aligned}$$

After the integration on the segment $[\xi, t]$, $t > \xi$, where $\xi \ge 0$ is arbitrary, taking into account the conditions 2) of the Theorem, we have

$$\begin{aligned}
&\int_{\xi}^{t} \{\beta_4^2(s)\,[\,1 - \alpha_1\,L\,]\,\|u^{\mathrm{iv}}(s)\|^2 + [\,\beta_3^2(s)\,(1 - \alpha_1\,L) - 2\,\beta_4(s)\,\beta_2(s)\,]\,r(s) +\\
&\quad [\,\beta_2^2(s)\,(1 - \alpha_1\,L) + \beta_4(s) - 2\,\beta_3(s) + 3\,\beta_4'(s)\,]\,q(s) +\\
&\quad [\,1 - \alpha_1\,L - \beta_2(s) + \frac{3}{2}\beta_3'(s) - 2\,\beta_4''(s)\,]\,p(s) +\\
&\quad [\,\beta_2''(s) - \beta_3'''(s) + \beta_4^{\mathrm{iv}}(s)\,]\,x(s, u_*)\,\}\,ds +\\
&\beta_4(t)\,\beta_3(t)\,r(t) + \beta_4(t)\,\beta_2(t)\,q'(t) +\\
&[\,\beta_3(t)\,\beta_2(t) - 3\,\beta_4(t)\,]\,q(t) + \beta_4(t)\,p''(t) +\\
&[\,\beta_3(t) - 2\,\beta_4(t) - \beta_4'(t)\,]\,p'(t) + [\,\beta_2(t) - \frac{3}{2}\beta_3(t) + 2\beta_4'(t)\,]\,p(t) +\\
&\beta_4(t)\,x'''(t, u_*) + [\,\beta_3(t) - \beta_4'(t)\,]\,x''(t, u_*) + [\,\beta_2(t) - \beta_3'(t) + \beta_4''(t)\,]\,x'(t, u_*) +\\
&x(t, u_*) \;\;\le\;\; C_0(\xi, u_*),\\
&t > \xi \ge 0, \qquad u_* \in U_*,
\end{aligned} \qquad (11)$$

where

$$\begin{aligned} C_0(\xi,u_*) = {} & \beta_4(\xi)\,\beta_3(\xi)\,r(\xi)+\beta_4(\xi)\,\beta_2(\xi)\,q'(\xi)+ \\ & [\,\beta_3(\xi)\,\beta_2(\xi)-3\,\beta_4(\xi)-(\beta_2(\xi)\,\beta_4(\xi))'\,]\,q(\xi)+\beta_4(\xi)\,p''(\xi)+ \\ & [\,\beta_3(\xi)-2\,\beta_4(\xi)-\dot\beta_4(\xi)\,]\,p'(\xi)+ \\ & [\,\beta_2(\xi)-\frac{3}{2}\,\beta_3(\xi)+2\,\dot\beta_4(\xi)-\dot\beta_3(\xi)+\ddot\beta_4(\xi)\,]\,p(\xi)+ \\ & \beta_4(\xi)\,x'''(\xi,u_*)+[\,\beta_3(\xi)-\dot\beta_4(\xi)\,]\,x''(\xi,u_*)+ \\ & [\,\beta_2(\xi)-\dot\beta_3(\xi)+\ddot\beta_4(\xi)\,]\,x'(\xi,u_*)+ \\ & [\,1-\dot\beta_2(\xi)+\ddot\beta_3(\xi)-\dddot\beta_4(\xi)\,]\,x(\xi,u_*). \end{aligned} \tag{12}$$

