Yugoslav Journal of Operations Research
5 (1995), Number 1, 111-121

ECONOMICALLY OPTIMUM DESIGN OF CUSUM CHARTS TO CONTROL NORMAL MEANS

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Abstract: In 1974, Chiu employed a two-dimensional Fibonacci search to get optimum design of Cusum charts. Subsequently, in 1982, Lashkari and Rahim presented an algorithm to find the design parameters of Cusum charts as well. Their algorithms are rather complicated. The main purpose of this paper is to propose a simplified, accurate and efficient solution algorithm for the economic design of Cusum charts.

Keywords: Control charts, Cusum charts, economic design, optimization, process control.

1. INTRODUCTION

The basic rule in using Shewhart charts is to take action when a point falls outside the control limits. Other rules can be applied to Shewhart charts, such as tests for runs and the use of warning limits, which attempt to incorporate information from the entire set of points into the decision procedure. The rule has also been suggested of taking two points in succession outside the warning limits as a signal for action. The use of a run of 7 above the central line on the chart as an action signal has been noted. From these developments in the use of the Shewhart chart, it was a natural step to adopt a rule for action that was based on all the data and not only the last few samples. This was done by the use of a cumulative sum chart. Such charts were proposed by Page [25].

Cusum charts differ from the Shewhart charts regarding the control limits or decision criteria. A common practice is to use a V-mask consisting of a vertex, a location point at a lead distance from the vertex, and two sloping lines serving as the decision lines. Another practice is to tabulate a cumulative statistic in relation to a reference value, with designated restarts and with action based on violation of a decision interval. Its advantage over the ordinary Shewhart chart is that it may be

Communicated by S.Dajović

equally effective at less expense. This stems from the possibility of the Cusum chart picking up a sudden and persistent change in the process average more rapidly than a comparable Shewhart chart, especially if the change is not large. It may also possibly locate the time of change more sharply. In general, Shewhart charts are especially effective for detecting changes in the mean level of the universe of more than 2σ, they are not too effective in detecting changes of the magnitude of 0.5 to 1.5σ. In this respect, Cusum charts can be created that significantly improve the chances of such detection.

The operation of a Cusum chart for controlling the mean of a process consists of taking samples of size n at regular intervals of g hours and plotting the cumulative sums $S_r = \sum_{j=1}^{r}(x_j - k)$ versus a sample number r, where x_j is the sample mean and k is the reference value. If a plotted point rises a distance h or more above the lowest previous point, it is assumed that an upward shift in the process mean has occurred. Thus the sample size n, sampling interval g, reference value k and decision limit h are the parameters required for operating a one-sided Cusum chart. If both positive and negative deviations are to be controlled, two one-sided charts with reference values k_1, k_2 ($k_1 > k_2$) and respective decision limits h and $-h$ or a V-mask with lead distance d and half angle ϕ can be employed.

Taylor [30] first investigates the economic design of Cusum charts for a normal mean. The model allows for the process shut-down during the search for the assignable cause. However, the model assumes that n and g are specified and that the effect of the assignable cause is a function of the sample size. The cost of sampling is also omitted. Chiu [2] gives four criticisms of it and presents a study of the economic design of Cusum charts for controlling normal process means, which satisfactorily overcomes the four drawbacks. Lashkari and Rahim [18] extend Chiu's study to the economic design of Cusum charts to control non-normal process means. If the cumulative sum exceeds the decision interval h, Chiu [2] and Lashkari and Rahim [18] assume that the process is stopped and a search for the assignable cause is undertaken. However, Goel and Wu [10] assume that the process is not shut down while a search is being made for the assignable cause and present a procedure for the economic design of Cusum charts to control the mean of a process with a normally quality characteristic.

