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A NEW ANALYTICAL APPROXIMATE SOLUTION OF FRACTIONAL COUPLED KORTEWEG-DE VRIES SYSTEM

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Abstract: The main objective of this work is to present a modification of the Mittag-Leffler function to deduce a relatively new analytical approximate method (for short MMLFM) able to solve time-fractional nonlinear partial differential equations (PDEs). Moreover, we employ the MMLFM to solve the time-fractional coupled Korteweg–de Vries (KdV) model described by two nonlinear fractional partial differential equations (FPDEs) based upon Caputo fractional derivative (CFD). The simulation of projected results is presented in some figures and tables. Furthermore, we compare our solutions when $\alpha = 1$ with known exact solutions which indicate a good agreement, in addition, we compare our outcomes with the results obtained by other methods in the literature such as the Natural decomposing method (NDM) and homotopy decomposition method (HDM) in order to prove the reliability and efficiency of our used method. Also, we display solutions with different values of α to present the effect of the fractional order on the proposed problem. The results of this article reveal the advantages of the MMLFM, which is simple, reliable, accurate, needs simple mathematical computations, is rapidly convergent to the exact solution, have a straightforward and easy algorithm compared to other analytical methods to study linear and nonlinear FPDEs, which makes this technique suited for real industrial or medical applications.

Keywords: Fractional coupled Korteweg-de Vries equation, fractional partial differential equations, Mittag-Leffler function, nonlinear problems, approximate solutions.

MSC: 35R11, 33E12, 35Q92, 35C10.

1. INTRODUCTION

The study of fractional calculus (FC) which includes fractional derivatives and integrals is one of the most fundamental and applicable branches. Where it has played a constructive role in many applications in different fields such as physics, engineering, chemistry and so on [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Moreover, there are several mathematical problems successfully modeled by linear or nonlinear FPDEs which has some important features such as non-locality, memory effect, kernel, non-singularity and others [11, 12, 13, 14, 15, 16, 17]. So, there are many well-known methods in the literature concerning the solution of FPDEs such as the spectral collection method (SCM) [18], HDM [19], variational iteration method (VIM) [20], Elzaki transform decomposition method [21], NDM [22], homotopy analysis method (HAM) [23, 24, 25], residual power series method [26], homotopy perturbation method (HPM) [27, 28], Fourier spectral method [29], finite difference method [30] and many others (see e.g., [31, 32, 33, 34, 35]).

The KdV model has been considered a key to many applications in physical phenomena, like long internal waves in a density-stratified ocean, shallow water waves, acoustic waves on a crystal lattice, ion-acoustic waves in plasma and many others. The historical ground behind the KdV model is fascinating this remarkable discovery date back to over 150 years by a young Scottish engineer named John Scott Russel [36], while conducting experiments to determine the most efficient design for canal boats, he came across a singular and beautiful phenomenon which he called the wave of translation. Considerable effort has been concentrated on the solution for the KdV model. For example, in [22] the authors used NDM to solve fractional KdV equations and compared their results with various methods in the literature, we will prove our method more efficient than NDM. In [19] the HDM has been applied to get the approximate solution for the fractional KdV model and demonstrated that this method is more efficient than other methods such as VIM, HAM, HPM and adomian decomposition method, we will prove our method more efficient than HDM, therefore, our method is more efficient than all of the aforementioned methods. The fractional VIM has been used to solve the fractional KdV equations with Jumarie fractional derivative [37]. In [38] the authors applied a coupled fractional transform to obtain the solutions of the fractional KdV model. A shifted Legendre polynomials are used to present a numerical technique for solving fractional KdV model in [39].

The motivation of this work is to investigate the analytic-approximate solutions of the time-fractional coupled KdV model by a new analytical-approximate method (i.e., MMLFM). To confirm the eligibility and efficacy of the proposed method, we compare the outcomes with known exact solutions and solutions obtained by other methods. Moreover, analytical solutions of the time-fractional coupled KdV model by the suggested method have not been considered before, which strongly motivated us to complete this work.

