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NUMERICAL INTEGRATION APPROXIMATIONS TO ESTIMATE THE WEITZMAN OVERLAPPING MEASURE: WEIBULL DISTRIBUTIONS

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Abstract: This paper deals with the problem of estimating the overlapping (OVL) Weitzman measure (Δ) when two independent random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ are given by two-parameter Weibull distribution. The measure Δ has been studied in the literature in the case of two Weibull distributions under the assumption that the two shape parameters are equal. In this work, a general expression for the Weitzmans measure is provided under the Weibull distribution without using any assumptions about the distribution parameters. Some new estimators for Δ are developed depending on three numerical integration rules known as trapezoidal, Simpson 1/3 and Simpson 3/8 rules. The performance of the proposed estimators were investigated and compared with some existing estimators via simulation technique and real data. The results demonstrated the superiority of the proposed estimators over the existing one in almost all considered cases.

Keywords: Overlapping measure, maximum likelihood method, numerical integration methods, Weibull distribution, relative bias, relative mean square error.

MSC: 62E17.

1. INTRODUCTION

The well-known overlapping (OVL) Weitzman coefficient (Δ) is defined as the intersection area between two probability density functions. Some authors defined it as a measure of similarity, agreement or closeness between two probability distributions. Let $f_X(x)$ and $f_Y(x)$ be two continuous probability density functions for the two random variables X and Y respectively, then the OVL measure Δ is defined by [1],

$$\Delta = \int min\{f_X(x), f_Y(x)\}dx$$

The above formula represents the common area under the two probability density functions $f_X(x)$ and $f_Y(x)$. The values of Δ ranged between 0 and 1. A value of zero for Δ indicates that the intersection of the support of the two variables X and Y is the empty set, while a value of 1 for Δ indicates a perfect agreement between two densities, which is equivalent to say, $f_X(x) = f_Y(x), \forall\, x$.

Other OVL measures have been studied in the literature, such as the Matusita and Morsita OVL measures [2]. In addition, the two OVL measures named, Pianka and Kullback-Leibler have also been studied by some authors [3] and [4]. OVL measures have many applications in various fields including ecology [5] and [6], reliability analysis [7] and genetics [8]. Sneath [9] used the OVL coefficient as a measure of disjunction and some authors used the OVL measure on income differences (see [10], [11] and [12]. Samawi *et al.* [13] suggested a new nonparametric test of symmetry based on OVL measure Δ and recently, Alodat *et al.* [14] showed the importance of OVL measure in goodness-of-fit test. Most studies have concerned the point and interval estimations of Δ under some specific pair distributions. Inman and Bradly [12] estimated Δ for two normal distributions with equal variances. The case of two normal distributions with equal means and different variances is considered by Mulekar and Mishra [10]. Reiser and Faraggi [15] constructed and investigated the confidence intervals for Δ in the case of two normal distributions with common variance. Mulekar and Mishra [16] suggested the use of Jackknife and Bootstrap methods to construct confidence intervals of Δ in the case of two normal distributions with common mean. Chaubey *et al.* [3] studied the point estimator of Δ when the two populations are assumed to be described by the inverse Gaussian distributions with equal means.

The case of two exponential distributions was studied by Samawi and Al-Saleh (2008), who also studied the effect of sampling scheme on Δ. Helu and Samawi [17] investigated the OVL measures for two Lomax distributions with different sampling procedures. Parallel to the work of Helu and Samawi [2], Dhaker *et al.* [18] considered the case of two inverse Lomax distributions to study the OVL measures. Finally, Wang and Tiana [19] also proposed methods for confidence interval estimation of Δ under a variety of distributions, including normal, gamma and mixture Gaussian.

Let X be a continuous random variable follow a Weibull distribution with a scale parameter α_1 and a shape parameter β_1, then the pdf of X is,

$$f_X(x; \alpha_1, \beta_1) = \frac{\beta_1}{\alpha_1}\left(\frac{x}{\alpha_1}\right)^{\beta_1 - 1} e^{-(x/\alpha_1)^{\beta_1}}, \qquad x > 0, \qquad \alpha_1, \beta_1 > 0.$$

We will denote it by, $X \sim We(\alpha_1,\ \beta_1)$. Thus, if $Y \sim We(\alpha_2,\ \beta_2)$ then the pdf of Y is,

$$f_Y(y;\alpha_2,\beta_2)=\frac{\beta_2}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1}e^{-(y/\alpha_2)^{\beta_2}},\qquad y>0,\qquad \alpha_2,\beta_2>0.$$