The economic design of statistical process control procedures has attracted the attention of the academic community for 30 years now, since the pioneering work by Duncan [3]. However, despite the significant progress in the development of analytical models for the optimal economic determination of control chart parameters under different assumptions, those models have not gained analogous popularity in industry. Montgomery [21] and Lorenzen and Vance [19] give the two main reasons for this surprising fact. First, the mathematical models used and the associated optimization schemes are relatively complex and presented in a manner that is difficult for a practitioner to understand and use. Second, there are practical difficulties in estimating cost and other parameters of the models heretofore proposed. The main purpose of this paper is to simplify the solution techniques of Chiu [2] and Lashkari and Rahim [18] for controlling normal means and lay the basis for the development of simple standards for economic process control, which would facilitate its application. The results and the execution times of all numerical examples show that our search algorithm is accurate and efficient.

2. PRODUCTION MODEL

The process is assumed to start in a state of in-control at an acceptable quality level (AQL), having a normal distribution $N(u_0, \sigma^2)$, where u_0 and σ^2 are known. There is a single assignable cause of variation which has the effect of shifting the process mean to $u_1 = u_0 + \delta\sigma$, where δ is known and positive. (The treatment of a negative δ is similar and we shall not consider it here.) The time until the assignable cause occurs has an exponential distribution with a mean of $1/\lambda$ hours. Samples of size n are drawn every g hours of production and the sample information is plotted on a Cusum chart, using a one-sided decision interval scheme, as explained in Ewan and Kemp [4]. Let the reference value be k and the decision interval be h. If the decision interval is exceeded, the production is stopped and a search for the assignable cause is made. In order to formulate the loss-cost function, the following notation will be used:

n = sample size
h = decision interval
H = standardized decision interval
g = sampling interval
k = reference value
τ_s = expected search time for false alarm
K_s = expected search cost for false alarm
τ_r = expected adjustment time for true alarm
K_r = expected adjustment cost for true alarm
P_0 = profit per hour when the process is in control
P_1 = profit per hour when the process is out of control
b = fixed cost of sampling
c = variable cost of sampling
F = average loss-cost per hour of the process.

Based on the model described in this section and the parameters just defined, Chiu [2] has shown that the expected loss-cost per hour of operation F is

$$F(n,k,H,g) = \frac{\lambda UB_1 + VB_0 + \lambda W + (b + cn)(1 + \lambda B_1)/g}{1 + \lambda B_1 + \tau_s B_0 + \lambda(\tau_r + \tau_s)} \tag{1}$$

where

$$B_0 = \frac{\lambda}{L_0(e^{\lambda g} - 1)} \tag{2}$$

$$B_1 = L_1 g - 1/\lambda + \frac{g}{e^{\lambda g} - 1} \tag{3}$$

$U = P_0 - P_1$

$V = K_s + P_0 \tau_s$

$W = K_r + K_s + P_0 (\tau_r + \tau_s)$

L_0 = the ARL of the Cusum chart plots for the acceptable quality level (AQL), u_0

L_1 = the ARL of the Cusum chart plots for the rejectable quality level (RQL), $u_1 = u_0 + \delta\sigma$.

Because L_0 and L_1 are functions of n, k and H, it should be noted that F is a function of four control scheme variables g, n, k and H where $H = h\sqrt{n}/\sigma$. The problem of optimization is the search for the optimum values of g, n, k and H for which F is a minimum. In this paper, we shall be mainly concerned with three optimization methods: the exact method by a two-dimensional Fibonacci search described in [2], the search algorithm described in [18] and our search algorithm.

3. DETERMINATION OF THE CONTROL PARAMETERS

3.1. DETERMINATION OF THE REFERENCE VALUE k

There is strong numerical and theoretical evidence in [4] that for given L_1, the value of L_0 approaches its maximum when k, the reference value, is chosen midway between the AQL and RQL:

$$k = u_0 + \frac{1}{2}\delta\sigma \tag{4}$$

and the curve of L_0 as a function of k is rather flat in the vicinity of this middle value. Since a larger L_0 means a smaller cost spent on false alarm and searches, we intuitively expect that equation (4) would be an approximately optimal choice of k.