The time-fractional coupled KdV model is considered as the following [19, 22]:

$$\begin{aligned} {}_0^C D_t^\alpha U(x,t) &= \mu \frac{\partial^3 U(x,t)}{\partial x^3} + \delta U(x,t) \frac{\partial U(x,t)}{\partial x} + \sigma V(x,t) \frac{\partial V(x,t)}{\partial x}, \\ {}_0^C D_t^\alpha V(x,t) &= \xi \frac{\partial^3 V(x,t)}{\partial x^3} + \rho U(x,t) \frac{\partial V(x,t)}{\partial x}, \end{aligned} \tag{1}$$

subject to initial conditions (ICs)

$$U(x,0) = \frac{\lambda}{a} sech^2 \left(\frac{1}{2} \sqrt{\frac{\lambda}{a}} x \right) = U_0, \tag{2}$$

$$V(x,0) = \frac{\lambda}{\sqrt{2a}} sech^2 \left(\frac{1}{2} \sqrt{\frac{\lambda}{a}} x \right) = V_0,$$

where ${}_0^C D_t^\alpha$ is CFD, μ, δ, σ, ξ, ρ, λ and a are constant parameters.

The main contribution of this paper is to introduce the basic idea and analysis of MMLFM in order to give an appropriate solution for general nonlinear FPDEs which is considered a new analytical-approximate method capable of solving many applications that contain FPDEs. Moreover, we applied this technique for evaluating the approximate solutions of the time-fractional coupled KdV model (1) via CFD. Finally, we compared our outcomes with the known exact solutions and the results obtained by other methods in the literature (i.e. NDM and HDM) where we obtained a good agreement which confirms the efficiency and eligibility of our method. The simulations of projected results are presented in some tables and figures for different values of fractional order α to display the impact of memory in the behaviour of obtained solutions. To the best of our knowledge, the time-fractional coupled KdV model (1) has not been solved before by this technique, which motivated us to conduct this research.

The remaining of this work is organized as follows. In Section 2, we introduce a brief history of FC and their properties. Section 3 explained the algorithm of the MMLFM for solving general nonlinear FPDEs. In Section 4, we discuss the solution procedure of the fractional coupled KdV model by MMLFM. Additionally, the simulation of our results is investigated by some figures and a comparison with the exact solution and two different methods is given in the tables to provide the validation of the proposed method. Section 5 included the conclusion and discussion.

2. PRELIMINARIES

Here, we present some basic concepts, definitions and properties of FC to achieve this work (see e.g. [40, 41, 42]).

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha > 0$, $t \in [0,T]$ for a function $\phi(x,t)$ is given by*

$$\begin{aligned} {}_0I_t^{\alpha}\phi(x,t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\phi(x,\xi)d\xi, \quad t>0, \\ {}_0I_t^{0}\phi(x,t) &= \phi(x,t). \end{aligned}$$

Definition 2. *The CFD of order $n-1 < \alpha \leq n \in \mathbb{N}$ on $[0,T]$ for absolutely continuous function $\phi(x,t)$ is given by*

$${}_0^CD_t^{\alpha}\phi(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1}\frac{\partial^n\phi(x,\xi)}{\partial\xi^n}ds, \quad t>0,$$

when $0<\alpha<1$, then we have

$${}_0^CD_t^{\alpha}\phi(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha}\frac{\partial\phi(x,\xi)}{\partial\xi}d\xi, \quad t>0.$$

Theorem 3. *let $\phi(x,t)$ be a differentiable function in $[0,T]$, $n-1<\alpha\leq n\in\mathbb{N}$ and $\beta>-1$, then*

$$\begin{aligned} {}_0^CD_t^{\alpha}\,{}_0I_t^{\alpha}\phi(x,t) &= \phi(x,t), \\ {}_0I_t^{\alpha}\,{}_0^CD_t^{\alpha}\phi(x,t) &= \phi(x,t) - \sum_{k=0}^{n-1}\frac{\partial^k\phi(x,t)}{\partial t^k}|_{t=0}\frac{t^k}{k!}. \end{aligned}$$

Also, we have the following properties:

$$\begin{aligned} {}_0^CD_t^{\alpha}t^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \\ {}_0I_t^{\alpha}t^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\beta+\alpha}. \end{aligned}$$

Definition 4. *The Mittag-Leffler function is given by*

$$E_{\alpha}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n\alpha+1)}, \qquad \alpha>0.$$