Under the assumption that X and Y are independent, Al-Saidy *et al*. [20] considered the case of two Weibull distributions with the same shape parameters and different scale parameters. Let $K=\alpha_1/\alpha_2$, $\beta_1=\beta_2(=\beta$, say) and $Q=(2\beta-1)/\beta$ they derived the formula of Δ, which is given by,

$$\Delta=\begin{cases}1-\left(K^{\beta}\right)^{\frac{1}{1-K^{\beta}}}\left|1-\frac{1}{K^{\beta}}\right| & ,\ K\neq 1\\ 1 & ,\ K=1\end{cases}$$

Most of the previous studied were accomplished by using some restrictions on the distributions parameters. For example, without using the assumption $\beta_1=\beta_2=\beta$ (Al-Saidy *et al*., 2005), the above formula of Δ is no longer true. The main goal of this work is to introduce a general expression for Δ under Weibull distributions without using any assumptions about the shape or scale parameters. The new expression is provided based on numerical integration methods; in particular the trapezoidal and Simpson rules that facilitate making inference on Δ. Accordingly, for each numerical integration rule, a new maximum likelihood estimator of Δ is obtained. The finite sample properties of the proposed estimators of Δ are studied and investigated via Monte-Carlo simulation technique and real data sets.

2. MAXIMUM LIKELIHOOD ESTIMATORES

Consider the two independent random samples, $X_1,X_2,\dots,X_{n_1}$ of size n_1 from $We(\alpha_1,\beta_1)$ and $Y_1,Y_2,\dots,Y_{n_2}$ is another random sample of size n_2 from $We(\alpha_2,\ \beta_2)$. The log-likelihood function is,

$$lnL(\alpha_1,\beta_1,\alpha_2,\beta_2)=n_1ln\beta_1+n_2ln\beta_2-(n_1\beta_1ln\alpha_1+n_2\beta_2ln\alpha_2)+$$
$$+(\beta_1-1)\sum_{i=1}^{n_1}lnx_i+(\beta_2-1)\sum_{i=1}^{n_2}lny_i-\frac{1}{{\alpha_1}^{\beta_1}}\sum_{i=1}^{n_1}x_i^{\beta_1}-\frac{1}{{\alpha_2}^{\beta_2}}\sum_{i=1}^{n_2}y_i^{\beta_2}$$

The ML estimators of $\alpha_1,\beta_1,\alpha_2$ and β_2 are obtained by solving the following equations simultaneously,

$$\frac{1}{\beta_1}+\frac{1}{n_1}\sum_{i=1}^{n_1}ln\,x_i-\frac{\sum_{i=1}^{n_1}\left[x_i^{\beta_1}\,ln\,x_i\right]}{\sum_{i=1}^{n_1}x_i^{\beta_1}}=0$$
$$\frac{1}{\beta_2}+\frac{1}{n_2}\sum_{i=1}^{n_2}ln\,y_i-\frac{\sum_{i=1}^{n_2}\left[y_i^{\beta_2}\,ln\,y_i\right]}{\sum_{i=1}^{n_2}y_i^{\beta_2}}=0$$
$$\alpha_1=\left(\frac{\sum_{i=1}^{n_1}x_i^{\beta_1}}{n_1}\right)^{\frac{1}{\beta_1}}$$

and

$$\alpha_2=\left(\frac{\sum_{i=1}^{n_2}y_i^{\beta_2}}{n_2}\right)^{\frac{1}{\beta_2}}$$

If $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are the ML estimators of $\alpha_1, \alpha_2, \beta_1$and β_2 respectively then the ML estimators of $f_X(x; \alpha_1, \beta_1)$ and $f_Y(x; \alpha_2, \beta_2)$ are $f_X\left(x; \hat{\alpha}_1, \hat{\beta}_1\right)$ and $f_Y\left(x; \hat{\alpha}_2, \hat{\beta}_2\right)$ respectively.

3. NUMERICAL INTEGRATION METHODS

Numerical integration method provides a way to solve problems quickly and easily with a satisfactory absolute upper error (see Chapra and Canale [21]). Trapezoidal, Simpson 1/3 and Simpson 3/8 rules to approximate the interested integral are adopted in this paper. Let $h(t)$ be a continuous function on $[a, b]$ and let $\Delta t = \frac{b-a}{k}$, a and b are finite real numbers. Suppose that the interval $[a, b]$ is divided into k subintervals each of length Δt as follows,

$$a = t_0 < t_1 < t_2 < \dots < t_k = b,$$

where $t_i = a + i\Delta t$, $i = 0, 1, \dots$, $k - 1$. The three interested numerical integration rules to approximate $\int_a^b h(t)dt$ are (Atkinson, 1989),

- The trapezoidal rule is given by,

$$\int_a^b h(t)dt = \frac{\Delta t}{2}[h(a) + 2h(t_1) + 2h(t_2) \dots + 2h(t_{k-1}) + h(b)],$$