3.2. DETERMINATION OF THE ARL

The ARL of a one-sided Cusum chart with horizontal boundaries at $(0, H)$ is defined as:

$$ARL = \frac{N(0)}{1 - P(0)} \tag{5}$$

where $P(0)$ and $N(0)$ are special cases at $z = 0$ of $P(z)$ and $N(z)$, which are defined as follows:

$$P(z) = \int_{-\infty}^{-z} f(x)\,dx + \int_0^H P(x) f(x-z)\,dx \qquad 0 \le z \le H \tag{6}$$

$$N(z)=1+\int_0^H N(x)f(x-z)dx \qquad 0\le z\le H \tag{7}$$

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\theta)^2}{2}} \tag{8}$$

where $f(x)$ is the probability density function of the standardized increments $(x_j-k)/(\sigma/\sqrt{n})$ in the cumulative sum for the normal process with $\theta=(u_0-k)/(\sigma/\sqrt{n})$ and H is the standardized decision interval defined as $H=h/(\sigma/\sqrt{n})$.

If we calculate $f(x)$ from equation (8) by substituting $\theta=-\delta\sqrt{n}/2$ and obtain the values of $P(0)$ and $N(0)$ from equations (6) and (7) for this value of $f(x)$, then L is given by equation (5). Similarly, L is obtained from these equations by substituting $\theta=\delta\sqrt{n}/2$ in equation (8) to get the value of $f(x)$. A detailed discussion of the solution of the Fredholm integral equations in (6) and (7) is given in [9], where it is indicated that the value of the *ARL* can be determined by a system of linear algebraic equations.

3.3. DETERMINATION OF g

It can be shown that Lemmas 1 and 2 hold.

LEMMA 1. $1>\frac{1}{\lambda g}-\frac{1}{e^{\lambda g}-1}>0$ for all $\lambda>0$ and $g>0$.

LEMMA 2. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\frac{1}{2}$

Lemma 1 tells us that the difference between $1/\lambda g$ and $1/(e^{\lambda g}-1)$ is within 0 and 1. McWilliams [20] thinks that examination of the numerical model literature shows that with very few exceptions, cost / risk parameters leading to small λg (0.4 or less) are of most practical interest. So, by Lemmas 1 and 2, we take 0.5 as a correction number. Replacing $1/(e^{\lambda g}-1)$ in equation (1) by $1/\lambda g - 1/2$, we shall get equation (9).

$$F_1=\frac{\lambda U\overline{B}_1+V\overline{B}_0+\lambda W+(b+cn)(1+\lambda\overline{B}_1)/g}{1+\lambda\overline{B}_1+\tau_s\overline{B}_0+\lambda(\tau_r+\tau_s)} \tag{9}$$

where

$\overline{B}_1=(L_1-\frac{1}{2})g$ and

$\overline{B}_0=\frac{\lambda}{L_0}(\frac{1}{\lambda g}-\frac{1}{2})$.

For a given (n, k, H), setting the partial derivatives of F_1 with respect to g equal to zero yields $g(\beta_1 g^2 + \beta_2 g + \beta_3) = 0$,

where

$$\beta_1 = (P_0 - P_1)(L_1 - \frac{1}{2})(\frac{1}{\lambda} + \tau_r + \tau_s - \frac{\tau_s}{2L_0}) - (L_1 - \frac{1}{2})$$
$$[K_r + K_s + P_0(\tau_r + \tau_s) + (b + cn)(L_1 - \frac{1}{2}) - \frac{K_s + P_0\tau_s}{2L_0}],$$

$$\beta_2 = \frac{2(L_1 - \frac{1}{2})}{\lambda}\{(P_0 - P_1)\frac{\tau_s}{L_0} - [\frac{K_s + P_0\tau_s}{L_0} + (b + cn)]\},$$

$$\beta_3 = -\frac{1}{\lambda}[\frac{K_s + P_0\tau_s}{L_0} + (b + cn)](\frac{1}{\lambda} + \tau_r + \tau_s - \frac{\tau_s}{2L_0}) +$$
$$\frac{\tau_s}{L_0\lambda}[K_r + K_s + P_0(\tau_r + \tau_s) + (b + cn)(L_1 - \frac{1}{2}) - \frac{K_s + P_0\tau_s}{2L_0}],$$

Since $g>0$, $\beta_1 g^2 + \beta_2 g + \beta_3 = 0$.

