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OPTIMAL SOLUTION OF PENTAGONAL FUZZY TRANSPORTATION PROBLEM USING A NEW RANKING TECHNIQUE

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Abstract: In this article, we propose a method based on a new ranking technique to find optimal solution for a pentagonal fuzzy transportation problem. Firstly, the proposed ranking method which is based on the centroid concept is applied. This transforms the pentagonal fuzzy transportation problem to crisp transportation problem and then the proposed algorithm is applied to find optimal solution of the problem in crisp form. Also, a new method to find initial basic feasible solution (IBFS) of crisp valued transportation problems is introduced in the paper. Further, we give two numerical illustrations for the newly proposed algorithm and compare the solution obtained with the solutions of existing methods. The proposed method can easily be understood and applied to real life transportation problems. Moreover, the proposed ranking method can be used to solve various other fuzzy problems of operations research.

Keywords: Fuzzy transportation problem, fuzzy number, Pentagonal fuzzy number, ranking function, optimal solution.

MSC: 03B52, 03E72, 49Q22, 90B06.

1. INTRODUCTION

The present world is competitive and full of industries and material goods. So, there is always a need to reduce the resultant cost of commodities as much as possible. One of the various factors that influence the final cost of commodity is the transportation cost of raw materials as well as finished goods. So, reduction in this cost leads to significant reduction in the final price of commodity. This need leads to formulation of transportation problems in decision making domain.

Transportation Problem (TP) [1] is one of the optimization problems which deals with finding the optimal cost of transporting commodities from various sources to different destinations. In classical problems of decision making, parameters are considered in crisp form. However, in real-life situations, due to various factors such as insufficient input information, bad statistical analysis, fluctuations in financial market, condition of roads, etc., difficulty arises in estimating the actual values of parameters of the problem such as transportation cost, demand and supply values. To deal with this, opinions of decision makers (DM) are sought to predict the values of parameters. Often, DMs give the values of parameters in linguistic terms. These terms are handled by considering parameters in the forms which are capable of handling and representing uncertainty, such as fuzzy numbers [2], generalized trapezoidal-valued intuitionistic fuzzy numbers [3]. This condition gives rise to various forms of decision making problems. One of these problems is Fuzzy Transportation Problem (FTP) in which atleast one parameter is considered in the form of fuzzy numbers.

As described by Kaur and Kumar [4], the following circumstances may lead to formulation of FTP: (i) there is some sort of uncertainty associated with unit transportation cost due to lack of information, fluctuation in fuel prices or some other reasons, (ii) decision maker cannot initially determine the exact value of availability at origins, (iii) market demand and hence demand at various destinations of a newly launched product cannot be crisply determined or it possess some sort of vagueness. There are many authors in the literature who have discussed FTPs and their solution methods in their work. ÓhÉigeartaigh [5] proposed an algorithm to solve FTP with crisp transportation cost and fuzzy demand and availability. Chanas et al. [6] used parametric programming approach to solve TPs with fuzzy demand and supply. Chanas and Kutcha [7] developed an algorithm to obtain crisp optimal solution of transportation problem with unit transportation costs as fuzzy numbers by transforming the problem to a bicriterial TP. Liu and Kao [8] used extension principle to develop a method which fuzzily determines the optimal transportation cost of a fully fuzzy TP. Gani and Razak [9] used parametric approach for two stage FTP with demand and availability in the form of trapezoidal fuzzy numbers. Li et al. [10] proposed goal programming approach for FTP with crisp demand and availability but fuzzy cost. Lin [11] introduced genetic algorithm for finding the best solution of TP with fuzzy demand and supply values. Their algorithm also includes ranking fuzzy numbers using sign-distance measurement to convert the fuzzy problem to defuzzified form. Dinagar and Palanivel [12] considered parameters of TP as trapezoidal fuzzy numbers and proposed fuzzy MODI method to obtain its optimal solution in fuzzy form. Pandian and Natarajan [13] developed fuzzy zero point method for FTPs. They [14] also introduced a new algorithm based on this method to find optimal more-for-less solution for a fully fuzzy TP with mixed constraints. Kaur and Kumar [15] proposed generalized fuzzy forms of north-west corner method, least-cost method, Vogel's approximation method (VAM) to find the IBFS followed by generalized fuzzy MODI method to obtain fuzzy optimal solution. They [16] also proposed a method using ranking function for generalized trapezoidal fuzzy TP with fuzzy cost and crisp demand and supply values. Later, Ebrahimnejad [17] introduced a computationally more efficient solution method for FTP of same kind which was simpler than the method proposed in [16]. Ebrahimnejad [18] introduced a new approach to obtain fuzzy optimal solution of bounded TP with fuzzy demand and availability. This method was based on bounded dual simplex method. Bisht and Srivastava [19] proposed one-point approach for fully fuzzy trapezoidal transportation problem. Their method involves point-wise breakup of trapezoidal fuzzy numbers, thereby converting the problem to four distinct crisp problems whose solutions are clubbed to get fuzzy optimal solution. Recently, Kane et al. [20] proposed a two-step method for fully FTP by first converting the problem to two interval transportation problems and then converting these obtained problems to crisp problems using mid-point value. Pratihar et al. [21] proposed modified VAM for fully fuzzy TP with parameters as interval type-2 fuzzy numbers.