- The Simpson 1/3 rule is given by,

$$\int_a^b h(t)dt = \frac{\Delta t}{3}[h(a) + 4h(t_1) + 2h(t_2) + \dots + 4h(t_{k-1}) + h(b)],$$

- The Simpson 3/8 rule is given by,

$$\int_a^b h(t)dt = \frac{3\Delta t}{8}[h(a) + 3h(t_1) + 3h(t_2) + 2h(t_3) + 3h(t_4) + 3h(t_5)$$
$$+2h(t_6) + \dots + 2h(t_{k-3}) + 3h(t_{k-2}) + 3h(t_{k-1}) + h(b)]$$

4. APPROXIMATION OF $\Delta(X, Y)$

The formula of Weitzman measure $\Delta(X, Y)$ between X and Y is,

$$\Delta(X, Y) = \int_0^\infty min\{ f_X(x) , f_Y(x)\}dx.$$

Let $X \sim We(\alpha_1, \beta_1)$ and $Y \sim We(\alpha_2, \beta_2)$, where X and Y are independent random variables. In this section, different approximations for $\Delta(X, Y)$ are given. These approximations are derived based on the trapezoidal, Simpson 1/3 and Simpson 3/8 rules. In the following three subsections, we assumed that the interval $[a, b]$ is partitioned into the k sub-intervals,

$[a = u_0, u_1), [u_1, u_2), [u_2, u_3), \dots, [u_{k-1}, u_k = b]$,

where $u_i = a + i\frac{b-a}{k}$, $i = 0,1,2, \dots, k-1$.

Accordingly, the main interest of this paper is to approximate $\Delta(X, Y)$ using numerical integration methods which are trapezoidal method and also Simpson methods. These methods enable us to estimate $\Delta(X, Y)$ without placing any restrictions on the parameters of Weibull distributions. These numerical methods require that the integral limits (bounds) to be finite. So as a first step, we need to find a suitable transformation that enables us to apply them.

In general, consider the transformation $t = W(x)$, where $W(x)$ is assumed to be continuous increasing function in x, such that, $W(0) = a$ and $W(\infty) = b$, where $a < b$ are two finite real numbers. Then $x = W^{-1}(t) = (V(t)\ say)$ and $\frac{dx}{dt} = V'(t)$. Therefore,

$$\Delta(X,Y) = \int_a^b min\{ f_X(V(t)),\ f_Y\ (V(t))\}V'(t)dt. \tag{1}$$

Now, let $\Delta(W(X), W(Y))$ be the Weitzman overlapping measure between $W(X)$ and $W(Y)$ then we state the following theorem.

Theorem 1. Let W be a continuous increasing (or decreasing) function then,

$$\Delta(W(X), W(Y)) = \Delta(X, Y).$$

Which indicates that Δ is invariant measure with respect to any continuous increasing or decreasing function.

Proof: Let $T_1 = W(X)$ be a continuous increasing (or decreasing) function of a random variable X. The pdf of T_1 is,

$$f_{T_1}(t_1) = f_X(V(t_1); \alpha_1, \beta_1)|V'(t_1)|$$

and the pdf of $T_2 = W(Y)$ is,

$$f_{T_2}(t_2) = f_Y(V(t_2); \alpha_2, \beta_2)|V'(t_2)|$$

Therefore,

$$\Delta(W(X), W(Y)) = \Delta(T_1, T_2)$$

$$= \int_a^b min\{f_{T_1}(t_1), f_{T_2}(t_1)\}dt_1$$

$$= \int_a^b min\{ f_X(V(t_1); \alpha_1, \beta_1)|V'(t_1)|,\ f_Y\ (V(t_1); \alpha_1, \beta_1)|V'(t_1)|\}dt_1$$

$$= \int_a^b min\{ f_X(V(t_1); \alpha_1, \beta_1),\ f_Y\ (V(t_1); \alpha_1, \beta_1)\}|V'(t_1)|dt_1$$

$$= \Delta(X, Y).$$

The last step is obtained based on Eq. (1). This completes the proof.