So $$g = \frac{-\beta_2 + [\beta_2^2 - 4\beta_1\beta_3]^{1/2}}{2\beta_1} = g(n, k, H) \qquad (10)$$

(In general, $\beta_1 > 0$ and $\beta_3 < 0$.)

4. THE SEARCH ALGORITHM

With equation (10), a search algorithm is developed based on the one-dimensional H pattern search technique of Hooke and Jeeves [14]. A computer program was written, coded in FORTRAN and employed on the loss-cost function to determine the optimal design parameters n, k, H that minimize equation (1).

Let $$F^*(n) = \underset{0<H}{\text{minimize}}\ \{F(n, k, H, g(n, k, H))\}$$

where $k = u_0 + \delta\sigma/2$. Then, all plots of $F^*(n)$ of all numerical examples considered in this paper are unimodal. Figure 1 which uses the data of representative Example 1 of Table 1 demonstrates this fact. The above observations form the basis of the search algorithm provided to solve the problem. It should be pointed out that the above observations are based on the numerical study and should not be interpreted to be true in general. However, because of the fairly large range of values studied, it seems appropriate to assume that they are representative of the situations that are expected to occur in practical applications.

The search algorithm is accomplished by the pattern search technique of Hooke and Jeeves [14]. In the first step, for a given sample size $n \geq 1$, we determine the optimal value of H and the resulting optimal value of $F^*(n)$. The optimal sampling interval g is given by equation (10). A one-dimensional pattern search H is used to find the optimal value of H. The search starts from a point $(n, k, H, g) = (n, u_0 + \delta\sigma/2, 1.10, g(n, u_0 + \delta\sigma/2, 1.10))$. Chiu [2] indicates that the economic design

Table 1. Comparisons among three optimization methods

Case NO	Given Parameters									
	δ	λ	P_0	P_1	K_r	K_s	τ_r	τ_s	b	c
1	2.0	0.01	150	50	20	10	0.2	0.1	0.5	0.1
2	2.0	0.05	150	50	20	10	0.2	0.1	0.5	0.1
3	2.0	0.01	150	140	20	10	0.2	0.1	0.5	0.1
4	2.0	0.01	2000	1000	20	10	0.2	0.1	0.5	0.1
5	2.0	0.01	150	50	50	25	0.2	0.1	0.5	0.1
6	2.0	0.01	150	50	20	10	1.0	0.5	0.5	0.1
7	2.0	0.01	150	50	200	10	2.0	0.1	0.5	0.1
8	2.0	0.01	150	50	2	1	0.2	0.1	0.5	0.1
9	2.0	0.01	150	50	20	10	0.2	0.1	5.0	0.1
10	2.0	0.01	150	50	20	10	0.2	0.1	0.5	1.0
11	1.0	0.01	150	50	20	10	0.2	0.1	0.5	0.1
12	1.0	0.01	150	130	20	10	0.2	0.1	0.5	0.1
13	1.0	0.05	150	50	20	10	0.2	0.1	0.5	0.1
14	0.5	0.01	150	50	20	10	0.2	0.1	0.5	0.1
15	0.5	0.01	150	145	20	10	0.2	0.1	0.5	0.1