Lemma 5. *The CFD of generalized Mittag-Leffler function is given by*

$$\begin{aligned} {}_0^CD_t^{\alpha}E_{\alpha}(\lambda t^{\alpha}) = {}_0^CD_t^{\alpha}(\sum_{n=0}^{\infty}\frac{\lambda^n t^{n\alpha}}{\Gamma(n\alpha+1)}) &= \sum_{n=1}^{\infty}\frac{\lambda^n t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} \\ &= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}t^{n\alpha}}{\Gamma(n\alpha+1)} = \lambda E_{\alpha}(\lambda t^{\alpha}). \end{aligned}$$

Theorem 6. *[43, 44] Let a function $\phi(x,t)=\sum_{k=0}^{\infty}\eta^k\phi_k(x,t)$, then a nonlinear operator $N(\phi)$ satisfies the following*

$$\frac{\partial^n}{\partial\eta^n}N(\phi)_{\eta=0} = \frac{\partial^n}{\partial\eta^n}N\left(\sum_{k=0}^{\infty}\eta^k\phi_k\right)_{\eta=0} = \frac{\partial^n}{\partial\eta^n}N\left(\sum_{k=0}^{n}\eta^k\phi_k\right)_{\eta=0}.$$

3. ANALYSIS OF THE PROPOSED METHOD

To clarify the basic idea and the algorithm of the MMLFM, we consider the following general nonlinear FPDEs:

$$ {}_0^C D_t^\alpha F(X,t) = L(\, F(X,t)) \, + N(\, F(X,t)) \,, \tag{3} $$

with ICs

$$ F(X,0) = G(X), \tag{4} $$

where L and N are the general linear and nonlinear differential operators, respectively, $F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$, $X = [\, x_1 \; x_2 \; \cdots \; x_n \,]$, $n, m \in \mathbb{N}$, and $G(X) = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}$.

The MMLFM represents the solution of $F(X,t)$ for Eq.(3) as follows:

$$ \begin{aligned} f_1(X,t) &= w_1(X)E_\alpha(\lambda_1 t^\alpha) = \sum_{k=0}^{\infty} w_1(X)\lambda_1^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ f_2(X,t) &= w_2(X)E_\alpha(\lambda_2 t^\alpha) = \sum_{k=0}^{\infty} w_2(X)\lambda_2^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ &\vdots \\ f_m(X,t) &= w_m(X)E_\alpha(\lambda_m t^\alpha) = \sum_{k=0}^{\infty} w_m(X)\lambda_m^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \end{aligned} \tag{5} $$

where $\lambda_1, \lambda_2, \cdots, \lambda_m$ are undetermined coefficient and from ICs (4) functions $w_1, w_2, \cdots, w_m$ satisfies $w_1 = g_1, w_2 = g_2, \cdots, w_m = g_m$. By using Lemma 5 and assumptions (5) the FPDEs (3) is given by

$$ \begin{aligned} \sum_{k=0}^{\infty} G(X)\lambda_m^{k+1} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = \;& L(\sum_{k=0}^{\infty} G(X)\lambda_m^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}) + \\ & N(\sum_{k=0}^{\infty} G(X)\lambda_m^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}), m = 1, 2, \cdots . \end{aligned} \tag{6} $$

Therefore, the linear operator L is given by

$$ \begin{aligned} L(F(X,t)) &= L(\sum_{k=0}^{\infty} G(X)\lambda_m^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}) = L(\, G(X)) \sum_{k=0}^{\infty} \lambda_m^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \\ &= \gamma G(X) \sum_{k=0}^{\infty} \lambda_m^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \end{aligned} \tag{7} $$

where γ is a constant. Following the Theorem 6 the nonlinear operator N can be written as

$$\begin{aligned} N(F(X,t)) &= N(\sum_{k=0}^{\infty} G(X)\lambda_m^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}) = N(\sum_{k=0}^{\infty} G(X)F_j(X,t)) \\ &= N(G(X))\,(N(F_0(X,t)) + \sum_{k=1}^{\infty}(N(\sum_{i=0}^{k} F_i(X,t)) - N(\sum_{i=0}^{k-1} F_i(X,t)))). \end{aligned} \tag{8}$$

By replacing the linear and nonlinear terms from Eq.(7) and Eq.(8) in Eq.(6), we obtain a general recursive formula in order to determine the coefficients λ_m. Then, we get a general solution of Eq.(3). For more details on the convergence of the Mittag-Leffler function can consult [14, 15, 45, 46].