Now, we can approximate $\Delta(X,Y)$ based on Eq. (1). The approximations of $\Delta(X,Y)$ by using trapezoidal, Simpson 1/3 and Simpson 3/8 rules are as follows:

Let $h(u) = min\{ f_X(V(t_1);\alpha_1,\beta_1),\ f_Y\ (V(t_1);\alpha_2,\beta_2)\}|V'(t_1)|$ then,

$$\Delta(X,Y) = \int_a^b min\{ f_X(V(t_1);\alpha_1,\beta_1)\ ,\ f_Y\ (V(t_1);\alpha_2,\beta_2)\}|V'(t_1)|dt_1$$

$$= \int_a^b h(t_1)dt_1.$$

Then the approximations of $\Delta(X,Y) = \int_a^b h(t_1)dt_1$ by using the numerical integration methods are:

- **Trapezoidal Approximation:** The approximation of $\Delta(X,Y)$ by using trapezoidal rule is,

$$\Delta_{Trap} \cong \frac{b-a}{2k}\left[h(a) + 2\sum_{j=1}^{k-1} h(u_j) + h(b)\right].$$

- **Simpson 1/3 Approximation:** The approximation of $\Delta(X,Y)$ by using Simpson 1/3 rule is (k is an integer positive number and a multiple of 2),

$$\Delta_{Simp1} \cong \frac{b-a}{3k}\left[h(a) + 4\sum_{j=1}^{k/2} h(u_{2j-1}) + 2\sum_{j=1}^{k/2-1} h(u_{2j}) + h(b)\right].$$

- **Simpson 3/8 Approximation:** The approximation of $\Delta(X,Y)$ by using Simpson 3/8 rule is (k is an integer positive number and a multiple of 3),

$$\Delta_{Simp2} \cong \frac{3(b-a)}{8k}\left\{h(a) + 3\sum_{\substack{j=1 \\ j\neq 3m}}^{k-1} h(u_j) + 2\sum_{j=1}^{k/3-1} h(u_{3j}) + h(b)\right\}, m \in N_0.$$

5. ESTIMATION OF $\Delta(X,Y)$

Let $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$ and $\hat{\beta}_2$ be the ML estimators of $\alpha_1, \alpha_2,\ \beta_1$ and β_2 respectively then the ML estimators of $f_X(x;\alpha_1,\beta_1)$ and $f_Y(x;\alpha_2,\beta_2)$ are $f_X(x;\hat{\alpha}_1,\hat{\beta}_1)$ and $f_Y(x;\hat{\alpha}_2,\hat{\beta}_2)$ respectively. Eidous and AL-Maqableh [22] suggested the following maximum likelihood estimator for $\Delta(X,Y)$. Their work depends entirely on writing the formula of $\Delta(X,Y)$ as an expected value of some function(s), and then they estimated the resulting expected value(s) by using the method of moments (See also Eidous and Al-Talafhah [23]). The estimator of $\Delta(X,Y)$ is,

$$\hat{\Delta}_{EM}=\frac{1}{2}\left[\frac{1}{n_1}\sum_{k=1}^{n_1}\left(\frac{min\{f_X(X_k;\hat{\alpha}_1,\hat{\beta}_1),f_Y(X_k;\hat{\alpha}_2,\hat{\beta}_2)\}}{f_X(X_k;\hat{\alpha}_1,\hat{\beta}_1)}\right)\right.$$
$$\left.+\frac{1}{n_2}\sum_{k=1}^{n_2}\left(\frac{min\{f_X(Y_k;\hat{\alpha}_1,\hat{\beta}_1),f_Y(Y_k;\hat{\alpha}_2,\hat{\beta}_2)\}}{f_Y(Y_k;\hat{\alpha}_1,\hat{\beta}_1)}\right)\right]$$

The proposed estimators of $\Delta(X,Y)$ can be obtained simply by substituting the ML estimators $f_X(.;\hat{\alpha}_1,\hat{\beta}_1)$ and $f_Y(.;\hat{\alpha}_2,\hat{\beta}_2)$ of $f_X(.;\alpha_1,\beta_1)$ and $f_Y(.;\alpha_2,\beta_2)$ back into $h(u)$ to obtain the corresponding estimators of Δ_{Trap}, Δ_{Simp1} and Δ_{Simp2}. Briefly, the proposed estimators of $\Delta(X,Y)$ are given as follows:

Let $\hat{h}(t)=min\{f_X(V(t_1);\hat{\alpha}_1,\hat{\beta}_1),\ f_Y\ (V(t_1);\hat{\alpha}_2,\hat{\beta}_2)\}V'(t_1)$ then,

- The proposed estimators of $\Delta(X,Y)$ that corresponding the trapezoidal approximation is,

$$\hat{\Delta}_{Trap}=\frac{b-a}{2k}\left[\hat{h}(a)+2\sum_{j=1}^{k-1}\hat{h}(u_j)+\hat{h}(b)\right].$$

- The proposed estimators of $\Delta(X,Y)$ that corresponding Simpson 1/3 approximation is

$$\hat{\Delta}_{Simp1}=\frac{b-a}{3k}\left[\hat{h}(a)+4\sum_{j=1}^{k/2}\hat{h}(u_{2j-1})+2\sum_{j=1}^{k/2-1}\hat{h}(u_{2j})+\hat{h}(b)\right].$$