Case NO	Chiu's Solution						Lashkari & Rahim's Solution						
	n	g	H	L_0	L_1	Cost1	n	g	H	L_0	L_1	Cost2	Execution Time: T1 (CPU sec.)*
1	5	1.41	0.58	382.40	1.05	2.281	5	1.640	0.550	370.88	1.05	2.2809	23.21
2	5	0.85	0.55	370.00	1.05	8.973	5	0.831	0.550	370.88	1.05	8.9739	23.06
3	5	4.88	0.55	370.90	1.05	1.208	4	4.247	0.850	242.73	1.09	1.2102	21.62
4	8	0.48	0.93	2708.00	1.07	11.410	7	0.488	0.850	4193.02	1.04	11.4245	26.72
5	5	1.40	0.88	521.40	1.08	2.729	8	1.473	0.800	865.87	1.03	2.7404	25.88
6	8	1.49	0.74	1387.00	1.04	4.054	5	1.378	0.850	982.84	1.09	4.0594	24.48
7	5	1.44	0.55	370.90	1.05	8.594	5	1.430	0.550	370.88	1.05	8.5945	23.42
8	5	1.41	0.45	274.80	1.04	1.975	4	1.311	0.550	182.53	1.08	1.9749	18.61
9	8	3.40	0.15	214.10	1.01	4.121	8	3.330	0.150	214.09	1.01	4.1217	13.94
10	2	2.17	0.49	34.00	1.21	3.721	2	2.131	0.500	34.77	1.21	3.7213	11.18
11	13	1.87	0.51	94.50	1.11	2.965	12	1.780	0.550	88.60	1.13	2.9564	39.17
12	13	4.29	0.51	94.50	1.11	1.713	12	4.028	0.550	88.60	1.13	1.7139	39.49
13	13	0.88	0.49	89.80	1.10	8.460	12	0.801	0.550	88.60	1.13	8.4620	38.27
14	30	2.82	0.37	23.90	1.18	4.208	30	2.597	0.350	22.97	1.18	4.2088	68.35
15	28	13.20	0.38	21.20	1.20	1.435	28	12.184	0.350	20.81	1.19	1.4378	84.93

Case NO	Our Solution							Ratios (**)		
	n	g	H	L_0	L_1	Cost3	Execution Time: T2 (CPU sec.)*	CR1	CR2	TR
1	5	1.428	0.500	318.79	1.04	2.2801	7.91	1.00040	1.00035	2.93426
2	5	0.853	0.500	318.79	1.04	8.9668	7.72	1.00092	1.00105	2.98705
3	5	4.753	0.500	318.79	1.04	1.2050	7.88	1.00083	1.00432	2.74385
4	8	0.459	0.931	2719.35	1.07	11.4044	9.25	1.00049	1.00178	2.88865
5	5	1.409	0.850	505.27	1.06	2.7277	6.75	1.00048	1.00468	3.87928
6	8	1.494	0.725	1317.78	1.04	4.0520	9.31	1.00049	1.00183	2.62943
7	5	1.457	0.500	318.79	1.04	8.5938	7.90	1.00008	1.00014	2.96458
8	5	1.422	0.425	255.07	1.04	1.9731	8.07	1.00098	1.00091	2.30607
9	7	3.449	0.125	357.52	1.01	4.1118	12.69	1.00224	1.00241	1.01826
10	2	2.184	0.481	33.39	1.20	3.7173	3.75	1.00100	1.00108	2.97800
11	13	1.879	0.500	92.09	1.10	2.9525	16.97	1.00085	1.00131	2.30819
12	13	4.329	0.500	92.09	1.10	1.7106	16.13	1.00140	1.00193	2.44823
13	13	0.884	0.500	92.09	1.10	8.4463	17.01	1.00182	1.00186	2.24985
14	30	2.852	0.350	22.97	1.18	4.2028	37.42	1.00128	1.00143	1.82656
15	29	11.860	0.312	29.29	1.17	1.4299	36.10	1.00357	1.00538	2.35263
The average values of ratios =								1.00111	1.00203	2.57418

*) All calculations were executed on a personal computer ENSONTECH (PC-386).

**) CR1 = Cost1 / Cost2 ; CR2 = Cost2 / Cost3 ; TR = T1 / T2 .

often requires evaluations of F for H in the range (0.2, 2.0). No published work provides adequate tables or charts for this purpose. So, we take H to be midway (=1.10) between 0.2 and 2.0 . Furthermore, using the method of approximately integral equations by systems of linear equations based on 3 Gaussian points, introduced by Goel and Wu [9], we calculate L_0 and L_1.