One of the methods to solve FTPs is by converting parameters of the problem from fuzzy form to crisp form with the aid of ranking techniques and then apply the classical methods for solving the problem. The concept of ranking fuzzy numbers was introduced by Jain [22]. Recently, Bisht and Dangwal [23], [24] proposed new ranking functions and applied them to solve game problem and interval valued transportation problems through fuzzy approach, respectively. Many defuzzification methods have been proposed by various authors and used in different decision making problems. Basirzadeh [25] utilized the ranking function defined by them for trapezoidal fuzzy numbers to convert FTP to classical TP and then applied the classical methods of solving TP to find the optimal solution. De and Beg [26] proposed defuzzification method for dense fuzzy sets. Mathur et al. [27] converted the fuzzy parameters to crisp form using ranking function and further applied minimum demand-supply followed by MODI method to attain optimal solution of trapezoidal FTP. Mitlif et al. [28] converted the problem to crisp form using proposed novel ranking function and then applied VAM followed by MODI method to obtain optimal solution. Bisht and Dangwal [29] proposed ranking function for octagonal fuzzy numbers and applied it to find optimal transportation cost for FTP.

As mentioned above, at initial stage, DMs usually give linguistic descriptions of parameters, which are intrinsically imprecise. So, the basic problem that arises is mathematical modeling of this imprecise information. Fuzzy set theory (FST) plays a vital role in representation of imprecision, specially, ambiguity related to natural language. As a result, fuzzy numbers and operators on them were developed, which form a mathematical foundation of application of FST. A fuzzy number refers to a connected set of possible values, where each possible value has its own weight (membership degree) between 0 and 1. Thus, it can be interpreted as generalization of a real number. Theoretical definition of fuzzy numbers and arithmetic operations on them are computationally complex for direct implementation. Hence, to reduce this complexity and encourage application of FST, special membership functions and hence different fuzzy numbers are introduced. One of them is pentagonal fuzzy number. As the name suggests, it has a pentagonal shape and can be defined by five numbers representing the vertices of pentagonal. Although triangular and trapezoidal fuzzy numbers are widely used but they are associated with three, four parameters respectively. However, DMs come across situations where real-life problems are concerned with five parameters. For example, if availability at some source (A_1) is "greater than 127 tons and less than 133 tons" i.e., the chances of availability being 127 or 133 are very low. Then, it can be written in the form of interval number $[127, 133]$. But, DM also knows that the chances of availability being 130 tons are very high and also roughly has idea that there are medium chances of availability being 129 tons or 132 tons. Then approximate value of $\tilde{\tilde{A}}_1$ can be expressed using values 127, 129, 130, 132, 137 tons by considering different degrees of membership. This indicates that availability can be described by a pentagonal fuzzy number $A_1 = (127, 129, 130, 132, 133)$. To handle such situations, we have considered parameters as PFNs in our paper. As a result, pentagonal fuzzy transportation problem is formulated.

In this paper, a new method is proposed to solve FTP with pentagonal fuzzy parameters, which is based on a new ranking technique and a new method to find IBFS. Numerical examples are solved using the proposed algorithm and the IBFS obtained is compared with the solutions obtained using existing methods to illustrate the advantage of this method. The main contributions of the paper are
(i) A new ranking function is proposed for PFNs.
(ii) A new simplified method to find IBFS is proposed, which leads to an IBFS, closer to the optimal solution in comparison to the IBFS obtained using some other methods in the literature.
(iii) In contrast to some methods existing in the literature, we have proposed solution method for problem in which all the parameters are considered in fuzzy form.
(iv) Using the existing relation between PFNs and trapezoidal, triangular fuzzy numbers, the proposed approach and the ranking function can also be used for TPs dealing with triangular and trapezoidal fuzzy numbers.

The outlay of rest of the paper is as follows: Section 2 presents some basic definitions. In Section 3, a new ranking technique based on centroid concept is proposed for pentagonal fuzzy numbers. A new method to find IBFS is introduced in Section 4. In Section 5, algorithm to find optimal solution of pentagonal fuzzy transportation problem is presented. Further, numerical examples are given in

Section 6. Comparison of solutions obtained using the proposed method as well as some existing methods is illustrated in Section 7. Conclusion is given in Section 8 along with merits of the method proposed in the paper.

