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# MULTI-SERVER MARKOVIAN HETEROGENEOUS ARRIVALS QUEUE WITH TWO KINDS OF WORKING VACATIONS AND IMPATIENT CUSTOMERS

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**Abstract:** This paper deals with multi-server queueing system with two kinds of Working Vacations (WVs) and impatient customers. A random timer is started whenever a customer comes into the system. The customer may abandon the system if the service is not completed before the impatience timer expires. Each time after serving all the customers, the system becomes empty and then the server begins 1st kind of vacation. On returning from 1st kind of WV, the server begins 2nd kind of WV whenever a system has no customers. When the server comes back from either 1st kind or 2nd kind of WV, if there is at least one customer in the system, the server switches to busy period. The steady state probabilities have been derived using the Probability Generating Functions (PGFs) method. Various measures of performance are presented and numerical illustrations are also provided.

**Keywords:** Busy state, steady state, impatient customer, $c$- server, performance measures.

**MSC:** 60K25, 60K30, 90B22.

## 1. INTRODUCTION

Queueing theory is used to identify and correct congestion points in a particular process. It is used to analyze existing processes and to map alternatives for better results. Examples of queuing theory being used in networks include sizing router or multiplexer buffers, and calculating end-to-end throughput in a network. In many real-world queueing systems, servers may be unavailable for a various reasons. The server may be working on additional tasks, having checked for maintenance, or simply taking a rest during this time of absence. Many authors have treated the impatience phenomenon under various assumptions. In past, several authors have considered the queuing models with differentiated vacations in queues.

Ibe and Isijola [1] study a single server multiple vacation queueing system with two kinds of vacations. Bouchentouf and Medjahri [2] deal with $M/M/1$ feedback with queueing system under balked customers and differentiated multiple vacations. Isijola and Ibe [3] study a multiple vacation queueing system with two kinds of vacations in which each kind of vacation can be interrupted when two predetermined thresholds are reached by the system's costomers. Bouchentouf et al. [4] deal with breakdowns, repairs, reneging, balking, Bernoulli feedback, and retention, under multiple synchronous WVs with a finite population Markovian multi-server machine system. Gupta et al. [5] study an $M/D/1$ queue with deterministic service time and the two vacation types are exponentially distributed. Yue et al. [6] analyse an single server queueing system with impatient timers which depends on the server's state and vacations.

Gray et al. [7] analyze a multiple-vacation queueing model, where the service station is subject to breakdown while in operation. Choudhury and Deka [8] deal with $M/G/1$ queue with two phases of heterogeneous service and unreliable server. Jain et al. [9] deal with the performance modeling of finite Markov $M/M/1/L/WV$ model for the fault-tolerant machining system with WV and working breakdown. Perel and Yechiali [10] study $M/M/c$ queues in a 2-phase (fast and slow) Markovian random environment with impatient customers. Bouchentouf et al. [11] establish a cost analysis for an $M/M/1/N$ queuing system with differentiated WVs, Bernoulli schedule vacation interruption, balking, and reneging.

Vijayashree and Janani [12] present a transient analysis of $M/M/1$ queueing system where the server is subject to two kinds of vacation. Bouchentouf and Guendouzi [13] study with an $M^X/M/c$ Bernoulli feedback queue with impatience timers and various multiple WVs. Bouchentouf and Guendouzi [14] study the sensitivity analysis of infinite-buffer queueing system with Bernoulli feedback, differentiated vacations with vacation interruptions and impatient customers. Bouchentouf et al. [15] deal with Bernoulli feedback, synchronous multiple vacation and customer's impatience with a limited capacity of c-server Markovian queueing model. Kumar and Sharma [16] analyse limited capacity of the Markovian multi-server queuing system with discouraged arrival, reneging and retention of reneging customers. Vijaya Laxmi and Edadasari [17] study a variant WV queueing system with second optional service, unreliable server and retention of reneged customers.

Bouchentouf et al. [18] analyse an $M/M/1$ feedback queue with variant of multiple vacation policy, server's states-dependent reneging, balking and retention of reneged customers. Ayyappan and Nirmala [19] study an unreliable single server bulk queueing model with overloading service, various rate of arrival and closedown under multiple vacations. Agrawal et al. analyses [20] the steady state probability distribution of the number of customers in $M/M/1$ queue which is obtained using matrix geometric approach. Economou et al. [21] study the customer strategic behavior concerning the join-or-balk dilemma in queueing systems with server vacations/failures.

Vadivukarasi et al. [22] examine the optimality of a single server queues where the server is permitted to take two kinds of vacations. Karpagam [23] discusses a bulk service queue with rework for the faculty item, possibility of breakdown, repair, and two types of multiple vacation with different threshold policy. Donthi [24] analyse the comparison between multi queue multi server and single queue multi server queueing system. Sinu Lal et al. [25] study a multi-server tandem queueing model with a specialist server operating with a vacation strategy.

In this paper, we consider a heterogeneous arrivals queue with two kinds of WVs and impatient customers. A comparison table of existing queueing model and our model are discussed below.

Table 1: Comparison with the existing queueing models

| **S. No** | **Model** | **Author** | **Methodology** |
|---|---|---|---|
| 1 | $M/M/1$ queueing model with working vacation and two type of server breakdown | Agrawal et al. [20] | Matrix Geometric Method |
| 2 | Analysis of unreliable bulk queueing system with overloading service, variant arrival rate, closedown under multiple vacation policy. | Ayyappan and Nirmala [19] | Supplementary variable technique |
| 3 | Variant vacation queueing system with Bernoulli feedback, balking and server's states-dependent reneging. | Bouchentouf et al. [18] | Probability Generating Function |
| 4 | A multi-station unreliable machine model with working vacation policy and customers' impatience. | Bouchentouf et al. [4] | Q-matrix method, direct search method, Quasi–Newton method |

Table 1 *Continued*

| | | | |
|---|---|---|---|
| 5 | Analysis and performance evaluation of Markovian feedback multi-server queueing model with vacation and impatience. | Bouchentouf et al. [15] | Recursive method |
| 6 | Sensitivity analysis of feedback multiple vacation queueing system with differentiated vacations, vacation interruptions and impatient customers. | Bouchentouf and Guendouzi [14] | Recursive method |
| 7 | The $M^X/M/c$ Bernoulli feedback queue with variant multiple working vacations and impatient customers: performance and economic analysis. | Bouchentouf and Guendouzi [13] | Probability Generating Function |
| 8 | On impatience in Markovian $M/M/1/N/DWV$ queue with vacation interruption. | Bouchentouf et al. [11] | Recursive technique, quadratic fit search method |
| 9 | Performance and economic evaluation of differentiated multiple vacation queueing system with feedback and balked customers. | Bouchentouf and Medjahri [2] | Recursive method |
| 10 | A single server queueing system with two phases of service subject to server breakdown and Bernoulli vacation. | Choudhury and Deka [8] | Probability Generating Function, Laplace Stieltjes Transform |
| 11 | The value of reneging for strategic customers in queueing systems with server vacations/failures. | Economou et al. [21] | Probability Generating Function |
| 12 | A vacation queueing model with service breakdowns. | Gray et al. [7] | Probability Generating Function, Matrix Geometric Method |
| 13 | $M/D/1$ Multiple vacation queueing systems with deterministic service time. | Gupta et al. [5] | Substitution method |

Table 1 *Continued*

| | | | |
|---|---|---|---|
| 14 | $M/M/1$ multiple vacation queueing systems with differentiated vacations. | Ibe and Isijola [1] | Substitution method |
| 15 | $M/M/1$ multiple vacation queueing systems with differentiated vacations and vacation interruptions | Isijola and Ibe [3] | Substitution method |
| 16 | Performance modelling of fault-tolerant machining system with working vacation and working breakdown. | Jain et al. [9] | Matrix Geometric Method |
| 17 | On a multi-server queueing system with customers' impatience until the end of service under single and multiple vacation policies. | Kadi et al. [26] | Probability Generating Function |
| 18 | Analysis of bulk service queuing system with rework, unreliable server, resuming service and two kinds of multiple vacation. | Karpagam [23] | Cumulative Distribution Functions, Probability Density Function and its Laplace-Stieltjes Transform |
| 19 | Transient analysis of a multi-server queuing model with discouraged arrivals and retention of reneging customers. | Kumar and Sharma [16] | 4th order Runge-Kutta method. |
| 20 | Queues with slow servers and impatient customers. | Perel and Yechiali [10] | Probability Generating Function |
| 21 | A Multi-server Tandem Queue with a Specialist Server Operating with a Vacation Strategy. | Sinu Lal et al. [25] | Matrix Geometric Method |
| 22 | Discussion on the optimization of finite buffer Markovian queue with differentiated vacations. | Vadivukarasi et al. [22] | Probability Generating Function |
| 23 | Transient analysis of an $M/M/1$ queueing system subject to differentiated vacations. | Vijayashree and Janani [12] | Probability Generating Function, Laplace transform techniques. |