4. APPLICATION AND RESULUTS

4.1. Implementing MMLFM on Fractional KdV Model

Here, we extend the MMLFM to solve the time-fractional KdV model, if we put $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$ and $\xi = -1$ in Eq.(1), we have fractional KdV model as follows [22]:

$$\begin{aligned} &{}_0^C D_t^{\alpha} U(x,t) + \frac{\partial^3 U(x,t)}{\partial x^3} + 6U(x,t)\frac{\partial U(x,t)}{\partial x} - 6V(x,t)\frac{\partial V(x,t)}{\partial x} = 0, \\ &{}_0^C D_t^{\alpha} V(x,t) + \frac{\partial^3 V(x,t)}{\partial x^3} + 3U(x,t)\frac{\partial V(x,t)}{\partial x} = 0, \end{aligned} \tag{9}$$

subject to ICs

$$\begin{aligned} U(x,0) = U_0 &= \frac{\lambda}{a} sech^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}x\right), \\ V(x,0) = V_0 &= \frac{\lambda}{\sqrt{2a}} sech^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}x\right). \end{aligned} \tag{10}$$

To prove the validity of the MMLFM we compare obtained results when $\alpha = 1$ with the following known exact solution [19, 22]:

$$\begin{aligned} U(x,t) &= \frac{\lambda}{a} sech^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}(x-\lambda t)\right), \\ V(x,t) &= \frac{\lambda}{\sqrt{2a}} sech^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{a}}(x-\lambda t)\right). \end{aligned} \tag{11}$$

To implement the MMLFM on KdV model (9), let the solutions in the following form

$$\begin{aligned} U(x,t) &= f(x)E_\alpha(At^\alpha) = \sum_{n=0}^{\infty} f(x)\lambda_1^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ V(x,t) &= g(x)E_\alpha(Bt^\alpha) = \sum_{n=0}^{\infty} g(x)\lambda_2^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{aligned} \tag{12}$$

where λ_1 and λ_2 are undetermined coefficient. From Eq.(10), we have $f(x) = U_0$ and $g(x) = V_0$.

Using Lemma (5) and Eq.(12), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(U_0\lambda_1^{n+1} + \lambda_1^n s_1 + 6U_0 s_2 d^n \Gamma(n\alpha+1) - 6V_0 s_3 E^n \Gamma(n\alpha+1)\right) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = 0, \\ &\sum_{n=0}^{\infty} \left(V_0\lambda_2^{n+1} + \lambda_2^n s_4 + 3U_0 s_3 C^n \Gamma(n\alpha+1)\right) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = 0, \end{aligned} \tag{13}$$

where

$$d^n = \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_1^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}, \qquad E^n = \sum_{k=0}^{n} \frac{\lambda_2^k \lambda_2^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)},$$

$$C^n = \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)},$$

$$s_1 = \frac{2\lambda^2\sqrt{\frac{\lambda}{a}}sech^4(\theta)\,tanh(\theta) - \lambda^2\sqrt{\frac{\lambda}{a}}sech^2(\theta)\,tanh^3(\theta)}{a^2},$$

$$s_2 = \frac{\lambda\sqrt{\frac{\lambda}{a}}sech^2(\theta)\,tanh(\theta)}{a}, \qquad s_3 = -\frac{\lambda\sqrt{\frac{\lambda}{a}}sech^2(\theta)\,tanh(\theta)}{\sqrt{2a}},$$

$$s_4 = \frac{\sqrt{2}\lambda^2\sqrt{\frac{\lambda}{a}}sech^4(\theta)\,tanh(\theta)}{\sqrt{a^3}} - \frac{\lambda^2\sqrt{\frac{\lambda}{a}}sech^2(\theta)\,tanh^3(\theta)}{\sqrt{2a^3}}; \qquad \theta = \frac{1}{2}\sqrt{\frac{\lambda}{a}}x.$$