- The proposed estimators of $\Delta(X,Y)$ that corresponding Simpson 3/8 approximation is,

$$\hat{\Delta}_{Simp2}=\frac{3(b-a)}{8k}\left\{\hat{h}(a)+3\sum_{\substack{j=1\\ j\neq 3m}}^{k-1}\hat{h}(u_j)+2\sum_{j=1}^{k/3-1}\hat{h}(u_{3j})+\hat{h}(b)\right\},m\in N_0.$$

6. NUMBER OF PARTITIONS AND TRANSFORMATION

To implement the different proposed estimators of $\Delta(X,Y)$ that stated in the previous section in practice, two quantities are to be determined. The first quantity is the number of partitions (subintervals) k. In this study, we suggest to take its value to be $k=min\{n_1,n_2\}$. It is well known that the maximum absolute error of the different numerical integral approximation decreases as the number of partitions k increases [21]. However, a preliminary simulation study was performed for different values of k greater than $min\{n_1,n_2\}$, it is found that there is no significant improvement in the estimation process for the different proposed estimators by taking $k>\min\{n_1,n_2\}$.

The second quantity is the transformation function W. Let Z be a continuous random variable with cumulative distribution function $F_Z(z),\ z \geq 0$ then our special interest is to take $W(x) = F_Z(x)$. In this case, $a = W(0) = 0$ and $b = W(\infty) = 1$. More specifically, we consider the following transformation in our simulation study in the next section,

$$F_Z(x) = 1 - e^{-x},\ \ 0 \leq x < \infty.$$

That is, $Z \sim \exp(1) = We(1,1)$. In addition, and to study the effect of the selected transformation on the estimation process we also take $Z \sim We(1,2)$. In general, let $Z \sim We(1,\theta)$ then,

$$F_Z(x) = 1 - e^{-x^{\theta}},\ \ 0 \leq x < \infty.$$

In this case, $t = W(x) = F_Z(x) = 1 - e^{-x^{\theta}}$. The inverse transformation is $x = V(t) = (-ln(1-t))^{1/\theta}$ and $dx = V'(t)dt = \frac{(-\ln(1-t))^{1/\theta}}{\theta(1-t)} dt$.

7. SIMULATION STUDY AND RESULTS

In this section, a Monte Carlo simulation study is conducted to justify the performances of the proposed estimators $\widehat{\Delta}_{Trap}$, $\widehat{\Delta}_{Simp1}$ and $\widehat{\Delta}_{Simp2}$ of Δ. The transformation $F_Z(x) = 1 - e^{-x^{\theta}}$ with $\theta = 1, 2$ is used for each of the proposed estimator. For sake of comparison, the estimator $\widehat{\Delta}_{EM}$ that suggested by Eidous and AL-Maqableh [22] is also considered.

Two random samples are simulated from two Weibull distributions. The first sample $x_1, x_2, \ldots, x_{n_1}$ is simulated from $f_X(x) = We(\alpha_1, \beta_1)$, while the second sample $y_1, y_2, \ldots, y_{n_2}$ is generated from $f_Y(y) = We(\alpha_2, \beta_2)$, where $f_X(x)$ and $f_Y(y)$ and the corresponding choosing parameters are given in Table (1). Also, the exact values of Δ for each pair of selection are also provided. Despite that the process of selection parameters seems to be arbitrary, we take into account that our selection should vary the exact values of Δ from 0 to 1. That is, the selection gives the values of Δ to be small (near zero), moderate (around one-half) and large (near one). For each pair of densities, the size of simulated data are $(n_1, n_2) = (12,12), (24,30),\ (96,180)$. All simulation results are calculated by using Mathematica Version 7.

Table 1: The simulated pair of distributions and the corresponding exact values of Δ.

Weibull Distributions	$f_X(x)$	$f_Y(y)$	Exact Δ
Case (1)	We(2,1)	We(1.9, 1.1)	0.929
Case (2)	We(4,1)	We(2.1, 1.8)	0.4370
Case (3)	We(2,1)	We(6.0, 5.0)	0.0144

The empirical results were calculated based on one thousand replication (Rep = 1000). For each estimator, we computed the Relative Bias (RB), Relative Mean Square Error (RMSE) and Efficiency (EFF), which are defined by,

$$RB = \frac{\hat{E}(estimator) - exact}{exact},$$

$$RMSE = \frac{\sqrt{\widehat{MSE}(estimator)}}{exact}$$

and

$$EFF = \frac{\widehat{MSE}(\hat{\Delta}_{EM})}{\widehat{MSE}(proposed\ estimator)}$$