Table 1 *Continued*

| | | | |
|---|---|---|---|
| 24 | Variant working vacation Markovian queue with second optional service, unreliable server and retention of reneged customers. | Vijaya Laxmi and Edadasari [17] | Probability Generating Function |
| 25 | Analysis of an $M/M/1$ queue with vacations and impatience timers which depend on the server's states. | Yue et al. [6] | Probability Generating Function |
| 26 | $M/M/c$ queueing system with two kinds of working vacation and impatience customer | Proposed Model | Probability Generating Function |

From this Table 1 most of the authors followed the PGF method. The role of PGF method in engineering is to provide the formal basis for analyzing uncertainties. In civil engineering, risk assessment, reliability analysis, cost benefit analysis are common applications. The structure of the paper is as follows: Section 2 defines the model description. Section 3 presents the solutions to the differential equations, Section 4 presents applications of our proposed model, Section 5 describes performance measures, Section 6 gives numerical analysis, cost analysis and optimization and Section 7 presents conclusion.

## 2. THE MODEL DESCRIPTION

We consider a $M/M/c$ queuing system with two kinds of WVs and impatience customers.

- The arrival of customers according to a Poisson process with heterogeneous arrival rate $\lambda_i$, where
$$\lambda_i = \begin{cases} \lambda_0, & \text{arrival rate during busy period} \\ \lambda_1, & \text{arrival rate during 1st kind of WV} \\ \lambda_2, & \text{arrival rate during 2nd kind of WV} \end{cases}$$
- Each server has an independently and identically distributed exponential service time distribution with rate $\mu_b$ for busy period.
- Two kind of WVs are considered.
(i) 1st kind of WV: The server can only go on WV each time the system becomes empty.
(ii) 2nd kind of WV: If no customer is waiting in the system for service when he returns from a 1st kind of WV.
- During a 1st and 2nd kind of WV, the arriving customers are served at the rates of $\mu_{v_1}$ and $\mu_{v_2}$.

- We assume that the durations of both kinds of WVs are exponentially distributed with parameters $\phi_1$ and $\phi_2$.
- A customer who arrives and finds at least one customer (i.e c customers) in the system, when all the servers are on busy period (1st and 2nd kind of WV) either decides to enter the queue with probabilities $\beta_0$ ($\beta_1$ and $\beta_2$) or balk with probabilities $\beta'_0$ ($\beta'_1$ and $\beta'_2$) respectively.
- The servers activate an impatience timer, which is exponentially distributed with parameters $\epsilon_0, \epsilon_1$ and $\epsilon_2$, when a customer enters the system and realizes that the servers are busy, on 1st and 2nd kinds of WV.

The flow chart for our model is given in Figure 1.

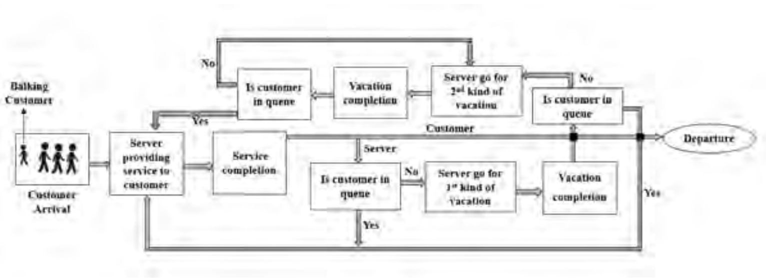


Figure 1: Flow chart diagram

Let $\mathcal{N}(t)$ be the number of customers in the system at time $t$, and $\mathcal{J}(t)$ represents the servers state at time $t$, where

$$\mathcal{J}(t) = \begin{cases} 0, \text{all the servers are in busy} \\ 1, \text{all the servers are in 1st kind of vacation} \\ 2, \text{all the servers are in 2nd kind of vacation} \end{cases}$$

The process $\{\mathcal{N}(t), \mathcal{J}(t); t \geq 0\}$ is defined as a continuous-time Markov process with a state space $\Omega = \{(0, j), j = 1, 2\} \bigcup \{(n, j), j = 0, 1, 2; n \geq 1\}$.
Let us define the steady state probabilities as,

$$P_{n,0} = \lim_{t \to \infty} P[\mathcal{N}(t) = n, \mathcal{J}(t) = j], n \geq 1$$

$$P_{n,j} = \lim_{t \to \infty} P[\mathcal{N}(t) = n, \mathcal{J}(t) = j], n \geq 0, j = 1, 2$$

The model can be described with the help of state-transition diagram which is given in Figure 2.

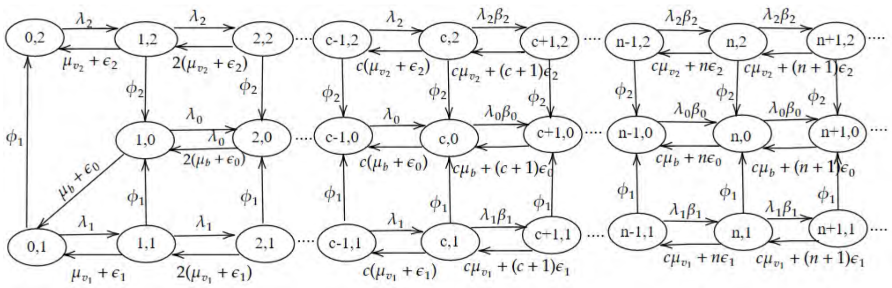


Figure 2: State-transition diagram

The steady state balance equations are presented as follows:

$$(\lambda_0+\mu_b+\epsilon_0)P_{1,0} = \phi_1P_{1,1}+\phi_2P_{1,2}+2(\mu_b+\epsilon_0)P_{2,0}, \tag{1}$$

$$\begin{aligned}(\lambda_0+n(\mu_b+\epsilon_0))P_{n,0} &= \phi_1P_{n,1}+\phi_2P_{n,2}+\lambda_0P_{n-1,0}\\ &\quad+(n+1)(\mu_b+\epsilon_0)P_{n+1,0}, 2\le n\le c-1\end{aligned} \tag{2}$$

$$\begin{aligned}(\lambda_0\beta_0+n(\mu_b+\epsilon_0))P_{n,0} &= \phi_1P_{n,1}+\phi_2P_{n,2}+\lambda_0P_{n-1,0}\\ &\quad+(c\mu_b+(n+1)\epsilon_0)P_{n+1,0}, n=c\end{aligned} \tag{3}$$

$$\begin{aligned}(\lambda_0\beta_0+c\mu_b+n\epsilon_0)P_{n,0} &= \phi_1P_{n,1}+\phi_2P_{n,2}+\lambda_0\beta_0P_{n-1,0}\\ &\quad+(c\mu_b+(n+1)\epsilon_0)P_{n+1,0}, n\ge c+1\end{aligned} \tag{4}$$

$$(\lambda_1+\phi_1)P_{0,1} = (\mu_b+\epsilon_0)P_{1,0}+(\mu_{v_1}+\epsilon_1)P_{1,1} \tag{5}$$

$$(\lambda_1+\mu_{v_1}+\epsilon_1+\phi_1)P_{1,1} = \lambda_1P_{0,1}+2(\mu_{v_1}+\epsilon_1)P_{2,1} \tag{6}$$

$$\begin{aligned}(\lambda_1+n(\mu_{v_1}+\epsilon_1)+\phi_1)P_{n,1} &= \lambda_1P_{n-1,1}+(n+1)(\mu_{v_1}+\epsilon_1)P_{n+1,1},\\ &\quad 2\le n\le c-1\end{aligned} \tag{7}$$