From the condition 2) of the Theorem and (4), (5), it follows

$$\beta_4^2(s)\,[\,1-\alpha_1\,L\,]>0, \qquad s\ge 0,$$

$$\begin{aligned} & \lim_{s\to\infty}\,[\,\beta_3^2(s)\,(1-\alpha_1\,L)-2\,\beta_4(s)\,\beta_2(s)\,]= \\ & \qquad \beta_{3\infty}^2\,(1-\alpha_1\,L)-2\,\beta_{4\infty}\,\beta_{2\infty} \;>\; 0, \\ & \lim_{s\to\infty}\,[\,\beta_2^2(s)\,(1-\alpha_1\,L)+\beta_4(s)-2\,\beta_3(s)+3\,\dot\beta_4(s)\,]= \\ & \qquad \beta_{2\infty}^2\,(1-\alpha_1\,L)+\beta_{4\infty}-2\,\beta_{3\infty} \;>\; 0, \\ & \lim_{s\to\infty}\,[\,1-\alpha_1\,L-\beta_2(s)+\frac{3}{2}\,\dot\beta_3(s)-2\,\ddot\beta_4(s)\,]= \\ & \qquad 1-\alpha_1\,L-\beta_{2\infty} \;>\; 0. \end{aligned}$$

Therefore there are $\varepsilon$, $0<\varepsilon<1/2$ and $t_0\ge 0$, such that:

$$\begin{aligned} & \beta_4^2(s)\,[\,1-\alpha_1\,L\,]\ge\varepsilon, \\ & \beta_3^2(s)\,(1-\alpha_1\,L)-2\,\beta_4(s)\,\beta_2(s)\ge\varepsilon, \\ & \beta_2^2(s)\,(1-\alpha_1\,L)+\beta_4(s)-2\,\beta_3(s)+3\,\dot\beta_4(s)\ge\varepsilon, \\ & 1-\alpha_1\,L-\beta_2(s)+\frac{3}{2}\,\dot\beta_3(s)-2\,\ddot\beta_4(s)\ge\varepsilon, \\ & s\ge t_0\,. \end{aligned}$$

Then from (11) we have

$$\begin{aligned} & \varepsilon\int_{\xi}^{t}\,\{\,\sum_{i=1}^{4}\,\|u^{(i)}(s)\|^2+f(s)\,\|u(s)-u_\infty\|^2\,\}\,ds+ \\ & \beta_4(t)\,\beta_3(t)\,r(t)+\beta_4(t)\,\beta_2(t)\,q'(t)+ \\ & [\,\beta_3(t)\,\beta_2(t)-3\,\beta_4(t)\,]\,q(t)+\beta_4(t)\,p''(t)+ \\ & [\,\beta_3(t)-2\,\beta_4(t)-\dot\beta_4(t)\,]\,p'(t)+[\,\beta_2(t)-\frac{3}{2}\,\beta_3(t)+2\dot\beta_4(t)\,]\,p(t)+ \\ & \beta_4(t)\,x'''(t,u_*)+[\,\beta_3(t)-\dot\beta_4(t)\,]\,x''(t,u_*)+[\,\beta_2(t)-\dot\beta_3(t)+\ddot\beta_4(t)\,]\,x'(t,u_*)+ \\ & x(t,u_*) \;\le\; C_0(\xi,u_*), \end{aligned} \tag{13}$$

for all $t > \xi \geq 0$, $u_* \in U_*$. Let $h(t) = \exp\{\int_0^t \beta_4(s)\,ds\} > 0$, $t \geq 0$. In (13) the integral is positive and $\beta_4(t)\,\beta_3(t)\,r(t) \geq 0$, $t \geq 0$, so they can be omitted. After that we will multiply (13) by $h(t)$ and integrate it on the segment $[\xi, t]$, for $t > \xi \geq t_0$. In this way we will get