For example, if $\hat{\Delta}$ is the estimator of Δ and if $\hat{\Delta}_{(j)}$ is the value of $\hat{\Delta}$ computed based on a sample of iteration $j, j = 1, 2, \ldots, Rep = 1000$ then,

$$\hat{E}(\hat{\Delta}) = \frac{\sum_{j=1}^{Rep} \hat{\Delta}_{(j)}}{Rep},$$

$$\widehat{MSE}(\hat{\Delta}) = \frac{\sum_{j=1}^{Rep} \left(\hat{\Delta}_{(j)} - E(\hat{\Delta})\right)^2}{Rep}.$$

and the EFF of $\hat{\Delta}$ with respect to $\hat{\Delta}_{EM}$ is,

$$EFF = \frac{\widehat{MSE}(\hat{\Delta}_{EM})}{\widehat{MSE}(\hat{\rho})}.$$

All computations and outputs of the simulation study are showed in Table (2). Based on these outputs we can summarize the results as follows.

- **The effect of transformation selection:** To study the effect of the selected transformation on the performance of the proposed estimators of Δ, the two transformations $F_Z(x) = 1 - e^{-x^{\theta}}$, $\theta = 1, 2$ were studied. The two selected transformations (i.e. $\theta = 1, 2$) give acceptable results for our proposed estimators. It is clear from the simulation results that these estimators are sensitive to the transformation selection. This can be inferred by examining and comparing the values of RMSEs and then the values of EFFs values associated with the proposed estimators when $\theta = 1$ and when $\theta = 2$ for different sample sizes. However, based on the simulation results, we recommend taking $\theta = 1$.
- **The effect of the selected numerical rule:** The three proposed estimators $\hat{\Delta}_{Trap}$, $\hat{\Delta}_{Simp1}$ and $\hat{\Delta}_{Simp2}$ are obtained based on trapezoidal, Simpson 1/3 and Simpson 3/8 rules respectively. By examining the simulation results related to these rules and by comparing between them, it is clear that the three rules give similar results for different sample sizes. In general, their results coincide when the sample sizes

get larger. This indicates that using any of these three rules is equivalent to use the other to estimate Δ.

- **Properties and performances of the various estimators:**

1. It is clear that the MSE values of the different estimators of Δ decrease with increasing sample sizes. This indicates the various estimators are consistent.
2. Most of the $|RB|s$ values of the proposed estimators are small for different sample sizes. As the sample sizes increase the associated $|RB|$ of the different estimators become negligible for most considered cases.
3. For the three simulated cases and based on the MSE and EFF values of all estimators, it is clear that the performances of the proposed estimators are better than that of Eidous and Al-Magableh [22] estimator for almost all considered cases. This feature becomes more evident when the exact value of Δ is small.

Table 2: The RB, RMSE and EFF of the estimators $\hat{\Delta}_{EM}, \hat{\Delta}_{Trap}, \hat{\Delta}_{Simp1}$ and $\hat{\Delta}_{Simp2}$ when the data are simulated from pair Weibull distributions as given in Table (1).

		$\theta = 1$				5			
(n_1, n_2)		$\hat{\Delta}_{EM}$	$\hat{\Delta}_{Trap}$	$\hat{\Delta}_{Simp1}$	$\hat{\Delta}_{Simp2}$	$\hat{\Delta}_{EM}$	$\hat{\Delta}_{Trap}$	$\hat{\Delta}_{Simp1}$	$\hat{\Delta}_{Simp2}$
Case 1									
(12,12)	RB	-0.120	-0.118	-0.118	-0.119	-0.118	-0.182	-0.160	-0.165
	RMSE	0.1809	0.1767	0.1768	0.1777	0.1830	0.2176	0.2025	0.2058
	EFF	1.000	**1.048**	1.047	1.036	1.000	0.707	0.816	0.790
(24,30)	RB	-0.0596	-0.0594	-0.0591	-0.0592	-0.0614	-0.102	-0.0893	-0.0924
	RMSE	0.1123	0.1112	0.1112	0.1112	0.1157	0.1324	0.1248	0.1266
	EFF	1.000	1.019	**1.020**	1.020	1.000	0.764	0.859	0.836
(96,180)	RB	-0.0118	-0.0117	-0.0116	-0.0116	-0.0116	-0.0236	-0.0199	-0.0209
	RMSE	0.0488	0.0484	0.0484	0.0484	0.0482	0.0497	0.0486	0.0489
	EFF	1.000	1.016	**1.016**	1.016	1.000	0.940	0.982	0.973
Case 2									
(12,12)	RB	-0.0858	-0.113	-0.113	-0.117	-0.0836	-0.0959	-0.0977	-0.0957
	RMSE	0.2880	0.2874	0.2881	0.2798	0.2829	0.2776	0.2780	0.2822
	EFF	1.000	1.005	1.000	**1.060**	1.000	1.038	1.035	1.004
(24,30)	RB	-0.0455	-0.0529	-0.0506	-0.0527	-0.0319	-0.0429	-0.0423	-0.0412
	RMSE	0.1855	0.1773	0.1736	0.1737	0.1784	0.1707	0.1706	0.1703
	EFF	1.000	1.095	**1.143**	1.141	1.000	1.093	1.094	1.098
(96,180)	RB	-0.0123	-0.0131	-0.0132	-0.0131	-0.0077	-0.0091	-0.0091	-0.0091
	RMSE	0.0799	0.0746	0.0746	0.0747	0.0770	0.0708	0.0708	0.0709
	EFF	1.000	1.146	1.146	1.144	1.000	**1.182**	1.182	1.181