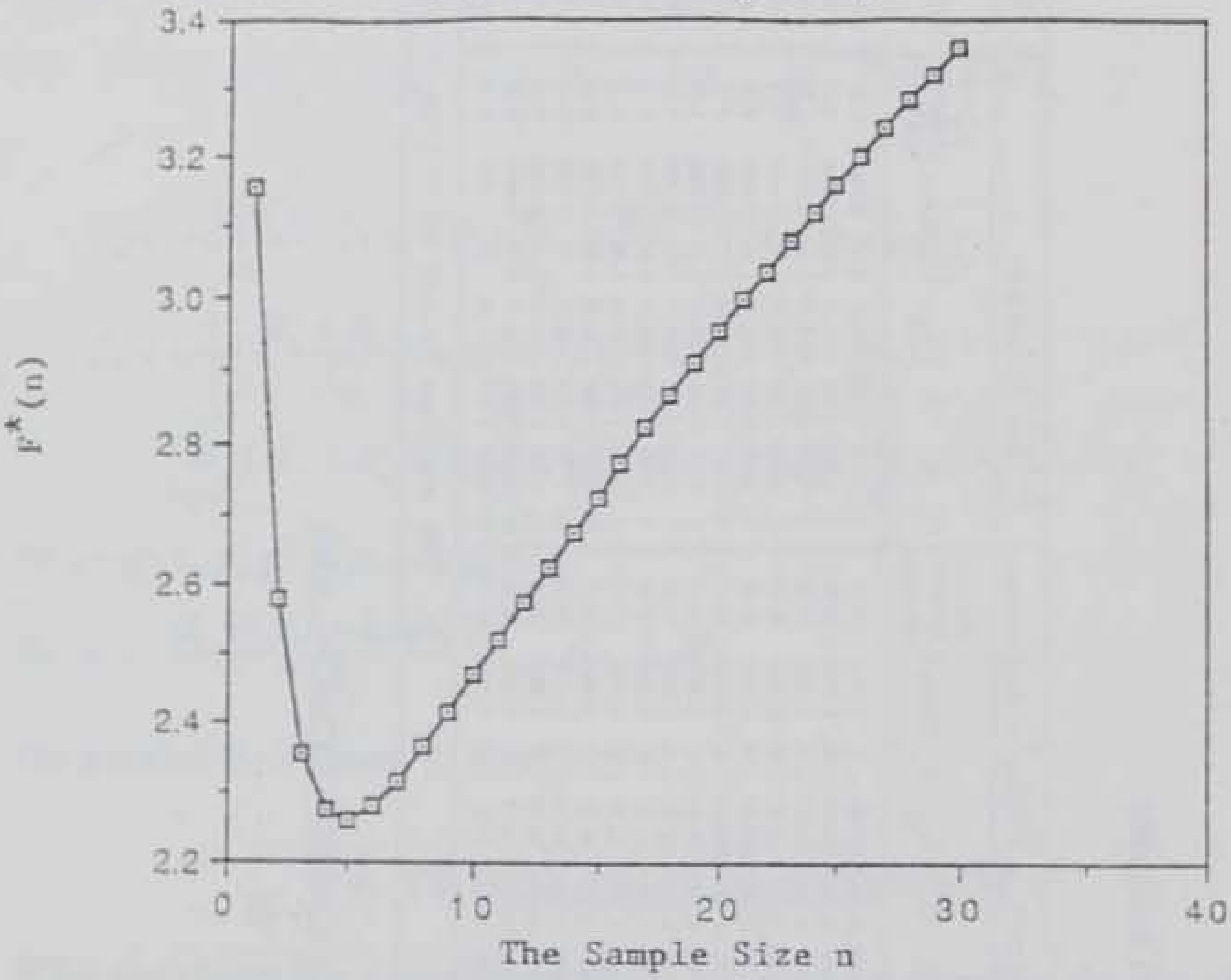

Figure 1. The plot of F^* for Example 1

The next step then aims at finding the n that minimizes $F^*(n)$. The value of n has no upper bound. The stopping rule is based on the unimodality on $F^*(n)$ and is described as follows.

The stopping rule on n: Let n^* be the smallest positive integer such that the following holds

$$F^*(n^* - 1) > F^*(n^*) < F^*(n^* + 1).$$

Then the search algorithm will terminate at $n = n^* + 1$.