$$\begin{aligned}(\lambda_1\beta_1+n(\mu_{v_1}+\epsilon_1)+\phi_1)P_{n,1} &= \lambda_1P_{n-1,1}+((n+1)\epsilon_1+c\mu_{v_1})P_{n+1,1},\\ &\quad n=c\end{aligned} \tag{8}$$

$$\begin{aligned}(\lambda_1\beta_1+(c\mu_{v_1}+n\epsilon_1)+\phi_1)P_{n,1} &= \lambda_1\beta_1P_{n-1,1}+((n+1)\epsilon_1+c\mu_{v_1})P_{n+1,1},\\ &\quad n\ge c+1\end{aligned} \tag{9}$$

$$\lambda_2P_{0,2} = (\mu_{v_2}+\epsilon_2)P_{1,2}+\phi_1P_{0,1} \tag{10}$$

$$(\lambda_2+(\mu_{v_2}+\epsilon_2)+\phi_2)P_{1,2} = \lambda_2P_{0,2}+2(\mu_{v_2}+\epsilon_2)P_{2,2} \tag{11}$$

$$\begin{aligned}(\lambda_2+n(\mu_{v_2}+\epsilon_2)+\phi_2)P_{n,2} &= \lambda_2P_{n-1,2}+(n+1)\mu_{v_2}+\epsilon_2)P_{n+1,2},\\ &\quad 2\le n\le c-1\end{aligned} \tag{12}$$

$$\begin{aligned}(\lambda_2\beta_2+n(\mu_{v_2}+\epsilon_2)+\phi_2)P_{n,2} &= \lambda_2P_{n-1,2}+((n+1)\epsilon_2+c\mu_{v_2})P_{n+1,2},\\ &\quad n=c\end{aligned} \tag{13}$$

$$\begin{aligned}(\lambda_2\beta_2+c\mu_{v_2}+n\epsilon_2+\phi_2)P_{n,2} &= \lambda_2\beta_2P_{n-1,2}+((n+1)\epsilon_2+c\mu_{v_2})P_{n+1,2},\\ &\quad n\ge c+1\end{aligned} \tag{14}$$

The normalizing condition is as follows,

$$\sum_{n=1}^{\infty} P_{n,0} + \sum_{n=0}^{\infty} P_{n,1} + \sum_{n=0}^{\infty} P_{n,2} = 1$$

Define the PGFs as follows:
$G_0(z) = \sum_{n=1}^{\infty} P_{n,0}z^n$, $G_i(z) = \sum_{n=0}^{\infty} P_{n,i}z^n, i = 1, 2$
Multiplying (1) to (4) by $z^n$, summing all possible values of $n$, and we get

$$\begin{aligned}(1-z)(\lambda_0\beta_0 z - c\mu_b)G_0(z) - \epsilon_0 z(1-z)G_0'(z) &= z\phi_1 G_1(z) + z\phi_2 G_2(z)\\ &- [\phi_1 P_{0,1} + \phi_2 P_{0,2} + (\epsilon_0 + \mu_b)P_{1,0}]z\\ &+ \lambda_0 z(z-1)R_2(z)(1-\beta_0) + \mu_b(1-z)R_1(z)\end{aligned} \tag{15}$$

where $R_1(z) = \sum_{n=1}^{c-1}(n-c)P_{n,0}z^n$ and $R_2(z) = \sum_{n=1}^{c-1} P_{n,0}z^n$
In a similar way, we get from equations (5)-(14)

$$\begin{aligned}\epsilon_1 z(1-z)G_1'(z) - [(\lambda_1\beta_1 z - c\mu_{v_1})(1-z) + \phi_1 z]G_1(z) &= -(\mu_b + \epsilon_0)zP_{1,0}\\ &+ \lambda_1 zR_4(z)(1-z)(1-\beta_1) - \mu_{v_1}(1-z)R_3(z)\end{aligned} \tag{16}$$

$$\begin{aligned}\epsilon_2 z(1-z)G_2'(z) - [(\lambda_2\beta_2 z - c\mu_{v_2})(1-z) + \phi_2 z]G_2(z) &= -\mu_{v_2}(1-z)R_5(z)\\ &+ \lambda_2 zR_6(z)(1-z)(1-\beta_2) - z\phi_1 P_{0,1} - z\phi_2 P_{0,2}\end{aligned} \tag{17}$$

where $R_3(z) = \sum_{n=0}^{c-1}(n-c)P_{n,1}z^n$, $R_4(z) = \sum_{n=0}^{c-1} P_{n,1}z^n$,
$R_5(z) = \sum_{n=0}^{c-1}(n-c)P_{n,2}z^n$ and $R_6(z) = \sum_{n=0}^{c-1} P_{n,2}z^n$.

## 3. THE SOLUTIONS OF DIFFERENTIAL EQUATIONS

For $z \neq 0$ and $z \neq 1$, equation (15) can be written as follows:

$$\begin{aligned}G_0'(z) - \left[\frac{\lambda_0\beta_0}{\epsilon_0} - \frac{c\mu_b}{\epsilon_0 z}\right]G_0(z) &= \frac{A - \phi_1 G_1(z) - \phi_2 G_2(z)}{\epsilon_0(1-z)} - \frac{\mu_b R_1(z)}{\epsilon_0 z}\\ &+ \frac{\lambda_0 R_2(z)(1-\beta_0)}{\epsilon_0}\end{aligned} \tag{18}$$

where $A = \phi_1 P_{0,1} + \phi_2 P_{0,2} + (\epsilon_0 + \mu_b)P_{1,0}$
To solve the first order linear differential equation (18), we obtain an IF as $e^{\frac{-\lambda_0\beta_0 z}{\epsilon_0}} z^{\frac{c\mu_b}{\epsilon_0}}$
Multiplying both sides of (18) by IF, we get

$$\begin{aligned}\frac{d}{dz}\left[e^{\frac{-\lambda_0\beta_0 z}{\epsilon_0}} z^{\frac{c\mu_b}{\epsilon_0}} G_0(z)\right] = \Bigg[&\frac{A - \phi_1 G_1(z) - \phi_2 G_2(z)}{\epsilon_0(1-z)} + \frac{\lambda_0 R_2(z)(1-\beta_0)}{\epsilon_0}\\ &- \frac{\mu_b R_1(z)}{\epsilon_0 z}\Bigg] e^{\frac{-\lambda_0\beta_0 z}{\epsilon_0}} z^{\frac{c\mu_b}{\epsilon_0}}\end{aligned}$$

Integrating both sides of the above equation from 0 to $z$, we get

$$G_0(z) = e^{\frac{\lambda_0\beta_0 z}{\epsilon_0}} z^{\frac{-c\mu_b}{\epsilon_0}} \int_0^z \left[\frac{A - \phi_1 G_1(s) - \phi_2 G_2(s)}{\epsilon_0(1-s)} + \frac{\lambda_0 R_2(s)(1-\beta_0)}{\epsilon_0} - \frac{\mu_b R_1(s)}{\epsilon_0 s}\right] e^{\frac{-\lambda_0\beta_0 s}{\epsilon_0}} s^{\frac{c\mu_b}{\epsilon_0}} ds$$

For $z \neq 0$ and $z \neq 1$, equation (16) can be written as follows:

$$G_1'(z) - \left(\frac{\lambda_1\beta_1}{\epsilon_1} - \frac{c\mu_{v_1}}{\epsilon_1 z} + \frac{\phi_1}{\epsilon_1(1-z)}\right)G_1(z) = \frac{-(\mu_b+\epsilon_0)P_{1,0}}{\epsilon_1(1-z)} - \frac{\mu_{v_1}R_3(z)}{\epsilon_1 z} + \frac{\lambda_1 R_4(z)(1-\beta_1)}{\epsilon_1} \tag{19}$$

In order to solve the differential equation (19), we obtain an integrating factor (IF) as $e^{\frac{-\lambda_1\beta_1 z}{\epsilon_1}} z^{\frac{c\mu_{v_1}}{\epsilon_1}} (1-z)^{\frac{\phi_1}{\epsilon_1}}$.

$$\frac{d}{dz}\left[e^{\frac{-\lambda_1\beta_1 z}{\epsilon_1}} z^{\frac{c\mu_{v_1}}{\epsilon_1}} (1-z)^{\frac{\phi_1}{\epsilon_1}} G_1(z)\right] = \left[\frac{-(\mu_b+\epsilon_0)P_{1,0}}{\epsilon_1(1-z)} + \frac{\lambda_1 R_4(z)(1-\beta_1)}{\epsilon_1} - \frac{\mu_{v_1}R_3(z)}{\epsilon_1 z}\right] e^{\frac{-\lambda_1\beta_1 z}{\epsilon_1}} z^{\frac{c\mu_{v_1}}{\epsilon_1}} (1-z)^{\frac{\phi_1}{\epsilon_1}}$$