Shortcoming of existing methods

- In the existing methods [30, 31], for fully fuzzy TP with PFNs, negative numbers in values of x_{ij}s exist. But, negative quantity of commodity has no physical meaning.
- In the existing methods [32, 33, 13], negative numbers exists in fuzzy optimal cost and also in quantity to be shipped from origins to destinations, which again has no physical interpretation.
- In the existing method [19], the value of one of the allocations in the example considered by them is $x_{33} = (2, 0, 6, 8)$. Here, first value i.e., 2 is greater than the second value, which implies that it is not a trapezoidal fuzzy number. Therefore, the result obtained using their method cannot be well interpreted by decision maker.
- The existing methods [34, 15, 35] can be applied to solve only those FTPs in which demand and supply values are crisp numbers and only unit transportation costs are in fuzzy form.
- The existing method [9] can used for only those FTPs in which demand and supply values are fuzzy numbers whereas unit transportation costs are in crisp form.

Advantages of proposed method

- In our solution approach, we do not use goal programming or parametric approach, which cannot be applied easily in real-life situations.
- The approach proposed in this paper can easily be coded in any programming language.
- The proposed approach can be understood and applied easily.
- The proposed approach does not involve arithmetic operations as well as comparison of fuzzy numbers, which are complex. Thus, the proposed approach reduces computational complexity.
- The optimal cost obtained by the proposed method is in crisp form, which can be compared easily.

2. PRELIMINARIES

Definition 1 (Fuzzy Number [29]). *A fuzzy number $\tilde{\tilde{A}}$ is a normal and convex fuzzy subset of real line $\mathbb{R}$ such that it's membership function $\mu_{\tilde{\tilde{A}}} : \mathbb{R} \to [0, 1]$ is piece-wise continuous in its domain.*

Definition 2 (Pentagonal Fuzzy Number (PFN) [24]). *A fuzzy number* $\tilde{\tilde{A}}^P = (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5; \omega_1, \omega_2)$ *is said to be a PFN (see Figure 1) if it satisfies the following properties:*
(i) $\mu_{\tilde{\tilde{A}}^P}(x)$ *is a function which is continuous in [0,1].*
(ii) $\mu_{\tilde{\tilde{A}}^P}(x)$ *is continuous and strictly increasing function in intervals* $[\varrho_1, \varrho_2]$ *and* $[\varrho_2, \varrho_3]$.
(iii) $\mu_{\tilde{\tilde{A}}^P}(x)$ *is continuous and strictly decreasing function in intervals* $[\varrho_3, \varrho_4]$ *and* $[\varrho_4, \varrho_5]$.
Here, $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ *and* ϱ_5 *are real numbers such that* $\varrho_1 \leq \varrho_2 \leq \varrho_3 \leq \varrho_4 \leq \varrho_5$. ω_1 *and* ω_2 *are the grades of points* ϱ_2 *and* ϱ_4 *respectively,* $\mu_{\tilde{\tilde{A}}^P}(x)$ *is the membership function of PFN and is defined as:*

$$\mu_{\tilde{\tilde{A}}^P}(x; \omega_1, \omega_2) = \begin{cases} \omega_1 \left(\dfrac{x - \varrho_1}{\varrho_2 - \varrho_1} \right), & \textit{if } \varrho_1 \leq x \leq \varrho_2 \\ 1 - (1 - \omega_1) \left(\dfrac{x - \varrho_3}{\varrho_2 - \varrho_3} \right), & \textit{if } \varrho_2 \leq x \leq \varrho_3 \\ 1, & \textit{if } x = \varrho_3 \\ 1 - (1 - \omega_2) \left(\dfrac{x - \varrho_3}{\varrho_4 - \varrho_3} \right), & \textit{if } \varrho_3 \leq x \leq \varrho_4 \\ \omega_2 \left(\dfrac{x - \varrho_5}{\varrho_4 - \varrho_5} \right), & \textit{if } \varrho_4 \leq x \leq \varrho_5 \\ 0, & \textit{otherwise} \end{cases}$$

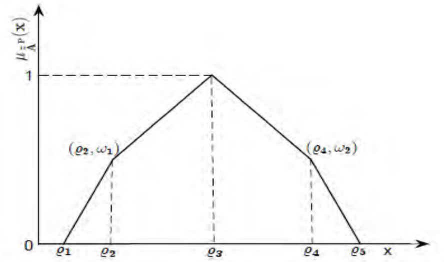

Figure 1: Pentagonal Fuzzy Number

From this generalized form of pentagonal fuzzy number, two other types of fuzzy numbers, namely, trapezoidal fuzzy number (TrFN) and triangular fuzzy number (TFN) can be conceptualized as:

Case I: If $\omega_1 = \omega_2 = 0$, then PFN reduces to **TFN** (fig. 2) i.e.; $\tilde{\tilde{A}}^P = (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) \equiv \tilde{\tilde{A}}^T = (\varrho_2, \varrho_3, \varrho_4)$ and

$$\mu_{\tilde{\tilde{A}}^T}(x) = \begin{cases} 1 - \left(\dfrac{x - \varrho_3}{\varrho_2 - \varrho_3} \right), & \text{if } \varrho_2 \leq x \leq \varrho_3 \\ 1, & \text{if } x = \varrho_3 \\ 1 - \left(\dfrac{x - \varrho_3}{\varrho_4 - \varrho_3} \right), & \text{if } \varrho_3 \leq x \leq \varrho_4 \\ 0, & \text{otherwise} \end{cases}$$

Case II: If $\omega_1 = \omega_2 = 1$, then PFN reduces to **TrFN** (fig 3) i.e; $\tilde{\tilde{\varrho}} = (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) \equiv \tilde{\tilde{A}}^{Tr} = (\varrho_1, \varrho_2, \varrho_4, \varrho_5)$ and

$$\mu_{\tilde{\tilde{A}}^{Tr}}(x) = \begin{cases} \left(\dfrac{x-\varrho_1}{\varrho_2-\varrho_1}\right), & \text{if } \varrho_1 \le x \le \varrho_2 \\ 1, & \text{if } \varrho_2 \le x \le \varrho_4 \\ \left(\dfrac{x-\varrho_5}{\varrho_4-\varrho_5}\right), & \text{if } \varrho_4 \le x \le \varrho_5 \\ 0, & \text{otherwise} \end{cases}$$

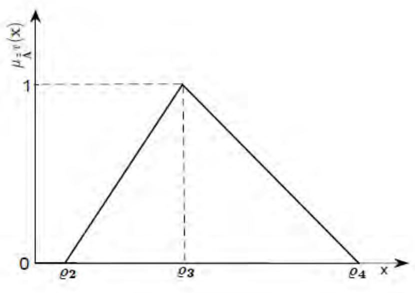

Figure 2: Triangular Fuzzy Number

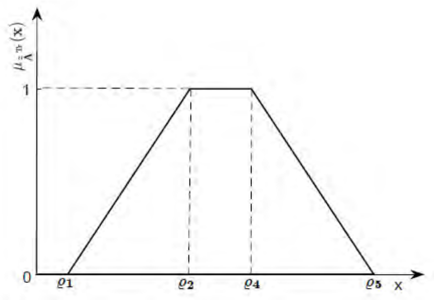

Figure 3: Trapezoidal Fuzzy Number

Definition 3 (Pentagonal Fuzzy Transportation Problem (PFTP)). *In real-world problems, the data available is not always in the crisp form due to various reasons, such as inaccuracy in measurement, change in cost with time, weather conditions etc. Rather, it may possess some fuzziness. A TP in which atleast one parameter is in the form of PFNs is called PFTP and is formulated as:*

$$Min\ Z = \sum_{i=1}^{M}\sum_{j=1}^{N} \tilde{\tilde{C}}_{ij}^{P} X_{ij}$$

$$\textit{subject to} \qquad \sum_{j=1}^{N} X_{ij} \le \tilde{\tilde{A}}_i^P; \qquad i = 1, 2, 3, ..., M \tag{1}$$

$$\sum_{i=1}^{M} X_{ij} \ge \tilde{\tilde{D}}_j^P; \qquad j = 1, 2, 3, ..., N \tag{2}$$

$$\textit{and} \qquad X_{ij} \ge 0\ ; i = 1, 2, 3, ..., M \ \ \textit{and} \ \ j = 1, 2, 3, ..., N \tag{3}$$

Here,

- M*: number of sources;*
- N*: number of destinations;*
- $\tilde{\tilde{A}}_i^P$*: pentagonal fuzzy supply at* i^{th} *origin;*
- $\tilde{\tilde{D}}_j$*: pentagonal fuzzy demand at* j^{th} *destination;*

$\tilde{\tilde{C}}^P_{ij}$: *pentagonal fuzzy cost of transportation of unit product from* i^{th} *origin to* j^{th} *destination;*
X_{ij}: *amount to be transported from* i^{th} *origin to* j^{th} *destination such that the total transportation cost is minimized;*
$\sum_{j=1}^{N} \tilde{\tilde{A}}^P_i$: *total pentagonal fuzzy availability of the product;*
$\sum_{j=1}^{M} \tilde{\tilde{D}}^P_j$: *total pentagonal fuzzy demand of the product.*

A necessary and sufficient condition for existence of solution is $\sum_{i=1}^{M} \tilde{\tilde{A}}^P_i = \sum_{j=1}^{N} \tilde{\tilde{D}}^P_j$ *i.e, the problem must be balanced. If problem is unbalanced, then it must be converted to balanced problem by introducing dummy source or origin.*