$$\begin{aligned}
&\beta_4(t)\,\beta_2(t)\,h(t)\,q(t) + \\
&\int_\xi^t [\,\beta_3(s)\,\beta_2(s) - 3\,\beta_4(s) - (\beta_2(s)\,\beta_4(s))' - \beta_2(s)\,\beta_4^2(s)\,]\,h(s)\,q(s)\,ds + \\
&\beta_4(t)\,h(t)\,p'(t) + [\,\beta_3(t) - 2\,\beta_4(t) - 2\,\beta_4'(t) - \beta_4^2(t)\,]\,h(t)\,p(t) + \\
&\int_\xi^t \{[\,\beta_4(s)\,h(s)\,]'' - [\,(\beta_3(s) - 2\,\beta_4(s) - \beta_4'(s))\,h(s)\,]' + \\
&\qquad [\,\beta_2(s) - \frac{3}{2}\,\beta_3(s) + 2\beta_4'(s)\,]\,h(s)\}\;p(s)\,ds + \\
&\beta_4(t)\,h(t)\,x'''(t, u_*) + [\,\beta_3(t) - 2\,\beta_4'(t) - \beta_4^2(t)\,]\,h(t)\,x'(t, u_*) + \\
&\{[\,\beta_2(t) - \beta_3'(t) + \beta_4''(t)\,]\,h(t) - [\,(\beta_3(t) - \beta_4'(t))\,h(t)\,]' + [\,\beta_4(t)\,h(t)\,]''\}\;x(t, u_*) + \\
&\int_\xi^t \{h(s) - [\,(\beta_2(s) - \beta_3'(s) + \beta_4''(s))\,h(s)\,]' + \\
&\qquad [\,(\beta_3(s) - \beta_4'(s))\,h(s)\,]'' - [\,\beta_4(s)\,h(s)\,]'''\}\;x(s, u_*)\,ds \quad \leq \\
&C_0(\xi, u_*)\int_\xi^t h(s)\,ds + C_1(\xi, u_*), \\
&t > \xi \geq t_0, \qquad\qquad u_* \in U_*.
\end{aligned}$$

where

$$\begin{aligned}
C_1(\xi, u_*) = {}&\beta_4(\xi)\,\beta_2(\xi)\,h(\xi)\,q(\xi) + \beta_4(\xi)\,h(\xi)\,p'(\xi) + \\
&[\,\beta_3(\xi) - 2\,\beta_4(\xi) - 2\,\beta_4'(\xi) - \beta_4^2(\xi)\,]\,h(\xi)\,p(\xi) + \\
&\beta_4(\xi)\,h(\xi)\,x''(\xi, u_*) + [\,\beta_3(\xi) - 2\,\beta_4'(\xi) - \beta_4^2(\xi)\,]\,h(\xi)\,x'(\xi, u_*) + \\
&\{[\,\beta_2(\xi) - \beta_3'(\xi) + \beta_4''(\xi)\,]\,h(\xi) - [\,(\beta_3(\xi) - \beta_4'(\xi))\,h(\xi)\,]' + \\
&[\,\beta_4(\xi)\,h(\xi)\,]''\}\;x(\xi, u_*).
\end{aligned}$$

From the conditions 2) of the Theorem, (4) and (5) it can be shown that all integrals on the left hand side of the above inequality are non–negative for some $t_1 \geq t_0$ and every $t > \xi \geq t_1$. Besides that we have: $\beta_4'(t) \leq 0$, $\beta_3'(t) \leq 0$, $\beta_4''(t) \geq 0$, so the last inequality becomes

$$\begin{aligned}
&\beta_4(t)\,\beta_2(t)\,h(t)\,q(t)+\beta_4(t)\,h(t)\,p'(t)+\\
&[\,\beta_3(t)-2\,\beta_4(t)-\beta_4^2(t)\,]\,h(t)\,p(t)+\\
&\beta_4(t)\,h(t)\,x''(t,u_*)+[\,\beta_3(t)-2\,\dot{\beta}_4(t)-\beta_4^2(t)\,]\,h(t)\,x'(t,u_*)+\\
&[\,\beta_2(t)-\beta_3(t)\,\beta_4(t)+4\,\beta_4(t)\,\dot{\beta}_4(t)\,]\,h(t)\,x(t,u_*)\quad\le\\
&C_0(\xi,u_*)\int_{\xi}^{t}h(s)\,ds+C_1(\xi,u_*),
\end{aligned}\tag{14}$$