		$\theta = 1$				5			
(n_1, n_2)		$\hat{\Delta}_{EM}$	$\hat{\Delta}_{Trap}$	$\hat{\Delta}_{Simp1}$	$\hat{\Delta}_{Simp2}$	$\hat{\Delta}_{EM}$	$\hat{\Delta}_{Trap}$	$\hat{\Delta}_{Simp1}$	$\hat{\Delta}_{Simp2}$
Case 3									
(12,12)	RB	-0.229	-0.0334	0.0963	0.0612	-0.204	-0.827	-0.799	-0.812
	RMSE	1.412	1.095	1.302	1.207	1.402	0.8700	0.8564	0.8627
	EFF	1.000	1.663	1.177	1.368	1.000	2.60	**2.68**	2.64
(24,30)	RB	-0.0894	-0.0031	-0.0306	-0.0063	-0.171	-0.739	-0.701	-0.719
	RMSE	0.9889	0.7476	0.7229	0.7696	0.9160	0.7771	0.7513	0.7631
	EFF	1.000	1.750	**1.872**	1.651	1.000	1.389	1.486	1.441
(96,180)	RB	-0.0625	-0.0392	-0.0391	-0.0397	-0.0071	-0.413	-0.345	-0.375
	RMSE	0.4389	0.3177	0.3182	0.3168	0.4546	0.4498	0.3969	0.4200
	EFF	1.000	**1.908**	1.902	1.919	1.000	1.021	1.312	1.171

8. REAL DATA ANALYSIS

In this section, two sets of data have been taken to be used as an application to calculate and investigate the properties of various estimators of Δ. The first set (Data set I) consists of 69 values representing the measured strength in the GPA of a single carbon fiber with a length of 20 mm. The second set (data set II) consists of 63 observations and represents the measured strength in GPA of single carbon fibers with a length of 50 mm. The measurements of the two sets are reproduced here and are as follows:

Data set I $(x_i, i = 1,2, \ldots, 69)$:
1.312 1.314 1.479 1.552 1.700 1.803 1.861 1.865 1.944 1.958 1.966 1.997 2.006 2.021 2.027 2.055 2.063 2.098 2.140 2.179 2.224 2.240 2.253 2.270 2.272 2.274 2.301 2.301 2.359 2.382 2.382 2.426 2.434 2.435 2.478 2.490 2.511 2.514 2.535 2.554 2.566 2.570 2.586 2.629 2.633 2.642 2.648 2.684 2.697 2.726 2.770 2.773 2.800 2.809 2.818 2.821 2.848 2.880 2.954 3.012 3.067 3.084 3.090 3.096 3.128 3.233 3.433 3.585 3.585.

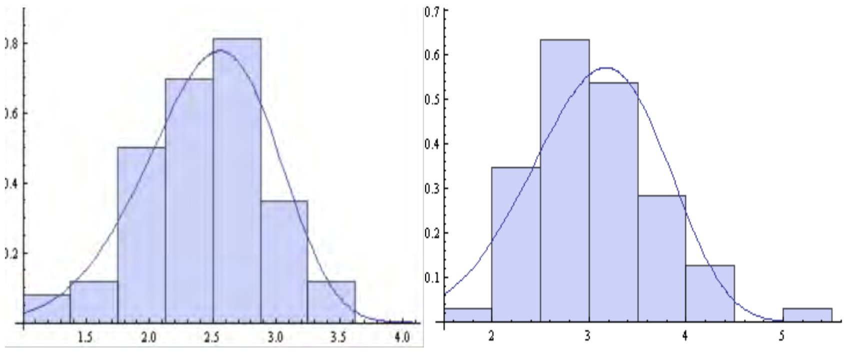
Data set II $(y_i, i = 1,2, \ldots, 63)$:
1.901 2.132 2.203 2.228 2.257 2.350 2.361 2.396 2.397 2.445 2.454 2.474 2.518 2.522 2.525 2.532 2.575 2.614 2.616 2.618 2.624 2.659 2.675 2.738 2.740 2.856 2.917 2.928 2.937 2.937 2.977 2.996 3.030 3.125 3.139 3.145 3.220 3.223 3.235 3.243 3.264 3.272 3.294 3.332 3.346 3.377 3.408 3.435 3.493 3.501 3.537 3.554 3.562 3.628 3.852 3.871 3.886 3.971 4.024 4.027 4.225 4.395 5.020.