Integrating both sides of above from 0 to $z$, we get

$$G_1(z) = e^{\frac{\lambda_1\beta_1 z}{\epsilon_1}} z^{\frac{-c\mu_{v_1}}{\epsilon_1}} (1-z)^{\frac{-\phi_1}{\epsilon_1}} \left[-\frac{\mu_{v_1}}{\epsilon_1}\int_0^z R_3(z) e^{-\frac{\lambda_1\beta_1 s}{\epsilon_1}} s^{\frac{c\mu_{v_1}}{\epsilon_1}-1} (1-s)^{\frac{\phi_1}{\epsilon_1}} ds + \frac{\lambda_1(1-\beta_1)}{\epsilon_1}\int_0^z R_4(s) e^{-\frac{\lambda_1\beta_1 s}{\epsilon_1}} s^{\frac{c\mu_{v_1}}{\epsilon_1}} (1-s)^{\frac{\phi_1}{\epsilon_1}} ds - \frac{(\mu_b+\epsilon_0)P_{1,0}}{\epsilon_1} K_1(z)\right] \tag{20}$$

where $K_1(z) = \int_0^z e^{-\frac{\lambda_1\beta_1 s}{\epsilon_1}} s^{\frac{c\mu_{v_1}}{\epsilon_1}} (1-s)^{\frac{\phi_1}{\epsilon_1}-1} ds$

If $z \neq 0$ and $z \neq 1$, then equation (17) can be written as

$$G_2'(z) - \left(\frac{\lambda_2\beta_2}{\epsilon_2} - \frac{c\mu_{v_2}}{\epsilon_2 z} + \frac{\phi_2}{\epsilon_2(1-z)}\right)G_2(z) = \frac{\lambda_2 R_6(z)(1-\beta_2)}{\epsilon_2} - \frac{B}{\epsilon_2(1-z)} - \frac{\mu_{v_2}R_5(z)}{\epsilon_2 z} \tag{21}$$

where $B = \phi_1 P_{0,1} + \phi_2 P_{0,2}$.

In order to solve the differential equation (21), we obtain an integrating factor

(IF) as $e^{\frac{-\lambda_2\beta_2 z}{\epsilon_2}} z^{\frac{c\mu_{v2}}{\epsilon_2}} (1-z)^{\frac{\phi_2}{\epsilon_2}}$.
Integrating both sides of the above equation from 0 to $z$, we get

$$G_2(z) = \frac{e^{\frac{\lambda_2\beta_2 z}{\epsilon_2}} z^{\frac{-c\mu_{v2}}{\epsilon_2}} (1-z)^{\frac{-\phi_2}{\epsilon_2}}}{-\epsilon_2}\Bigg[\mu_{v_2}\int_0^z R_5(s)e^{-\frac{\lambda_2\beta_2 s}{\epsilon_2}} s^{\frac{c\mu_{v2}}{\epsilon_2}-1}(1-s)^{\frac{\phi_2}{\epsilon_2}} ds$$

$$-\lambda_2(1-\beta_2)\int_0^z R_6(s)e^{-\frac{\lambda_2\beta_2 s}{\epsilon_2}} s^{\frac{c\mu_{v2}}{\epsilon_2}} (1-s)^{\frac{\phi_2}{\epsilon_2}} ds + BK_6(z)\Bigg] \tag{22}$$

where $K_6(z) = \int_0^z e^{-\frac{\lambda_2\beta_2 s}{\epsilon_2}} s^{\frac{c\mu_{v2}}{\epsilon_2}} (1-s)^{\frac{\phi_2}{\epsilon_2}-1} ds$
By taking vertical cuts from State-transition diagram, we get
$P_{n,0} = \dfrac{\lambda_0^{n-1} P_{1,0}}{n!(\mu_b+\epsilon_0)^{n-1}}, n = 1, 2, ...c-1$
$P_{n,1} = \dfrac{\lambda_1^{n} P_{0,1}}{n!(\mu_{v1}+\epsilon_1)^{n}}, n = 0, 1, 2, ...c-1$
$P_{n,2} = \dfrac{\lambda_2^{n} P_{0,2}}{n!(\mu_{v2}+\epsilon_2)^{n}}, n = 0, 1, 2, ...c-1$
Substituting the above terms in $R_1(z), R_2(z), R_3(z), R_4(z), R_5(z)$ and $R_6(z)$.
$R_1(z) = \sum_{n=1}^{c-1}(n-c)\dfrac{\lambda_0^{n-1} P_{1,0}}{n!(\mu_b+\epsilon_0)^{n-1}} z^n$, $R_2(z) = \sum_{n=1}^{c-1}\dfrac{\lambda_0^{n-1} P_{1,0}}{n!(\mu_b+\epsilon_0)^{n-1}} z^n$
$R_3(z) = \sum_{n=0}^{c-1}(n-c)\dfrac{\lambda_1^{n} P_{0,1}}{n!(\mu_{v_1}+\epsilon_1)^{n}} z^n$, $R_4(z) = \sum_{n=0}^{c-1}\dfrac{\lambda_1^{n} P_{0,1}}{n!(\mu_{v_1}+\epsilon_1)^{n}} z^n$
$R_5(z) = \sum_{n=0}^{c-1}(n-c)\dfrac{\lambda_2^{n} P_{0,2}}{n!(\mu_{v_2}+\epsilon_2)^{n}} z^n$, $R_6(z) = \sum_{n=0}^{c-1}\dfrac{\lambda_2^{n} P_{0,2}}{n!(\mu_{v_2}+\epsilon_2)^{n}} z^n$
Substituting the $R_i(z), i = 1, 2...6$ in equation (20) and (22), we get

$$G_1(z) = e^{\frac{\lambda_1\beta_1 z}{\epsilon_1}} z^{\frac{-c\mu_{v_1}}{\epsilon_1}} (1-z)^{\frac{-\phi_1}{\epsilon_1}}\Bigg[\frac{-(\mu_b+\epsilon_0)P_{1,0}}{\epsilon_1}K_1(z) - \frac{\mu_{v_1}}{\epsilon_1}K_3(z)P_{0,1}$$

$$+\frac{\lambda_1(1-\beta_1)P_{0,1}}{\epsilon_1}K_2(z)\Bigg] \tag{23}$$

$$G_2(z) = \frac{e^{\frac{\lambda_2\beta_2 z}{\epsilon_2}} z^{-\frac{c\mu_{v2}}{\epsilon_2}} (1-z)^{\frac{-\phi_2}{\epsilon_2}}}{\epsilon_2}\Bigg[(\lambda_2(1-\beta_2)K_4(z) - \mu_{v_2}K_5(z))P_{0,2} - BK_6(z)\Bigg] \tag{24}$$

where
$K_2(z) = \int_0^z \sum_{n=0}^{c-1} \dfrac{\lambda_1^n}{n!(\mu_{v_1}+\epsilon_1)^n} s^n e^{-\frac{\lambda_1\beta_1 s}{\epsilon_1}} s^{\frac{c\mu_{v1}}{\epsilon_1}} (1-s)^{\frac{\phi_1}{\epsilon_1}} ds$

$$K_3(z) = \int_0^z \sum_{n=0}^{c-1} (n-c)\frac{\lambda_1^n}{n!(\mu_{v_1}+\epsilon_1)^n} s^n e^{-\frac{\lambda_1\beta_1 s}{\epsilon_1}} s^{\frac{c\mu_{v_1}}{\epsilon_1}-1}(1-s)^{\frac{\phi_1}{\epsilon_1}} ds$$

$$K_4(z) = \int_0^z \sum_{n=0}^{c-1} \frac{(\lambda_2)^n s^n}{n!(\mu_{v_2}+\epsilon_2)^n} e^{-\frac{\lambda_2\beta_2 s}{\epsilon_2}} s^{\frac{c\mu_{v_2}}{\epsilon_2}}(1-s)^{\frac{\phi_2}{\epsilon_2}} ds$$

$$K_5(z) = \int_0^z \sum_{n=0}^{c-1} \frac{(n-c)(\lambda_2)^n s^n}{n!(\mu_{v_2}+\epsilon_2)^n} e^{-\frac{\lambda_2\beta_2 s}{\epsilon_2}} s^{\frac{c\mu_{v_2}}{\epsilon_2}-1}(1-s)^{\frac{\phi_2}{\epsilon_2}} ds$$