It is remarkable that in Eq.(13) the part $t^{n\alpha}$ is not able to occur equal zero. So, their coefficients are equal to zero. Then, the recurrence relation is given by:

$$\begin{aligned} \lambda_1^{n+1} &= \frac{-\lambda_1^n s_1 - 6U_0 s_2 d^n \Gamma(n\alpha+1) + 6V_0 s_3 E^n \Gamma(n\alpha+1)}{U_0}, \\ \lambda_2^{n+1} &= \frac{-\lambda_2^n s_4 - 3U_0 s_3 C^n \Gamma(n\alpha+1)}{V_0}. \end{aligned} \tag{14}$$

By substituting different values of $n \geq 0$ in Eq.(14), we get

$$\lambda_1^1 = \frac{-\lambda_1^0 s_1 - 6U_0 s_2 d^0 + 6V_0 s_3 E^0}{U_0},$$

$$\lambda_2^1 = \frac{-\lambda_2^0 s_4 - 3U_0 s_3 C^0}{V_0},$$

where $\lambda_1^0 = \lambda_2^0 = 1$, $d^0 = E^0 = C^0 = 1$. When $n = 1$, we have

$$\lambda_1^2 = \frac{-\lambda_1^1 s_1 - 6U_0 s_2 d^1 \Gamma(\alpha+1) + 6V_0 s_3 E^1 \Gamma(\alpha+1)}{U_0},$$

$$\lambda_2^2 = \frac{-\lambda_2^1 s_4 - 3U_0 s_3 C^1 \Gamma(\alpha+1)}{V_0},$$

where $d^1 = \frac{2\lambda_1^1\lambda_1^0}{\Gamma(\alpha+1)}$, $E^1 = \frac{2\lambda_2^0\lambda_2^1}{\Gamma(\alpha+1)}$ and $C^1 = \frac{\lambda_1^0\lambda_2^1+\lambda_1^1\lambda_2^0}{\Gamma(\alpha+1)}$.

In a similar way, by replacing different values of $n = 2, 3, \cdots$, we be able to acquire other factors for λ_1 and λ_2. Then, interchange these gained factors in the following power series which perform to the approximate solution Eq.(12):

$$U(x,t) = U_0(\lambda_1^0 + \lambda_1^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + \lambda_1^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \lambda_1^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots),$$
$$V(x,t) = V_0(\lambda_2^0 + \lambda_2^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + \lambda_2^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \lambda_2^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots).$$

4.2. Simulation of the Results

In this part, we show the graphical representation of the obtained results by MMLFM for the time-fractional KdV model (9). Moreover, a comparison between the obtained approximate solution with the exact solution (11) and two other methods (i.e., NDM [22] and HDM [19]) are reported in tables to test the accuracy of the proposed method. The presented results are computed by substituting the 5th-terms of the power series solution, where we find the solution converges to the exact solutions and the absolute error is significantly small.

In Figs.1 and 3, we display the behaviour of the solutions obtained by our method for the time-fractional KdV model (1) compared with the exact solutions Eq.(11) .

In Fig.2 and 4, we show the impact of changing the fractional order α on the approximate solutions for $U(x,t)$ and $V(x,t)$, respectively.

In Fig.5, we present the effect of fractional order α of the approximate solutions for model (1) w.s.t. space variable x by selecting the fixed time $t = 0.2$. Furthermore, we highlight the effect of α of the approximate solutions for model (1) w.s.t. time t by selecting the fixed space variable $x = 5$ in Fig. 6.

The results presented in these figures indicate that the solutions obtained by the MMLFM are consistent with the exact solutions for the time-fractional KdV model.