Table 1: Tabular form of transportation problem

Warehouses / Factories	W_1	W_2	W_3	.....	W_N	Availability
F_1	$\tilde{\tilde{C}}^P_{11}$	$\tilde{\tilde{C}}^P_{12}$	$\tilde{\tilde{C}}^P_{13}$	.....	$\tilde{\tilde{C}}^P_{1N}$	$\tilde{\tilde{A}}^P_1$
F_2	$\tilde{\tilde{C}}^P_{21}$	$\tilde{\tilde{C}}^P_{22}$	$\tilde{\tilde{C}}^P_{23}$	.....	$\tilde{\tilde{C}}^P_{2N}$	$\tilde{\tilde{A}}^P_2$
. . .	. . .	. . .	. . .	. . .	. . .	. . .
F_M	$\tilde{\tilde{C}}^P_{M1}$	$\tilde{\tilde{C}}^P_{M2}$	$\tilde{\tilde{C}}^P_{M3}$	.....	$\tilde{\tilde{C}}^P_{MN}$	$\tilde{\tilde{A}}^P_M$
Demand	$\tilde{\tilde{D}}^P_1$	$\tilde{\tilde{D}}^P_2$	$\tilde{\tilde{D}}^P_3$	.....	$\tilde{\tilde{D}}^P_N$	

3. PROPOSED RANKING TECHNIQUE

Ranking fuzzy numbers is a primary as well as an essential problem of fuzzy arithmetic, especially in the field of decision making. When parameters of the problem are treated as fuzzy numbers, often, it is required to quantify and compare the data before taking any decision. Using a proper ranking method facilitates appropriate results, whereas an improper ranking can mislead the solutions. Because of this, ranking becomes an important component of the decision making process. Thus, many methods of ranking fuzzy numbers have been introduced by various authors. But, ordering and comparison of fuzzy numbers is challenging. Since, natural order exists in real numbers, it was suggested to extend this ordering to fuzzy numbers by converting them to real numbers. A ranking function is a function, say, $R^{\text{PFN}} : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$, where $\mathcal{F}(\mathbb{R})$ denotes the set of fuzzy numbers defined on real line $\mathbb{R}$. It maps fuzzy number to a unique real number. In this section, we propose a new ranking function for PFNs.

Consider a pentagonal fuzzy number $\tilde{\tilde{A}}^P = (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5; \omega_1, \omega_2)$ as shown in Figure 1. Extend the line joining $A(\varrho_1, 0)$ and $B(\varrho_2, w_1)$ and also the line joining $E(\varrho_5, 0)$ and $D(\varrho_4, w_2)$. Let the intersection of these lines be $F(x, y)$ (see Figure 4). Then,

$$x = \frac{\omega_1(\varrho_1\varrho_5 - \varrho_1\varrho_4) - \omega_2(\varrho_1\varrho_5 - \varrho_2\varrho_5)}{\omega_1(\varrho_5 - \omega_2(\varrho_1 - \varrho_2)}, \quad y = \omega_1\left(\frac{x - \varrho_1}{\varrho_2 - \varrho_1}\right),$$

***Case: If $\boldsymbol{\omega_1 = \omega_2}$**, then

$$x = \frac{\varrho_2\varrho_5 - \varrho_1\varrho_4}{\varrho_5 - \varrho_4 - \varrho_1 + \varrho_2}, \qquad y = \frac{x - \varrho_1}{\varrho_2 - \varrho_1}$$

Remark 4. *We will consider $\omega_1 = \omega_2$ in the numerical examples.*

Now, join F to C and let $G(s,t)$ be the mid point of FC. Then,

$$s = \frac{x + \varrho_3}{2}\ , \ t = \frac{1 + y}{2}$$

Now, Ranking function $R^{PFN}(\tilde{\tilde{A}}^P)$ which is based on the concept of centroid of triangle GAE (see Figure 5) is defined as