for every $t>\xi\ge t_1$, $u_*\in U_*$. Integrating (14) on $[\xi,t]$, we have

$$\begin{aligned}
&\int_{\xi}^{t}\beta_4(s)\,\beta_2(s)\,h(s)\,q(s)\,ds+\int_{\xi}^{t}[\,\beta_3(s)-2\,\beta_4(s)-2\,\beta_4^2(s)-\dot{\beta}_4(s)\,]\,h(s)\,p(s)\,ds+\\
&\beta_4(t)\,h(t)\,p(t)+\beta_4(t)\,h(t)\,x'(t,u_*)+[\,\beta_3(t)-3\,\dot{\beta}_4(t)-2\,\beta_4^2(t)\,]\,h(t)\,x(t,u_*)+\\
&\int_{\xi}^{t}\{\,[\,\beta_2(s)-\beta_3(s)\,\beta_4(s)+4\,\beta_4(s)\,\dot{\beta}_4(s)\,]\,h(s)-\\
&\qquad[\,(\beta_3(s)-2\,\dot{\beta}_4(s)-\beta_4^2(s))\,h(s)\,]'+[\,\beta_4(s)\,h(s)\,]''\,\}\,x(s,u_*)\,ds\quad\le\\
&C_0(\xi,u_*)\int_{\xi}^{t}\int_{\xi}^{s}h(\theta)\,d\theta\,ds\;+C_1(\xi,u_*)\,(t-\xi)+C_2(\xi,u_*),
\end{aligned}$$

for $t>\xi\ge t_1$, $u_*\in U_*$ where

$$\begin{aligned}
C_2(\xi,u_*)=\;&\beta_4(\xi)\,h(\xi)\,p(\xi)+\beta_4(\xi)\,h(\xi)\,x'(\xi,u_*)+\\
&[\,\beta_3(\xi)-3\,\dot{\beta}_4(\xi)-2\,\beta_4^2(\xi)\,]\,h(\xi)\,x(\xi,u_*).
\end{aligned}$$

Taking into account the conditions 2) of the Theorem and relations (4), (5), we can find that there exists $t_2\ge t_1$, such that the integrals on the left hand side of the last inequality are non-negative for all $t>\xi\ge t_2$. Hence

$$\begin{aligned}
&\beta_4(t)\,h(t)\,p(t)+\beta_4(t)\,h(t)\,x'(t,u_*)+\\
&[\,\beta_3(t)-3\,\dot{\beta}_4(t)-2\,\beta_4^2(t)\,]\,h(t)\,x(t,u_*)\quad\le\\
&C_0(\xi,u_*)\int_{\xi}^{t}\int_{\xi}^{s}h(\theta)\,d\theta\,ds+C_1(\xi,u_*)+(t-\xi)+C_2(\xi,u_*),
\end{aligned}\tag{15}$$

$$t>\xi\ge t_2,\qquad u_*\in U_*.$$

After the integration in (15) on $[\xi,t]$, we get

$$\begin{aligned}
&\int_{\xi}^{t}\{\,\beta_4(s)\,p(s)+[\,\beta_3(s)-4\,\dot{\beta}_4(s)-3\,\beta_4^2(s)\,]\,x(t,u_*)\,\}\,h(s)\,ds+\beta_4(t)\,h(t)\,x(t,u_*)\quad\le\\
&C_0(\xi,u_*)\int_{\xi}^{t}\int_{\xi}^{s}\int_{\xi}^{v}h(\theta)\,d\theta\,dv\,ds+\frac{1}{2}C_1(\xi,u_*)(t-\xi)^2+C_2(\xi,u_*)(t-\xi)+C_3(\xi,u_*),
\end{aligned}$$

for $t>\xi\ge t_2$, $u_*\in U_*$, where

$$C_3=\beta_4(\xi)\,h(\xi)\,x(\xi,u_*).$$

From $\dot{\beta}_4 \le 0$ and (5) it follows that there is $t_3$, $t_3 \ge t_2$, such that the integrals on the left hand side of the above inequality are non-negative for all $t > \xi \ge t_3$, so that

$$\beta_4(t)\, h(t)\, x(t, u_*) \;\le\; C_0(\xi, u_*) \int_\xi^t \int_\xi^s \int_\xi^v h(\theta)\, d\theta\, dv\, ds +$$