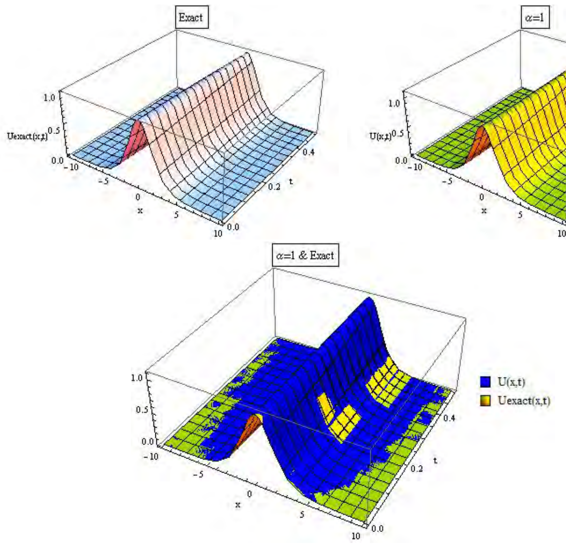

Figure 1: Comparison between the MMLFM solution for $U(x,t)$ when $\alpha = 1$, $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$, $\xi = -1$ and $a = 1$ with the exact solution for the time-fractional coupled KdV model.

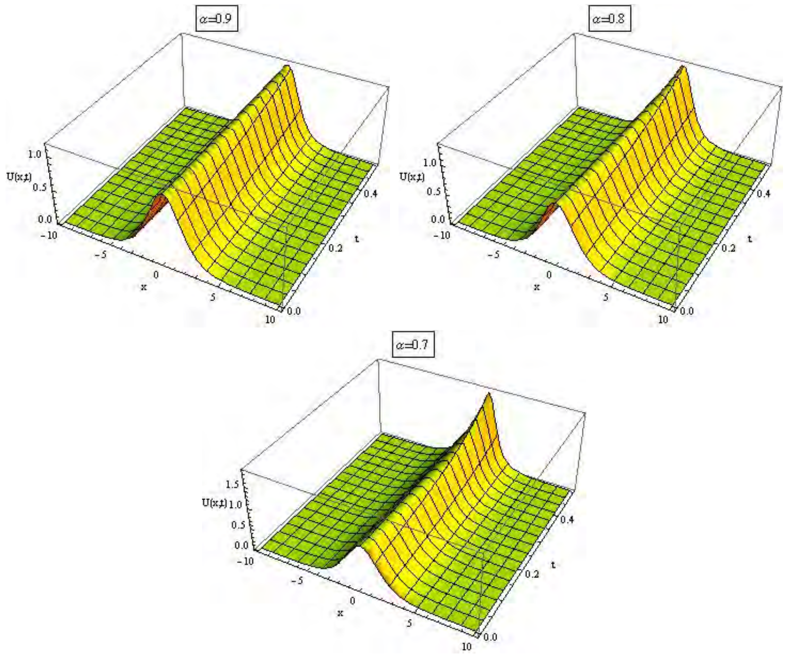

Figure 2: The MMLFM solution for $U(x,t)$ with different values of α with $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$ and $\xi = -1$.

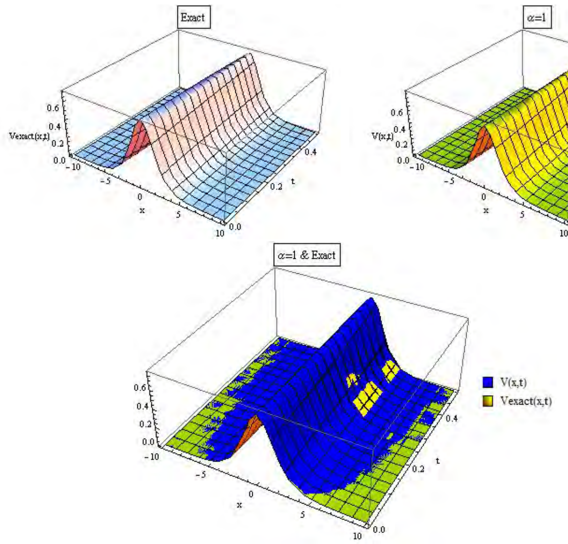

Figure 3: Comparison between the MMLFM solution for $V(x,t)$ when $\alpha = 1$, $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$, $\xi = -1$ and $a = 1$ with the exact solution for the time-fractional coupled KdV model.