From (23), put $z=1$ and we get,

$$P_{1,0} = \left[\frac{\lambda_1(1-\beta_1)K_2(1)-\mu_{v_1}K_3(1)}{(\epsilon_0+\mu_b)K_1(1)}\right]P_{0,1}$$

Similarly from (24), put $z=1$ and we get

$$P_{0,1} = \frac{[\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1)-\phi_2K_6(1)]P_{0,2}}{\phi_1K_6(1)}$$

Substitute (23) and (24) in $G_0(z)$,

$$G_0(z) = \frac{e^{\frac{\lambda_0\beta_0 z}{\epsilon_0}} z^{\frac{-c\mu_b}{\epsilon_0}}}{\epsilon_0}\Bigg[AK_7(z) + \frac{\phi_2P_{0,2}}{\epsilon_2}\left(\frac{\lambda_2(1-\beta_2)K_9(z)}{K_6(1)} + \frac{\mu_{v_2}K_{10}(z)}{K_6(1)}\right) + \frac{\phi_1P_{0,1}K_8(z)}{\epsilon_1} + \lambda_0(1-\beta_0)K_{11}(z)P_{1,0} - \mu_bK_{12}(z)P_{1,0}\Bigg] \quad (25)$$

where,

$$K_7(z) = \int_0^z e^{\frac{-\lambda_0\beta_0 s}{\epsilon_0}} s^{\frac{c\mu_b}{\epsilon_0}}(1-s)^{-1}ds$$

$$K_8(z) = \int_0^z \Bigg[\frac{(\lambda_1(1-\beta_1)K_2(1)-\mu_{v_1}K_3(1))K_1(s)}{K_1(1)} - \lambda_1(1-\beta_1)K_2(s) + \mu_{v_1}K_3(s)\Bigg] e^{\frac{\lambda_1\beta_1 s}{\epsilon_1}-\frac{\lambda_0\beta_0 s}{\epsilon_0}} s^{\frac{c\mu_b}{\epsilon_0}-\frac{c\mu_{v_1}}{\epsilon_1}}(1-s)^{\frac{-\phi_1}{\epsilon_1}-1}ds$$

$$K_9(z) = \int_0^z (-K_4(s)K_6(1)+K_4(1)K_6(s)) e^{\frac{\lambda_2\beta_2 s}{\epsilon_2}-\frac{\lambda_0\beta_0 s}{\epsilon_0}} s^{\frac{c\mu_b}{\epsilon_0}-\frac{c\mu_{v_2}}{\epsilon_2}}(1-s)^{\frac{-\phi_2}{\epsilon_2}-1}ds$$

$$K_{10}(z) = \int_0^z (K_5(s)K_6(1)-K_5(1)K_6(s)) e^{\frac{\lambda_2\beta_2 s}{\epsilon_2}-\frac{\lambda_0\beta_0 s}{\epsilon_0}} s^{\frac{c\mu_b}{\epsilon_0}-\frac{c\mu_{v2}}{\epsilon_2}}(1-s)^{\frac{-\phi_2}{\epsilon_2}-1}ds$$

$$K_{11}(z) = \int_0^z \sum_{n=1}^{c-1} \frac{\lambda_0^{(n-1)} s^n}{n!(\mu_b+\epsilon_0)^{(n-1)}} e^{-\frac{\lambda_0\beta_0 s}{\epsilon_0}} s^{\frac{c\mu_b}{\epsilon_0}} ds$$

$$K_{12}(z)=\int_0^z\sum_{n=1}^{c-1}\frac{(n-c)\lambda_0^{(n-1)}s^n}{n!(\mu_b+\epsilon_0)^{(n-1)}}e^{-\frac{\lambda_0\beta_0 s}{\epsilon_0}}s^{\frac{c\mu_b}{\epsilon_0}-1}ds$$

From (25), put $z=1$ and we get

$$P_{.,0}=G_0(1)=e^{\frac{\lambda_0\beta_0}{\epsilon_0}}\frac{P_{0,2}}{\epsilon_0}\Big[a_1K_7(1)+a_2K_8(1)+a_3+(\lambda_0(1-\beta_0)K_{11}(1)-\mu_bK_{12}(1))a_4\Big] \tag{26}$$

where

$$a_1=\left[\frac{\lambda_1(1-\beta_1)K_2(1)-\mu_{v_1}K_3(1)}{K_1(1)}\right]\left[\frac{\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1)-\phi_2K_6(1)}{\phi_1K_6(1)}\right]+\left[\frac{\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1)}{K_6(1)}\right]$$

$$a_2=\left[\frac{\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1)-\phi_2K_6(1)}{K_6(1)\epsilon_1}\right]$$

$$a_3=\frac{\phi_2}{\epsilon_2}\left[\frac{\lambda_2(1-\beta_2)K_9(1)}{K_6(1)}+\frac{\mu_{v_2}K_{10}(1)}{K_6(1)}\right]$$

$$a_4=\left[\frac{\lambda_1(1-\beta_1)K_2(1)-\mu_{v_1}K_3(1)}{K_1(1)(\epsilon_0+\mu_b)}\right]\left[\frac{\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1)-\phi_2K_6(1)}{\phi_1K_6(1)}\right]$$

From equation (16), put $z=1$ and using $P_{1,0}$ and $P_{0,1}$ value we get

$$P_{.,1}=G_1(1)=\frac{1}{\phi_1^2K_1(1)K_6(1)}[\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1)-\phi_2K_6(1)]P_{0,2}\times[\lambda_1(1-\beta_1)K_2(1)-\mu_{v_1}K_3(1)] \tag{27}$$

From (17), put $z=1$ and we get

$$P_{.,2}=G_2(1)=\frac{(\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1))P_{0,2}}{\phi_2K_6(1)} \tag{28}$$

Equations (26)-(28) are the probabilities of the servers on busy, on 1st and 2nd kinds of WVs respectively.
Adding equations (26)-(28) and using normalization condition, we get

$$P_{0,2}=\left\{\frac{e^{\frac{\lambda_0\beta_0}{\epsilon_0}}}{\epsilon_0}\Big[a_1K_7(1)+a_2K_8(1)+a_3+(\lambda_0(1-\beta_0)K_{11}(1)-\mu_bK_{12}(1))a_4\Big]+\Big[\frac{1}{\phi_1^2K_1(1)K_6(1)}(\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1)-\phi_2K_6(1))(\lambda_1(1-\beta_1)K_2(1)-\mu_{v_1}K_3(1))\Big]+\frac{\lambda_2(1-\beta_2)K_4(1)-\mu_{v_2}K_5(1)}{\phi_2K_6(1)}\right\}^{-1} \tag{29}$$

We use the condition to find the stability condition, $0 < P_{0,2} < 1$.
Substitute equation (29) into the above condition and then performing some algebraic manipulations, we get
$0 < \epsilon_0 \phi_1^2 \phi_2 K_6(1) K_1(1) < 1$.

## 4. APPLICATION OF THE MODEL

This model can be used in a variety of queueing systems found in real-world, including information transmission systems, flexible production systems, airports, toll booths and others.