$$R^{\text{PFN}}(\tilde{\tilde{A}}^P) = \frac{\varrho_1 + \varrho_5 + s}{3}$$

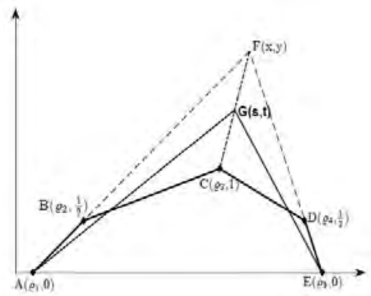

Figure 4: Normal Pentagonal Fuzzy Number

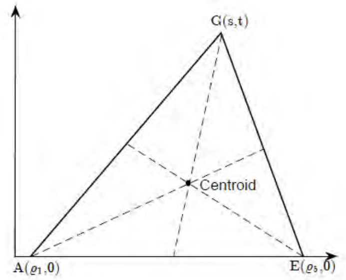

Figure 5: Centroid of triangle GAE

Using the above ranking function, comparison of two PFNs $\tilde{\tilde{Y}}^P$ and $\tilde{\tilde{Z}}^P$ can be done in the following way:
(i) If $R^{\text{PFN}}(\tilde{\tilde{Y}}^P) < R^{\text{PFN}}(\tilde{\tilde{Z}}^P)$ then $\tilde{\tilde{Y}}^P \tilde{<} \tilde{\tilde{Z}}^P$.
(ii) If $R^{\text{PFN}}(\tilde{\tilde{Y}}^P) > R^{\text{PFN}}(\tilde{\tilde{Z}}^P)$ then $\tilde{\tilde{Y}}^P \tilde{>} \tilde{\tilde{Z}}^P$.
(iii) If $R^{\text{PFN}}(\tilde{\tilde{Y}}^P) = R^{\text{PFN}}(\tilde{\tilde{Z}}^P)$ then $\tilde{\tilde{Y}}^P \tilde{=} \tilde{\tilde{Z}}^P$.

Wang and Kerre [36] listed some axioms as reasonable properties of ordering approaches for ordering fuzzy properties. They further examined these properties in respect of various ranking approaches proposed in the literature and compiled the results in the form of a table. It is worthwhile to note that only one ordering procedure of all the procedures considered by them satisfies all the axioms. However, most of them fail to satisfy one or more axioms. The ranking procedure proposed by us also satisfies most of the axioms stated by them and it can be compared to other ranking approaches proposed for PFNs as depicted in Table 2.

Table 2: Comparison of ordering using different approaches

Example Sets	Removal Area method	Alpha-cut method	Centroid method	Vidhya and Ganesan method [37]	Chakraborty et al. method [38]	Selvam et al. method [39]	Proposed method
(1) $\tilde{\tilde{S}}^P=(1,2,3,4,5)$ $\tilde{\tilde{T}}^P=(-2,-1,0,1,2)$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$
(2) $\tilde{\tilde{S}}^P = (1,3,4,6,7)$ $\tilde{\tilde{T}}^P = (0,2,5,7,8)$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \simeq \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$
(3) $\tilde{\tilde{S}}^P = (1,3,4,6,7)$ $\tilde{\tilde{T}}^P = (1,2,3,6,10)$	$\tilde{\tilde{S}}^P \simeq \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$
(4) $\tilde{\tilde{S}}^P = (1,3,4,6,7)$ $\tilde{\tilde{T}}^P = (1,2,4,5,7)$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \simeq \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$
(5) $\tilde{\tilde{S}}^P = (1,3,4,6,7)$ $\tilde{\tilde{T}}^P = (0,1,3,8,9)$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \simeq \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{>} \tilde{\tilde{T}}^P$	$\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$

Remark 1 : From Table 2, it can be noted that Alpha-cut method, Removal-area method, Vidhya & Ganesan method, Chakraborty et al. method give equal values for Example sets (2), (3), (4) and (5) respectively. Therefore, these methods fail to rank PFNs mentioned in the above examples. However, the method proposed by us can rank these PFNs.

Remark 2 :The ranking method proposed by us has consistency in ranking fuzzy numbers and their images, i.e., $R^{\text{PFN}}(-\tilde{\tilde{S}}) = -R^{\text{PFN}}(\tilde{\tilde{S}})$. Thus, if $\tilde{\tilde{S}} \tilde{<} \tilde{\tilde{T}}$, then $(-\tilde{\tilde{S}}) \tilde{>} (-\tilde{\tilde{T}})$. However, this is not true in case of Sevlam et al. proposed method. For example, let $\tilde{\tilde{S}}^P = (1,2,3,4,5)$, and $\tilde{\tilde{T}}^P = (1,3,4,6,7)$. Then, by our ranking approach $R^{\text{PFN}}(\tilde{\tilde{S}}^P) = 3$ and $R^{\text{PFN}}(\tilde{\tilde{T}}^P) = 4.16$. Thus, $\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$. Slso, $R^{\text{PFN}}(-\tilde{\tilde{S}}^P) = -3$ and $R^{\text{PFN}}(-\tilde{\tilde{T}}^P) = -4.16$. Thus, $-\tilde{\tilde{S}}^P \tilde{>} -\tilde{\tilde{T}}^P$.
However, by ranking approach of Sevlam et al. [39], $R_{\tilde{\tilde{S}}^P} = 3.017$ and $R_{\tilde{\tilde{T}}^P} = 4.26$. Thus, $\tilde{\tilde{S}}^P \tilde{<} \tilde{\tilde{T}}^P$. Tut, $R^{\text{PFN}}(-\tilde{\tilde{S}}^P) = 3.017$ and $R^{\text{PFN}}(-\tilde{\tilde{T}}^P) = 4.26$. Thus, $-\tilde{\tilde{S}}^P \tilde{<} -\tilde{\tilde{T}}^P$.

4. A NEW METHOD TO FIND IBFS (PRODUCT METHOD)

The following are the steps to find IBFS:

Step 1: Examine the problem for balanced condition.

Step 2: If the problem is balanced then go to step 4. If it is unbalanced, transform it to balanced TP.