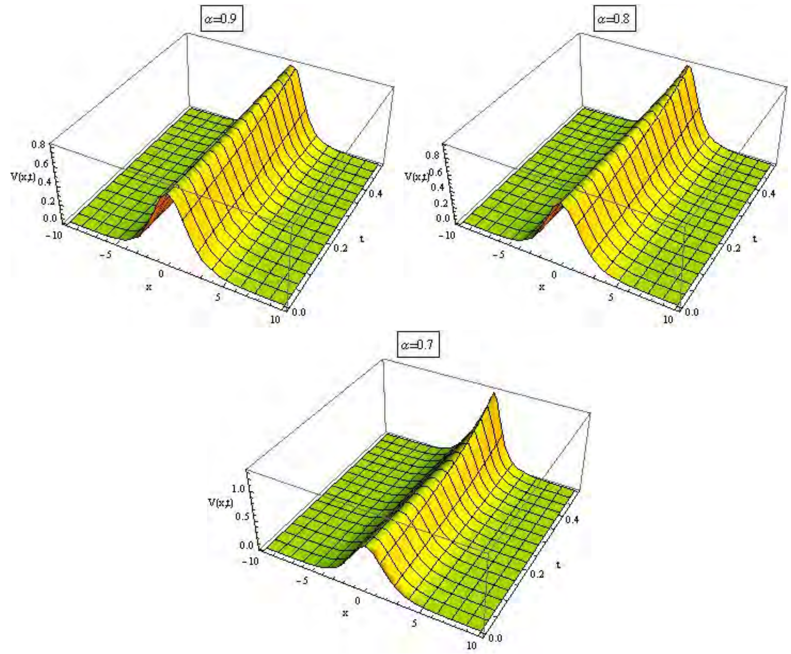

Figure 4: The MMLFM solution for $V(x,t)$ with different values of α with $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$ and $\xi = -1$.

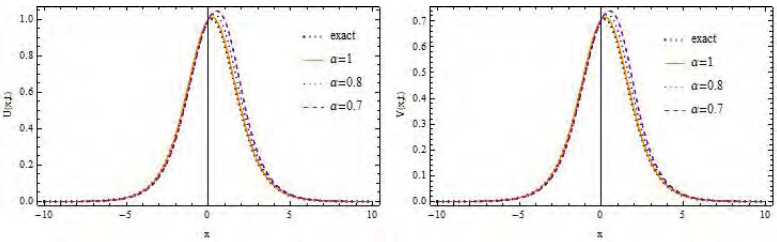

Figure 5: The obtained solutions by MMLFM for $U(x,t)$ and $V(x,t)$ with x for the time-fractional coupled KdV model with different values α when $t = 0.2$, $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$ and $\xi = -1$.

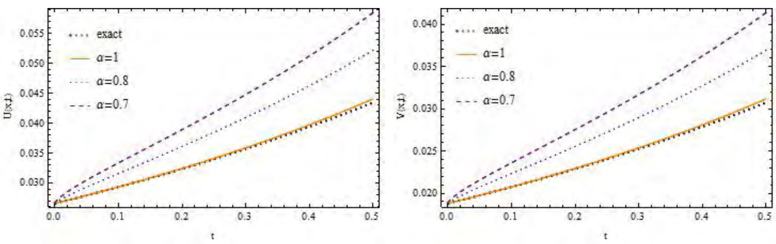

Figure 6: The obtained solutions by MMLFM for $U(x,t)$ and $V(x,t)$ with t for the time-fractional coupled KdV model with different values α when $x = 5$, $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$ and $\xi = -1$.

In Tables 1 and 2, we report the obtained numerical values of $U(x,t)$ and $V(x,t)$, respectively, with different values of the state x and time t when fractional order $\alpha = 1$, in order to be compared with the exact values, NDM [22] and HDM [19].

From these tables, we note that the obtained values when $\alpha = 1$ are nearly identical to the exact solutions and the absolute values are very small. Also, it can be observed that the obtained solutions by the MMLFM are more accurate than those obtained in [19] and [22].

From the presented results in the tables and figures, we confirm that the validity, exactness and efficiency of our used method.