$$\frac{1}{2} C_1(\xi, u_*)(t-\xi)^2 + C_2(\xi, u_*)(t-\xi) + C_3(\xi, u_*),$$

for $t > \xi \ge t_3$, $u_* \in U_*$. Consequently

$$\overline{\lim_{t\to\infty}} \| u(t) - u_* \|^2 \le 2 \overline{\lim_{t\to\infty}} [\, \beta_4(t)\, h(t) \,]^{-1} \{ C_3(\xi, u_*) + C_2(\xi, u_*)(t-\xi) +$$

$$\frac{1}{2} C_1(\xi, u_*)(t-\xi)^2 + C_0(\xi, u_*) \int_\xi^t \int_\xi^s \int_\xi^v h(\theta)\, d\theta\, dv\, ds \} \;=$$

$$2 \beta_{4\infty}^{-1} \lim_{t\to\infty} h^{-1}(t) \{ C_3(\xi, u_*) + C_2(\xi, u_*)(t-\xi) +$$

$$\frac{1}{2} C_1(\xi, u_*)(t-\xi)^2 + C_0(\xi, u_*) \int_\xi^t \int_\xi^s \int_\xi^v h(\theta)\, d\theta\, dv\, ds \}.$$

It is clear that $\lim_{t\to\infty} h(t) = +\infty$. In order to calculate the limit of the right hand side we will use the L'Hospital rule three times. After that we will get

$$\overline{\lim_{t\to\infty}} \| u(t) - u_* \|^2 \le b_0\, C_0(\xi, u_*), \tag{16}$$

$$\xi \ge t_3, \quad u_* \in U_*.$$

where $b_0 = 2 \beta_{4\infty}^{-4}$. Using (5) and $\dot{\beta}_4 \le 0$, it is not difficult to see that the third term on the left hand side of (15) is non-negative for some $t_4 \ge t_3$ and all $t \ge t_4$. From here and denotations (10) we have

$$\| u'(t) \|^2 + \langle u'(t), u(t) - u_* \rangle \;\le$$

$$2 \{ C_0(\xi, u_*) \int_\xi^t \int_\xi^s h(\theta)\, d\theta\, ds + C_1(\xi, u_*)(t-\xi) + C_2(\xi, u_*) \} [\, \beta_4(t)\, h(t) \,]^{-1},$$

$$t > \xi \ge t_4, \qquad u_* \in U_*.$$

Therefore from (16) and the inequality

$$2 | \langle a, b \rangle | \le \| a \|^2 + \| b \|^2, \qquad a, b \in \mathbf{E}^n, \tag{17}$$

it follows

$$\overline{\lim_{t\to\infty}}\, \|u'(t)\|^2 \ \le \ b_0\, C_0(\xi, u_*) +$$

$$2\,\overline{\lim_{t\to\infty}}\, [\,\beta_4(t)\, h(t)\,]^{-1}\, \{\, C_2(\xi, u_*) + C_1(\xi, u_*)\,(t-\xi) + C_0(\xi, u_*) \int_\xi^t \int_\xi^s h(\theta)\, d\theta\, ds\,\} \ =$$

$$b_0\, C_0(\xi, u_*) +$$

$$\frac{2}{\beta_{4\infty}} \lim_{t\to\infty} \{\, C_0(\xi, u_*) \int_\xi^t \int_\xi^s h(\theta)\, d\theta\, ds + C_1(\xi, u_*)\,(t-\xi) + C_2(\xi, u_*)\,\}\, h^{-1}(t)\,.$$

Using the L'Hospital rule two times, it can be obtained

$$\overline{\lim_{t\to\infty}}\, \|u'(t)\|^2 \le b_1\, C_0(\xi, u_*) \tag{18}$$

$$\xi \ge t_4, \quad u_* \in U_*,$$

where $b_1 = b_0 + 2/\beta^3_{4\infty}$. From (4), (5) it is clear that there is $t_5 \ge t_4$, such that the third and the sixth terms on the left hand side of (14) are non–negative. Taking into account (10) and (17) in a similar way as before, we can get

$$\overline{\lim_{t\to\infty}}\, \beta_4(t)\,[\,\beta_2(t) - \frac{3}{2}\beta_4(t)\,]\,\|u''(t)\|^2 \ \le$$