Case I: If total availability is greater than total demand i.e., $\sum A_i > \sum D_j$, introduce a dummy destination having all costs zero and demand $\sum A_i - \sum D_j$.

Case II: If total demand is greater than total availability i.e., $\sum A_i < \sum D_j$, introduce a dummy source having all costs zero and availability $\sum D_j - \sum A_i$. Then, go to step 3.

Step 3: Do first allocation in one of the cells of dummy source or destination according to the following two cases:

Case I: If a dummy destination (say j^{th} destination) has been introduced in step 2, then write the second minimum cost for each row in the front of each row. Select maximum of these values. Let this value is corresponding to i^{th} row. Then allocate $x_{ij} = \min(A_i, B_j)$ in the ij^{th} cell.

Case II: If a dummy source (say i^{th} source) has been introduced in step 2, then write the second minimum cost for each column at the bottom of each column. Select maximum of these values. Let this value is corresponding to j^{th} column. Then allocate $X_{ij} = \min\{A_i, B_j\}$ in the ij^{th} cell.

After allocation, following three cases arise:

(i) If $\min\{A_i, D_j\} = A_i$, then replace D_j by $D_j - A_i$.

(ii) If $\min(A_i, D_j) = D_j$, then replace A_i by $A_i - D_j$.

(iii) If $A_i = B_j$, then follow any one of the above two cases.

Now go to step 4.

Step 4: For each row, compute the product of maximum and minimum cost in that row and write it in the front of each row.

Similarly, write the product for each column in the bottom of each column.

Find the row or column corresponding to which this product is maximum.

Choose the cell (say ij^{th} cell) with the minimum cost in the selected row or column and allocate $X_{ij} = \min(A_i, D_j)$ in that cell.

Replace D_j by $D_j - A_i$ or A_i by $A_i - D_j$ and obtain a new fuzzy transportation table by ignoring i^{th} row or j^{th} column accordingly as $\min(A_i, D_j)$= A_i or D_j respectively.

Step 5: Calculate the product for reduced fuzzy transportation table and repeat step 5 until all allocations are done and IBFS is obtained.

5. PROPOSED ALGORITHM

The following are the steps to find optimal solution of PFTP:

Step 1 : Write down the problem in the form of Table 1.
Step 2 : Use the proposed ranking technique to transform the fuzzy problem to crisp transportation problem.
Step 3 : Apply the above proposed product method to obtain IBFS of the problem.
Step 4 : Use MODI method to check if the IBFS obtained is optimal or not.
Step 5 : If not, repeat MODI method until we arrive at optimal solution.
Step 6 : Calculate optimum (minimum) transportation cost.

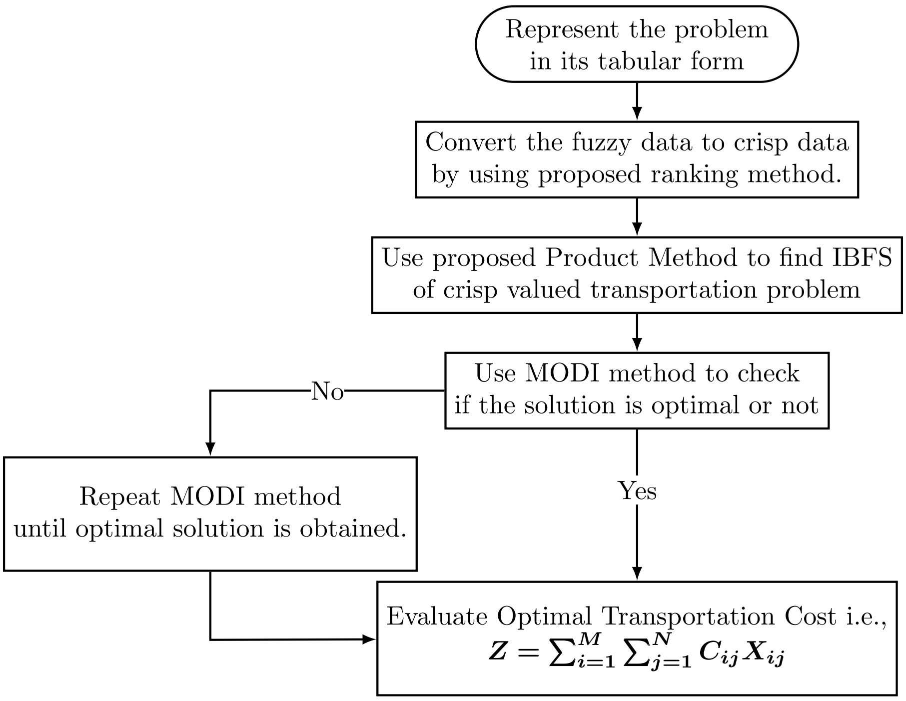

Figure 6: Flowchart for proposed algorithm

6. NUMERICAL ILLUSTRATIONS

6.1. Example 1

Three factories F_1, F_2, F_3 of a company has availabilities $(127, 129, 130, 132, 133)$; $(147, 148, 150, 151, 153)$ and $(167, 169, 170, 172, 173)$, respectively. These factories supply to four warehouses W_1, W_2, W_3, W_4 with demands $(87, 88, 90, 92, 93)$; $(97, 99, 100, 102, 103)$; $(137, 139, 140, 142, 143)$ and $(117, 118, 120, 121, 123)$, respectively. The transportation cost is given in Table 3.