- The model considered in this paper has applications in the manufacturing system as well. For example, consider the process of distributing a product (ghee) from manufacturing facility with impatient customers. The function of this center is to distribute ghee to fulfill customer orders. Here we assume that, there are many distributing agencies (c-servers). The manufacturing facility can produce ghee before demand in the form of stock making. However, the system administrator does not want to maintain a high inventory level because many of the items on the list result in an increase in hosting costs. If no orders are place at this time, the agency may decide to wait for the ghee orders (1st kind of WV). After checking the orders, if no one gives an order, then the agency will take a 2nd kind of WV. Upon arrival, the order may be fulfilled from inventory, if any production facility is available, or is temporarily out of stock. Customers whose orders are temporarily out of stock may become impatient and decide to cancel their orders if the customer's waiting period exceeds the customer's patience level.
- In this example, we assume that c check-in counters (c-servers) are available. An airline check-in counter where passengers line up in a single line and wait for one of several agents for service. Here we consider that the passenger service lines are not always busy. On that time the server may take a WV. When the server checks a passenger, other passengers wait for service. An impatient customers may either join a queue or balk and return at a later time. The server begins a 1st kind of WV, after checking all the passengers in the counter. On returning from this vacation, if no one is waiting for service, then the server will take a 2nd kind of WV. When the server comes back from either 1st kind or 2nd kind of WV, if any passengers comes to the counter for service continuously, the server changes to a busy period.
- Here, we consider that the scenario in hospital and assume that more than one doctors (c-doctors) are available. The patients are waiting for doctor consultation. Here we consider that the doctors are not busy for all the time. On that time the doctor may take a WV. When the doctor checks a patient, other patients wait for service. An impatient patients may either join a line or balk and return at a later time. The doctor begins a 1st kind of WV, after checking all the patients. On returning from this vacation, if no one is waiting for consultation, then the doctor will take a 2nd kind of WV. When the doctor comes back from either 1st kind or 2nd kind of WV, if any

patients comes to the line for consultation continuously, the doctor switches to a busy period.

## 5. PERFORMANCE MEASURES

Let $L_{s_b}$ be the average size of the system when all the servers are busy. Let $L_{s_{v_1}}$ and $L_{s_{v_2}}$ be the average size of the system when each server is on 1st and 2nd kinds of WV. We derive the average size of the systems $L_{s_b}, L_{s_{v_1}}$ and $L_{s_{v_2}}$. From equations (18), (19) and (21), we get

$$\begin{aligned} L_{s_b} &= \lim_{z\to 1} G_0'(z) \\ &= \frac{1}{\epsilon_0}\big[(\lambda_0\beta_0 - c\mu_b)G_0(1) + \phi_1 G_1'(1) + \phi_2 G_2'(1) + \lambda_0(1-\beta_0)R_2(1) \\ &\quad - \mu_b R_1(1)\big] \end{aligned} \tag{30}$$

$$\begin{aligned} L_{s_{v_1}} &= \lim_{z\to 1} G_1'(z) \\ &= \frac{1}{(\epsilon_1+\phi_1)(\phi_1^2 K_1(1)K_6(1))}\Big[(\lambda_2(1-\beta_2)K_4(1) - \mu_{v_2}K_5(1) - \phi_2 K_6(1)) \\ &\quad \times (\lambda_1\beta_1 - c\mu_{v_1})(\lambda_1(1-\beta_1)K_2(1) - \mu_{v_1}K_3(1))P_{0,2} \\ &\quad + (\phi_1^2 K_1(1)K_6(1))(\mu_{v_1}R_3(1))\Big] \end{aligned}$$

$$\begin{aligned} L_{s_{v_2}} &= \lim_{z\to 1} G_2'(z) \\ &= \frac{1}{(\epsilon_2+\phi_2)\phi_2 K_6(1)}\Big[(\lambda_2(1-\beta_2)K_4(1) - \mu_{v_2}K_5(1))P_{0,2} + (\phi_2 K_6(1)) \\ &\quad \times (\lambda_2\beta_2 - c\mu_{v_2})(\lambda_2(1-\beta_2)R_6(1) - \mu_{v_2}R_5(1))\Big] \end{aligned}$$

Substituting the values of $G_1'(1)$ and $G_2'(1)$ in (30), we get

$$\begin{aligned} \mathrm{L}_s = \tfrac{1}{\epsilon_0}\Bigg\{&(\lambda_0\beta_0 - c\mu_b)e^{\frac{\lambda_0\beta_0}{\epsilon_0}}\tfrac{P_{0,2}}{\epsilon_0}[a_1K_7(1) + a_2K_8(1) + a_3 + (\lambda_0(1-\beta_0)K_1(1) \\ &- \mu_b K_{12}(1))a_4] + \frac{1}{(\epsilon_1+\phi_1)(\phi_1 K_1(1)K_6(1))}\Big[(\lambda_1\beta_1 - c\mu_{v_1})(\lambda_2(1-\beta_2)K_4(1) \\ &- \mu_{v_2}K_5(1) - \phi_2K_6(1))(\lambda_1(1-\beta_1)K_2(1) - \mu_{v_1}K_3(1))P_{0,2} + (\phi_1^2K_1(1)K_6(1)) \\ &\times \mu_{v_1}R_3(1)\Big] + \frac{1}{(\epsilon_2+\phi_2)K_6(1)}\Big[(\lambda_2\beta_2 - c\mu_{v_2})(\lambda_2(1-\beta_2)K_4(1) - \mu_{v_2}K_5(1))P_{0,2} \\ &+ (\phi_2K_6(1))(\lambda_2(1-\beta_2)R_6(1) - \mu_{v_2}R_5(1))\Big] + \lambda_0(1-\beta_0)R_2(1) - \mu_b R_1(z)\Bigg\} \end{aligned}$$

Define $L_s = L_{s_{v_1}} + L_{s_{v_2}} + L_{s_b}$, where $L_s$ is the average size of the system.

**Special Case:**

Substituting $\lambda_0 = \lambda, \beta_0 = \theta, \mu_b = \beta\mu, \phi_1 = \phi, \phi_2 = 0$ and $\epsilon_0 = \sigma\epsilon_1$ in equation (30), we get the equation (67) of Kadi et al. [26].

## 6. NUMERICAL ANALYSIS

In this section we find performance measures numerically by using MATLAB software. We fix the parameters as $\lambda_0 = 12, \lambda_1 = 6, \lambda_2 = 5.5, \beta_0 = 0.9, \beta_1 = 0.2, \beta_2 = 0.1, \epsilon_0 = 25, \epsilon_1 = 9, \epsilon_2 = 3, \phi_1 = 0.8, \phi_2 = 0.5, \mu_b = 8.5, \mu_{v_1} = 7, \mu_{v_2} = 6.5$.

The impact of parameters $\lambda_0, \lambda_1$ and $\lambda_2$ on the mean system size $L_s$ with the variation of $\mu_b, \mu_{v_1}$ and $\mu_{v_2}$ are shown in Figures 3, 4 and 5. From Figures 3 to 5, we observe that if $\lambda_0, \lambda_1$ and $\lambda_2$ increases, the mean system size $L_s$ increases for lowering the values of $\mu_b, \mu_{v_1}$ and $\mu_{v_2}$. We observed from the Figures 3 to 5 that $L_s$ increases due to the increase in the arrival rate which is quite reasonable.

In Table 2, we increase the $\mu_b$ value 7.6 to 8.5. Then, the probabilities $P_{.,1}$ and $P_{.,2}$ increase and $P_{.,0}$ decreases. In Table 3, we increase the $\mu_{v_1}$ value from 7.1 to 8. Then, the probability $P_{.,2}$ increases and $P_{.,0}$ and $P_{.,1}$ decrease. In Table 4, we increase the $\mu_{v_2}$ value from 5.6 to 6.5. Then, the probability $P_{.,2}$ increases and $P_{.,0}$ and $P_{.,1}$ decrease. In Table 5, we increase the $\lambda_0$ value from 11.1 to 12. Then, the probability $P_{.,0}$ increases and $P_{.,1}$ and $P_{.,2}$ decrease. In Table 6, we increase the $\lambda_1$ value from 5.1 to 6. Then, the probability $P_{.,0}$ and $P_{.,1}$ increase and $P_{.,2}$ decreases. In Table 7, we increase the $\lambda_2$ value from 5.6 to 6.5. Then, the probability $P_{.,0}$ and $P_{.,2}$ increase and $P_{.,1}$ decreases.