These data were originally reported and analyzed by Badar and Priest [24]. Also, some authors analyzed these data in the system of reliability to estimate the well-known quantity $P(X < Y)$ subject to Weibull distribution (See, Gül Akgü and Şenoğlu [25] and Almarashi et al. [26]) .

The values of the parameters were calculated using the maximum likelihood method, which gave $\hat{\alpha}_1 = 2.65086$, $\hat{\alpha}_2 = 3.31472$, $\hat{\beta}_1 = 5.50485$ and $\hat{\beta}_2 = 5.04941$. Based on these values, we obtain,

$$\hat{\Delta}(X,Y) = \int_0^\infty min\left\{We(\hat{\alpha}_1,\hat{\beta}_1)\,, We(\hat{\alpha}_2,\hat{\beta}_2)\right\}dx$$
$$= 0.591011.$$

For illustration, the plot of the Weibull distribution for each data set together with its histogram is depicted in Figure (2). In addition, Figure (3) gives the overlapping area under the two Weibull distributions. The values of the three proposed estimators $\widehat{\Delta}_{Trap}$, $\widehat{\Delta}_{Simp1}$, $\widehat{\Delta}_{Simp2}$ and the estimator $\widehat{\Delta}_{EM}$ (Eidous and AL-Maqableh [22]) of Δ together with their biases, standard deviations (SD) and mean square errors (MSE) are given in Table (3). These statistical properties are computed by using the Bootstrap method with 500 replications (For the use of bootstrap method, see Mulekar and Mishra [16]). As we can see from Table (3), the MSEs of the proposed estimators are less than that of $\widehat{\Delta}_{EM}$, and this supports our conclusion based on the simulation results.



(a) (b)

Figure 1: a. Histogram and $We(2.65086,\ 5.50485)$ curve for the first data set and b. Histogram and $We(3.31472,\ 5.04941)$ curve for the second data set

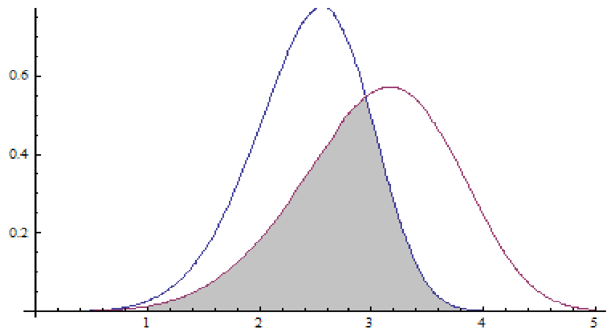


Figure 2: The overlapping area (shaded area) of the two data sets based on $We(2.65086,\ 5.50485)$ and $We(3.31472,\ 5.04941)$. The shaded area, $\widehat{\Delta}(X,Y) = 0.591011$

Table 3: The approximate Bias, SD and MSE for each considered estimators of Δ.

	Estimator	Bias	SD	MSE
$\hat{\Delta}_{EM}$	0.601533	-0.009152	0.062198	0.003952
$\hat{\Delta}_{Trap}$	0.590232	-0.008275	0.059551	0.003615
$\hat{\Delta}_{Simp1}$	0.589878	-0.007857	0.059125	0.003558
$\hat{\Delta}_{Simp2}$	0.589482	-0.007968	0.059297	0.003580

9. CONCLUSION

The purpose of this study was to establish a new technique for estimating the Weitzman coefficient $\Delta(X, Y)$ under a pair of Weibull distributions based on numerical integration methods. The benefit of the proposed technique is to estimate $\Delta(X, Y)$ without setting any conditions on the parameters of the Weibull distributions. The numerical results showed that the new technique is effective and that the performance of the produced estimators is better than the performance of the estimator developed by Eidous and Al-Maqableh [22]. Therefore, this technique can be used to estimate the other OVL coefficients mentioned in the literature, such as the Matusita coefficient (Eidous and Al-Shorman [27] and Eidous and Abu Al-Hayja`a [28]) and Pianka and Kullback-Leibler coefficients (Eidous and Abu Al-Hayja`a [29]) without specifying any conditions on the parameters of the distributions under study.

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