$$\overline{\lim_{t\to\infty}}\, \{\, \frac{1}{2}[\,1 + |f_1(t)|\,]\,\|u - u_*\|^2 + [\,1 + \frac{|f_1(t)|}{2}\,]\,\|u'(t)\|^2 +$$

$$[\,h(t)\,]^{-1}\,[\,C_0(\xi, u_*) \int_\xi^t h(s)\, ds + C_1(\xi, u_*)\,]\,\},$$

where $f_1(t) = \beta_3(t) - 2\beta_4'(t) - \beta_4^2(t)$, $t \ge 0$. Let $M_1 > 0$, such that $|f_1(t)| \le M_1$, $t \ge 0$. Then

$$\beta_{4\infty}[\,\beta_{2\infty} - \frac{3}{2}\beta_{4\infty}\,]\,\overline{\lim_{t\to\infty}}\, \|u''(t)\|^2 \ \le$$

$$\lim_{t\to\infty} \{\, \frac{1 + M_1}{2}\,\|u - u_*\|^2 + [\,1 + \frac{M_1}{2}\,]\,\|u'(t)\|^2 +$$

$$[\,h(t)\,]^{-1}\,[\,C_0(\xi, u_*) \int_\xi^t h(s)\, ds + C_1(\xi, u_*)\,]\,\}\,.$$

From (5) we have $\beta_{4\infty}[\,\beta_{2\infty} - (3/2)\,\beta_{4\infty}\,] > 0$. Using L'Hospital's rule on the right hand side of the above inequality and the estimates (16), (18) we will get

$$\overline{\lim_{t\to\infty}}\, \|u''(t)\|^2 \le b_2\, C_0(\xi, u_*), \tag{19}$$

$$\xi \ge t_5, \quad u_* \in U_*,$$

where $b_2$ does not depend on $\xi$ and $u_*$. Taking into account (4), (5), (17) and denotations (10), from (13) we can obtain

$$\begin{aligned}
&\varepsilon\int_{\xi}^{t} \{ \sum_{i=1}^{4} \|u^{(i)}(s)\|^2 + f(s)\,\|u(s)-u_\infty\|^2 \}\, ds + \\
&[\beta_4(t)\,\beta_3(t)-\beta_4^2(t)\,\beta_2^2(t)-\beta_4^2(t)]\,\|u'''\|^2 \;\leq \\
&|f_2(t)|\,\|u''\|^2+|f_3(t)|\,\|u'\|^2+|f_4(t)|\,\|u-u_\ast\|^2+C_0(\xi,u_\ast), \\
&t>\xi\geq t_5, \qquad u_\ast\in U_\ast,
\end{aligned}$$

where the functions $f_i$, $i=2,3,4$; are bounded and do not depend on $\xi$ and $u_\ast$. Let $M_2>0$, such that $|f_i(t)|\leq M_2$, $i=2,3,4$; $t\geq 0$. Since

$$\lim_{t\to\infty}[\beta_3(t)-\beta_4(t)\,\beta_2^2(t)-\beta_4(t)]=\beta_{3\infty}-\beta_{4\infty}\,\beta_{2\infty}^2-\beta_{4\infty}>0,$$

then there exist a moment $t_7\geq t_6$ and a number $\delta$, $0<\delta<\varepsilon$, such that for all $t\geq t_7$: $\beta_4(t)\,[\beta_3(t)-\beta_4(t)\,\beta_2^2(t)-\beta_4(t)]>\delta$,. Hence

$$\begin{aligned}
&\delta\{\int_{\xi}^{t}[\sum_{i=1}^{4}\|u^{(i)}(s)\|^2+f(s)\,\|u(s)-u_\infty\|^2]\,ds+\|u'''\|^2\} \;\leq \\
&M_2\,[\,\|u''\|^2+\|u'\|^2+\|u-u_\ast\|^2\,]+C_0(\xi,u_\ast),
\end{aligned}$$

for all $t>\xi\geq t_7$, $u_\ast\in U_\ast$. Consequently

$$\delta\{\int_{\xi}^{t}[\sum_{i=1}^{4}\|u^{(i)}(s)\|^2+f(s)\,\|u(s)-u_\infty\|^2]\,ds \;\leq\; b_3\,C_0(\xi,u_\ast), \tag{20}$$