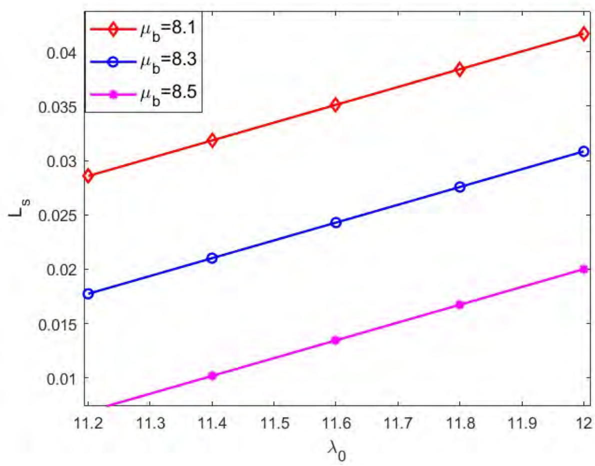


Figure 3: Average size of the system by varying the parameter $\mu_b$

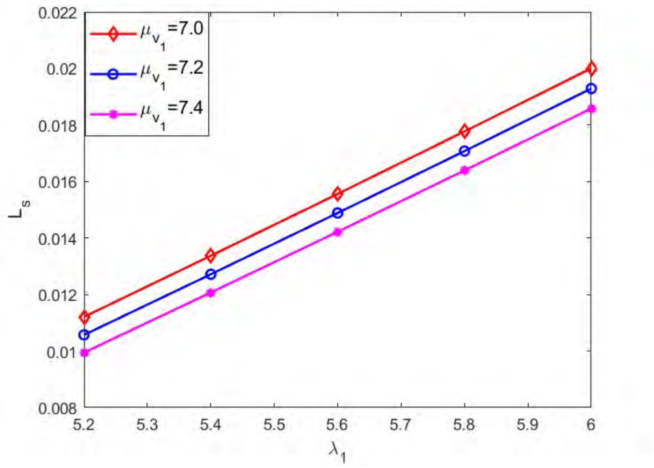


Figure 4: Average size of the system by varying the parameter $\mu_{v1}$

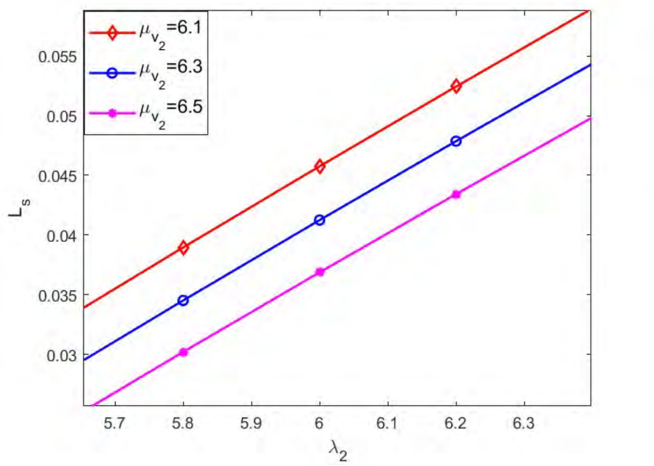


Figure 5: Average size of the system by varying the parameter $\mu_{v_2}$

### 6.1. Cost Analysis and Optimization

In this subsection, we develop a model for the costs obtained in this queueing system. Let us consider the below notations.

- $C_0$ - Cost per unit period whenever the servers are busy.
- $C_1$ - Cost per unit period whenever the servers are on 1st kind of WV.
- $C_2$ - Cost per unit period whenever the servers are on 2nd kind of WV.

Table 2: Effect of $\mu_b$ on probabilities

| $\mu_b$ | $P_{.,0}$ | $P_{.,1}$ | $P_{.,2}$ |
|---|---|---|---|
| 7.6 | 0.4594569 | 0.1477844 | 0.3927586 |
| 7.7 | 0.4593926 | 0.1478020 | 0.3928054 |
| 7.8 | 0.4593288 | 0.1478194 | 0.3928518 |
| 7.9 | 0.4592654 | 0.1478368 | 0.3928978 |
| 8.0 | 0.4592025 | 0.1478540 | 0.3929435 |
| 8.1 | 0.4591401 | 0.1478710 | 0.3929889 |
| 8.2 | 0.4590781 | 0.1478880 | 0.3930339 |
| 8.3 | 0.4590165 | 0.1479048 | 0.3930787 |
| 8.4 | 0.4589554 | 0.1479215 | 0.3931231 |
| 8.5 | 0.4588948 | 0.1479381 | 0.3931671 |

Table 3: Effect of $\mu_{v_1}$ on probabilities

| $\mu_{v_1}$ | $P_{.,0}$ | $P_{.,1}$ | $P_{.,2}$ |
|---|---|---|---|
| 7.1 | 0.4588546 | 0.1477825 | 0.3933629 |
| 7.2 | 0.4588148 | 0.1476285 | 0.3935567 |
| 7.3 | 0.4587754 | 0.1474761 | 0.3937485 |
| 7.4 | 0.4587364 | 0.1473252 | 0.3939384 |
| 7.5 | 0.4586978 | 0.1471759 | 0.3941263 |
| 7.6 | 0.4586596 | 0.1470281 | 0.3943123 |
| 7.7 | 0.4586218 | 0.1468818 | 0.3944964 |
| 7.8 | 0.4585844 | 0.1467369 | 0.3946787 |
| 7.9 | 0.4585473 | 0.1465936 | 0.3948591 |
| 8.0 | 0.4585107 | 0.1464516 | 0.3950377 |

- $C_q$ - Cost per unit period whenever a customer joins the queue and waits for service.
- $C_b$ - Cost per unit period whenever a customer balks.
- $C_r$ - Cost per unit period whenever a customer reneges, either during busy or both kinds of WV.
- $C_s$ - Cost per service per unit period.
- $C_F$ - Cost per unit to fixed server purchase.
- $TC$ - Expected total cost per unit period.

Table 4: Effect of $\mu_{v_2}$ on probabilities

| $\mu_{v_2}$ | $P_{.,0}$ | $P_{.,1}$ | $P_{.,2}$ |
|---|---|---|---|
| 5.6 | 0.4607195 | 0.1550000 | 0.3842805 |
| 5.7 | 0.4605089 | 0.1541850 | 0.3853060 |
| 5.8 | 0.4603004 | 0.1533779 | 0.3863218 |
| 5.9 | 0.4600938 | 0.1525784 | 0.3873279 |
| 6.0 | 0.4598892 | 0.1517865 | 0.3883244 |
| 6.1 | 0.4596865 | 0.1510021 | 0.3893114 |
| 6.2 | 0.4594857 | 0.1502251 | 0.3902891 |
| 6.3 | 0.4592869 | 0.1494555 | 0.3912576 |
| 6.4 | 0.4590899 | 0.1486932 | 0.3922169 |
| 6.5 | 0.4588948 | 0.1479381 | 0.3931671 |

Table 5: Effect of $\lambda_0$ on probabilities

| $\lambda_0$ | $P_{.,0}$ | $P_{.,1}$ | $P_{.,2}$ |
|---|---|---|---|
| 11.1 | 0.4587531 | 0.1479768 | 0.3932701 |
| 11.2 | 0.4587687 | 0.1479726 | 0.3932587 |
| 11.3 | 0.4587844 | 0.1479683 | 0.3932473 |
| 11.4 | 0.4588001 | 0.1479640 | 0.3932359 |
| 11.5 | 0.4588158 | 0.1479597 | 0.3932245 |
| 11.6 | 0.4588316 | 0.1479554 | 0.3932130 |
| 11.7 | 0.4588473 | 0.1479511 | 0.3932016 |
| 11.8 | 0.4588631 | 0.1479468 | 0.3931901 |
| 11.9 | 0.4588789 | 0.1479424 | 0.3931786 |
| 12.0 | 0.4588948 | 0.1479381 | 0.3931671 |

- $TR$ - Expected total revenue per unit period.
- $TP$ - Expected total profit per unit period.
- $E(L_q)$ - Average number of customers in the queue.
- $R_b$ - Average rate of balking.
- $R_r$ - Average rate of reneging.
- $R_a$ - Average rate of abandonment of a customer due to impatience.
- $E_s$ - Expected number of customers served per unit period.