Let $F^{**} = F^*(n^*)$. Thus F^{**} is the overall minimum value of the loss-cost function, and the value of L_0, L_1, g and H corresponding to F^{**} are the near-optimal values of the design parameters. The decision interval h is obtained from $h = H\sigma / \sqrt{n}$, and the reference value k from $k = u_0 + \delta\sigma/2$.

5. THREE OPTIMIZATION METHODS

Chiu's and Lashkari and Rahim's solution algorithms are described in [2] and in [18], respectively. Comparing their algorithms with ours, we find that theirs is fairly complicated and involved. For example, their algorithms require that the optimum value of g is given by equation (16) in [2] or equation (29) in [18], which is a quartic equation in g. However, in this paper, the optimum value of g is given by equation (10), which is a linear equation in g. On the other hand, to get equation (18) in [2] or equation (31) in [18], Chiu, Lashkari and Rahim omitted many terms in equation (16) in [2] or in equation (29) in [18]. However, equation (10) in this paper is free from those assumptions and approximations made by Chiu, Lashkari and Rahim. No terms are neglected for finding the optimum solution. So, our search algorithm simplifies the solution techniques of Chiu [2] and Lashkari and Rahim [18].

Optimal designs for 15 numerical examples from Table 3 in [2], were obtained by using Lashkari and Rahim's and our algorithms, respectively. Table 1 compares them with Chiu's solutions [2] and Lashkari and Rahim's solutions [18]. We find that in all cases considered our search algorithm yields lower loss-costs than those of Chiu, Lashkari and Rahim. The gains on costs from using our search algorithm are only marginal. However, Chiu [2] reports that each determination of the optimum plan needs 4-9 minutes (240-540 seconds) on an ICL 4-50 computer. So, it is anticipated that our search algorithm is more efficient than that of Chiu. Moreover, Table 1 tells us that our search algorithm is still more efficient than that of Lashkari and Rahim.

Chiu [2] presents a simplified economic scheme for the determination of the design parameters of a cumulative sum chart for controlling normal process means, which can be easily handled by a quality control practitioner at the workshop level. Although Chiu's scheme is reasonably good, it still has at least the following drawbacks. However, our search procedure is free from those drawbacks. Those drawbacks are described as follows.

(1) If the workshop supervisors decide to use the manual computation, they will encounter a lot of computations to complete all steps of Example 3 in [2]. In general, it takes about 20 minutes (=1200 seconds) to complete all steps of the same example. Mathematical computations sometimes are a barrier to the application of Chiu's scheme because some workshop supervisors will psychologically not like mathematical computations although mathematical computations are rather simple.

(2) The simplified procedure of Chiu is a semi-economic approach under the assumption that the probability of a true alarm and its selection are to be at least at a given level (typical values are 0.90 or 0.95). This assumption is intuitively reasonable because it enables the manufacturer to detect an assignable cause rather quickly on an average of about 1.1 or 1.05 samples after its occurrence. However, in cases where the true alarm is significantly lower than 0.90 the solution derived using the semi-economic scheme of Chiu may be very far from the optimum resulting in significant cost penalties. For the details, see Tagaras [2].

Lorenzen and Vance [19] and Montgomery [21] indicate that the main difficulties in the use of economic designs are the computations involved, the difficulty in specifying process parameters, and the fact that the sampling interval g is rarely a natural quantity of time. The second difficulty, fortunately, costs do not have to be estimated with high precision, although other model components, such as the magnitude of the shift, require relatively accurate determination. Sensitivity analysis of the specific model could help the practitioner decide which parameters are critical in the problem. The third difficulty, equation (10) can be used for this purpose. Finally, as the second difficulty, we propose a simplified computer program to solve it. It is the future trend that the personal computer is used at the workshop level. Lorenzen and Vance [19] point out the need for optimization techniques implementable in real time on a personal computer. When the computer program is set up already, using it only requires some basic knowledges of computer programming. It can be easily handled by a quality practitioner at the workshop level. All calculations in this paper were executed on a personal computer ENSONTECH (PC-386). Furthermore, the reported execution times are low. So, our computer program should be available to the prospective users. To sum up, our computer algorithm is simplified, accurate and efficient for the economic design of Cusum charts for controlling normal process means.

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