$$\overline{\lim_{t\to\infty}}\,\|u'''(t)\|^2 \;\leq\; b_3\,C_0(\xi,u_\ast), \tag{21}$$

$$\xi\geq t_7, \quad u_\ast\in U_\ast.$$

where $b_3=\delta^{-1}[\,M_2\,(b_2+b_1+b_0)+1\,]$. From (20) it follows that there is a sequence $\{s_j\}\subseteq[0,+\infty)$ such that

$$\underline{\lim_{s\to\infty}}\,[\sum_{i=1}^{4}\|u^{(i)}(s)\|^2\,] \;=\; \lim_{j\to\infty}[\sum_{i=1}^{4}\|u^{(i)}(s_j)\|^2\,] \;=\; 0,$$

i.e.

$$\lim_{j\to\infty}\|u^{(i)}(s_j)\|=0, \qquad i=1,2,3,4. \tag{22}$$

Then from (16) and the conditions 2) of the Theorem, it is obvious that there exist a subsequence $\{\hat{s}_j\}$, a point $u_\infty\in\mathbb{E}^n$ and a real number $\alpha_\infty>0$, such that

$$\lim_{j\to\infty}\|u(\hat{s}_j)-u_\infty\|=0, \qquad\qquad \lim_{j\to\infty}\alpha(\hat{s}_j)=\alpha_\infty. \tag{23}$$

Setting $t=\hat{s}_j$ in the differential equation (3), from (22) and (23), we can get

$$\begin{aligned}
&\lim_{j\to\infty}\|\beta_4(\hat{s}_j)\,u^{\mathrm{iv}}(\hat{s}_j)+\beta_3(\hat{s}_j)\,u'''(\hat{s}_j)+\beta_2(\hat{s}_j)\,u''(\hat{s}_j)+u'(\hat{s}_j)+u(\hat{s}_j)- \\
&P_U\,[\,u(\hat{s}_j)-\alpha(\hat{s}_j)\,J'(u(\hat{s}_j))\,]\,\| \;= \\
&\|u_\infty-P_U\,[\,u_\infty-\alpha_\infty\,J'(u_\infty)\,]\,\| \;=\; 0.
\end{aligned}$$

Consequently (see[4], pp.171) $u_\infty \in U_*$. From (12) where $\xi = \hat{s}_j$, $u_* = u_\infty \in U_*$, along with (22), (23) we have

$$\lim_{j\to\infty} C_0(\hat{s}_j, u_\infty) = 0.$$

Let $j_0 \ge 1$ be such a number that $\hat{s}_j \ge t_7$, for every $j \ge j_0$. Then the relations (16), (18)-(21) hold for $\xi = \hat{s}_j$, $j \ge j_0$ and $u_* = u_\infty$. Therefore

$$\overline{\lim_{t\to\infty}} \| u(t) - u_\infty \|^2 \le b_0 \lim_{j\to\infty} C_0(\hat{s}_j, u_\infty) = 0.$$

$$\overline{\lim_{t\to\infty}} \| u^{(i)}(t) \|^2 \le b_i \lim_{j\to\infty} C_0(\hat{s}_j, u_\infty) = 0, \qquad i = 1, 2, 3.$$

Hence

$$\lim_{t\to\infty} \| u(t) - u_\infty \| = 0, \qquad \lim_{t\to\infty} \| u^{(i)}(t) \| = 0, \qquad i = 1, 2, 3. \tag{24}$$

From the differential equation (3) and the relations, when $t \to \infty$, it can be obtained

$$\lim_{t\to\infty} \| u^{\mathrm{iv}}(t) \| = 0. \tag{25}$$

The inequality (20) and the relations (24), (25) give the statement of the theorem.

## 3. CONCLUSION

The projection-gradient methods of the higher order are important because of their higher rate of convergence in comparison with the methods of that type of the first order [1]. Besides that, the continuous methods give us a large choice of the numerical integration methods for solving the corresponding differential equations. These facts justify the investigation of the continuous methods of the higher order.

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