Table 6: Effect of $\lambda_1$ on probabilities

| $\lambda_1$ | $P_{.,0}$ | $P_{.,1}$ | $P_{.,2}$ |
|---|---|---|---|
| 5.1 | 0.4577247 | 0.1434100 | 0.3988653 |
| 5.2 | 0.4578529 | 0.1439059 | 0.3982412 |
| 5.3 | 0.4579815 | 0.1444036 | 0.3976149 |
| 5.4 | 0.4581106 | 0.1449031 | 0.3969863 |
| 5.5 | 0.4582401 | 0.1454045 | 0.3963554 |
| 5.6 | 0.4583701 | 0.1459076 | 0.3957223 |
| 5.7 | 0.4585006 | 0.1464126 | 0.3950869 |
| 5.8 | 0.4586315 | 0.1469193 | 0.3944492 |
| 5.9 | 0.4587629 | 0.1474278 | 0.3938093 |
| 6.0 | 0.4588948 | 0.1479381 | 0.3931671 |

Table 7: Effect of $\lambda_2$ on probabilities

| $\lambda_2$ | $P_{.,0}$ | $P_{.,1}$ | $P_{.,2}$ |
|---|---|---|---|
| 5.6 | 0.4592365 | 0.1600977 | 0.3778656 |
| 5.7 | 0.4595716 | 0.1589814 | 0.3792703 |
| 5.8 | 0.4599004 | 0.1578444 | 0.3807012 |
| 5.9 | 0.4602231 | 0.1566861 | 0.3821587 |
| 6.0 | 0.4605396 | 0.1555061 | 0.3836436 |
| 6.1 | 0.4608503 | 0.1543038 | 0.3851566 |
| 6.2 | 0.4611552 | 0.1530787 | 0.3866983 |
| 6.3 | 0.4614545 | 0.1518302 | 0.3882694 |
| 6.4 | 0.4617483 | 0.1505576 | 0.3898707 |
| 6.5 | 0.4620367 | 0.1492605 | 0.3915031 |

$TC = C_0P_{.,0}+C_1P_{.,1}+C_2P_{.,2}+C_qE(L_q)+C_bR_b+C_rR_r+c(\mu_b+\mu_{v_1}+\mu_{v_2})C_s+cC_F$
$TR = RE_s$
$TP = TR - TC$
where $E(L_q) = L_s - cG_0(1) - R_1(1)$,
$R_b = (\lambda_0(1-\beta_0)+\lambda_1(1-\beta_1)+\lambda_2(1-\beta_2))\left(1-\sum_{n=1}^{c-1}P_{n,0}-\sum_{n=0}^{c-1}P_{n,1}-\sum_{n=0}^{c-1}P_{n,2}\right)$
$R_r = \epsilon_0 L_{s_b} + \epsilon_1 L_{s_{v_1}} + \epsilon_2 L_{s_{v_2}}$
$R_a = R_b + R_r$
$E_s = \mu_0[cG_0(1)+R_2(1)] + \mu_{v_1}[cG_1(1)+R_4(1)] + \mu_{v_2}[cG_2(1)+R_6(1)]$
From Figure 6, We fix the parameters as $\lambda_0 = 12, \lambda_1 = 6, \lambda_2 = 5.5, \beta_0 = 0.9, \beta_1 =$

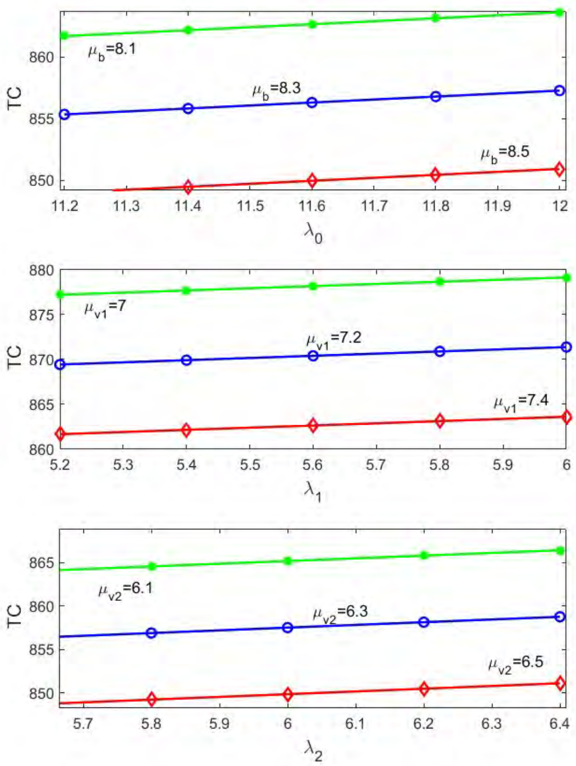


Figure 6: Total expected cost by varying the service rates

$0.2, \beta_2 = 0.1, \epsilon_0 = 25, \epsilon_1 = 9, \epsilon_2 = 3, \phi_1 = 0.8, \phi_2 = 0.5, \mu_b = 8.5, \mu_{v_1} = 7, \mu_{v_2} = 6.5, C_0 = 8, C_1 = 6, C_2 = 4, C_q = 8, C_b = 5, C_s = 4, C_r = 5, C_F = 4, R = 50.$

The impact of parameters $\lambda_0, \lambda_1$ and $\lambda_2$ on the total expected cost with the variation of $\mu_b, \mu_{v_1}$ and $\mu_{v_2}$ are shown in Figure 6. We know that, if the queue system size becomes large, the total expected cost per unit period of the system increase. However, in this Figure 6, we see that the total expected cost increase with $\lambda_0, \lambda_1$ and $\lambda_2$.

Here, we find the total expected cost function $(\mu_b)$ and $(\mu_{v_1})$ for this model. We obtain the optimal value for $\mu_b$ to minimize the cost. The expected cost function per unit period is given by,

$F(\mu_b) = C_0 P_{.,0} + C_1 P_{.,1} + C_2 P_{.,2} + C_q E(L_q) + C_b R_b + C_r R_r + c(\mu_b + \mu_{v_1} + \mu_{v_2})(C_s) + cC_F.$

The optimal cost can be formulated as $F(\mu_b^*) = minF(\mu_b)$.

Then, we develop the approximations to achieve the optimal values by direct search method.

From Table 8 and 9, we concluded that the minimum expected cost for $\mu_b$ and $\mu_{v_1}$ are given below.

$$F(4.2) = \begin{cases} 727.2481, \lambda_0 = 12.0 \\ 727.4922, \lambda_0 = 12.1 \\ 727.7364, \lambda_0 = 12.2 \\ 727.9806, \lambda_0 = 12.3 \\ 728.2248, \lambda_0 = 12.4 \end{cases}$$

$$F(8.57) = \begin{cases} 884.9760, \lambda_1 = 4.3 \\ 885.3760, \lambda_1 = 4.4 \\ 885.7760, \lambda_1 = 4.5 \\ 886.1760, \lambda_1 = 4.6 \\ 886.5760, \lambda_1 = 4.7 \end{cases}$$

Table 8: Effect of $\mu_b$ on cost function

| $\mu_b$ | 4.15 | 4.2 | 4.25 | 4.3 | 4.35 |
|---|---|---|---|---|---|
| $\lambda_0$=12 | 730.3220 | **727.2481** | 728.8317 | 730.4154 | 731.9991 |
| $\lambda_0$=12.1 | 730.8508 | **727.4922** | 729.0758 | 730.6595 | 732.2432 |
| $\lambda_0$=12.2 | 731.3796 | **727.7364** | 729.3200 | 730.9036 | 732.4872 |
| $\lambda_0$=12.3 | 731.9084 | **727.9806** | 729.5641 | 731.1477 | 732.7313 |
| $\lambda_0$=12.4 | 732.4372 | **728.2248** | 729.8082 | 731.3918 | 732.9754 |

Table 9: Effect of $\mu_{v_1}$ on cost function

| $\mu_{v_1}$ | 8.54 | 8.57 | 8.6 | 8.63 | 8.66 |
|---|---|---|---|---|---|
| $\lambda_1$=4.3 | 919.3741 | **884.9760** | 886.1460 | 887.3160 | 888.4860 |
| $\lambda_1$=4.4 | 919.6104 | **885.3760** | 886.5460 | 887.7160 | 888.8860 |
| $\lambda_1$=4.5 | 919.8463 | **885.7760** | 886.9460 | 888.1160 | 889.2860 |
| $\lambda_1$=4.6 | 920.0832 | **886.1760** | 887.3460 | 888.5160 | 889.6860 |
| $\lambda_1$=4.7 | 920.3206 | **886.5760** | 887.7460 | 923.8179 | 924.9838 |

## 7. CONCLUSION

We have analyzed an c-server queueing system with two kinds of WVs and impatient customers. Our queueing model approach is examined using PGFs. The minimum expected cost function is calculated using the MATLAB software. The steady state probabilities, various performance measure, cost analyses and some numerical analysis are presented in this paper. In future, this model can be develop by $c$-server with various kinds of vacations, customer feedback, breakdown and impatience customers.

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