Table 1: The outcomes of $U(x,t)$ for the exact and MMLFM solutions in the presence of solutions provided by other methods in [22, 19] with various values of t and x when $\alpha = 1$, $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$, $\xi = -1$ and $a = 1$.

x	t	Exact	MGMFM	NDM [22]	Error NDM [22]	Error HDM [19]	Error MGMFM
-10	0.1	0.000164305	0.000164305	0.000164334	2.95039 $\times 10^{-8}$	2.99039 $\times 10^{-8}$	5.0551 $\times 10^{-10}$
	0.2	0.00014867	0.000148672	0.000148901	2.30335 $\times 10^{-7}$	2.33335 $\times 10^{-7}$	1.76949$\times 10^{-9}$
-5	0.1	0.0240923	0.0241029	0.0240963	3.93592 $\times 10^{-6}$	3.96592 $\times 10^{-6}$	1.05846 $\times 10^{-6}$
	0.2	0.0218248	0.0218619	0.0218556	0.0000308049	0.0000338049	0.000037098
5	0.1	0.0293476	0.0293615	0.0293435	0.000202966	0.00000397592	0.0000138323
	0.2	0.0323838	0.0324471	0.0323501	0.000515586	0.0000378049	0.0000633656
10	0.1	0.000200679	0.000200679	0.000200648	2.11254 $\times 10^{-8}$	2.96039 $\times 10^{-8}$	6.59404 $\times 10^{-10}$
	0.2	0.000221782	0.000221785	0.000221527	2.28173 $\times 10^{-7}$	2.37335 $\times 10^{-7}$	3.01639 $\times 10^{-9}$

Table 2: The outcomes of $V(x,t)$ for the exact and MMLFM solutions in the presence of solutions provided by other methods in [22, 19] with various values of t and x when $\alpha = 1$, $\mu = -1$, $\eta = 6$, $\delta = -6$, $\sigma = 6$, $\rho = -3$, $\xi = -1$ and $a = 1$.

x	t	Exact	MGMFM	NDM [22]	Error NDM [22]	Error HDM [19]	Error MGMFM
-10	0.1	0.000116181	0.000116181	0.000116202	2.08624 $\times 10^{-8}$	2.18624 $\times 10^{-8}$	3.57125 $\times 10^{-10}$
	0.2	0.000105126	0.000105127	0.000105289	1.62872 $\times 10^{-7}$	1.64872 $\times 10^{-7}$	1.25122 $\times 10^{-9}$
-5	0.1	0.0170358	0.0170433	0.0170386	2.78312 $\times 10^{-6}$	2.88312 $\times 10^{-6}$	7.48447 $\times 10^{-6}$
	0.2	0.0154325	0.0154587	0.0154542	0.0000217824	0.0000287824	0.0000262322
5	0.1	0.0207519	0.0207617	0.020749	2.9094 $\times 10^{-6}$	2.98312 $\times 10^{-6}$	9.78092$\times 10^{-6}$
	0.2	0.0228988	0.0229436	0.022875	0.0000238036	0.0000247824	0.000044806
10	0.1	0.000141901	0.000141902	0.000141879	2.19313 $\times 10^{-8}$	2.09624 $\times 10^{-8}$	4.6629 $\times 10^{-10}$
	0.2	0.000156823	0.000156826	0.000156643	1.79991 $\times 10^{-7}$	1.72872 $\times 10^{-7}$	2.13291 $\times 10^{-9}$

5. CONCLUSION

In this article, we have successfully offered the approximate analytical solution for the nonlinear time-fractional coupled KdV model by using a relatively new method called MMLFM. The mathematical formalism of the proposed model consists of FPDEs based on Caputo differential operator. We presented the algorithm and basic idea of the MMLFM to solve general nonlinear FPDEs. In order to demonstrate that the used method is an efficient and powerful tool for FPDEs, we have tested this method in solving the nonlinear time-fractional KdV model (9) with convenient ICs (10). We have provided a numerical simulation for obtained results in some figures and a comparison between given exact solutions (11) and two different methods in some tables in order to illustrate the efficiency and validity of the used method. From the given results in this paper, we conclude that the MMLFM is a well-organized approximate analytical method for solving nonlinear FPDEs that have real-life applications in different fields.

Future recommendations are that the MMLFM can easily be implemented to obtain the analytic-approximate solution for several linear and nonlinear sophisticated systems of FPDEs facing researchers and has physical and biological applications in real life. Additionally, the mathematical formalism of the fractional order models can be based upon other fractional derivative operators.

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