Table 3: Fuzzy unit transportation cost: Example 1

Factories↓ Warehouses→	W_1	W_2	W_3	W_4
F_1	(7,8,9,12,13)	(9,11,12,13,15)	(12,13,15,16,18)	(5,6,8,9,11)
F_2	(11,12,13,15,17)	(8,9,11,13,14)	(6,7,9,10,12)	(7,8,10,11,13)
F_3	(17,19,20,21,23)	(2,4,6,7,8)	(4,5,7,9,10)	(15,18,19,20,21)

Solution:
Step 1: Problem is converted to tabular form (Table 4).

Table 4: Tabular Form: Example 1

	W_1	W_2	W_3	W_4	*Supply*
F_1	(7, 8, 9, 12, 13)	(9, 11, 12, 13, 15)	(12, 13, 15, 16, 18)	(5, 6, 8, 9, 11)	(127, 129, 130, 132, 133)
F_2	(11,12,13,15,17)	(8,9,11,13,14)	(6,7,9,10,12)	(7,8,10,11,13)	(147, 148, 150, 151, 153)
F_3	(17,19,20,21,23)	(2,4,6,7,8)	(4,5,7,9,10)	(15,18,19,20,21)	(167, 169, 170, 172, 173)
Demand	(87, 88, 90, 92, 93)	(97, 99, 100, 102, 103)	(137, 139, 140, 142, 143)	(117, 118, 120, 121, 123)	

Step 2: The reduced crisp TP on converting the fuzzy data to crisp values using proposed ranking technique is shown in Table 5.

Table 5: Defuzzified data: Example 1

Factories↓ Warehouses→	W_1	W_2	W_3	W_4	*Supply*
F_1	9.83	12	14.83	7.83	130.16
F_2	13.66	11	8.83	9.83	149.83
F_3	20	5.33	7	18.41	170.16
Demand	90	100.16	140.16	119.83	

Step 3: The problem is balanced. So, we go to step 4.
The IBFS obtained on application of **steps 4, 5, 6 and 7** of our algorithm is:
$\mathbf{x_{11} = 90}$, $\mathbf{x_{14} = 40.16}$, $\mathbf{x_{22} = 70.16}$, $\mathbf{x_{24} = 79.67}$, $\mathbf{x_{32} = 100.16}$, $\mathbf{x_{33} = 70}$.
We now apply MODI method to find optimal transportation problem.
Step 8: The solution obtained (Table 6) is examined for optimality condition. It comes out to be optimal.
Thus, the optimum transportation cost is

$$\mathbf{Z = 9.83x90 + 7.83x40.16 + 8.83x70.16 + 9.83x79.67 + 5.33x100.16 + 7x70} = 3625.67$$

Table 6: Optimal Solution: Example 1

Factories↓ Warehouses→	W_1	W_2	W_3	W_4	***Supply***
F_1	**90** / 9.83	– / 12	– / 14.83	**40.16** / 7.83	130.16
F_2	– / 13.66	– / 11	**70.16** / 8.83	**79.67** / 9.83	149.83
F_3	– / 20	**100.16** / 5.33	**70** / 7	– / 18.41	170.16
Demand	90	100.16	140.16	119.83	

6.2. Example 2

Consider the FTP taken by Geetha and Selvakumari [40].
A company has four factories A, B, C, D with production quantities 30, 27, 40, 50 respectively and four warehouses P, Q, R, S with demands 20, 40, 34 and 53 respectively. The unit transportation cost is given in Table 7. Find the optimal transportation cost.

Table 7: Unit cost of transportation: Example 2

Factories↓ Warehouses→	**P**	**Q**	**R**	**S**
A	$(2,4,6,8,9)$	$(3,5,7,8,9)$	$(2,4,5,6,7)$	$(3,4,6,7,12)$
B	$(0,2,5,6,8)$	$(4,5,6,8,11)$	$(2,3,5,7,11)$	$(1,5,6,9,11)$
C	$(1,2,3,4,5)$	$(2,3,4,6,8)$	$(4,5,6,8,9)$	$(6,7,8,9,13)$
D	$(3,5,6,7,8)$	$(1,5,6,7,8)$	$(2,7,8,9,10)$	$(3,3,4,5,9)$

Solution:

Step 1: Problem is converted to tabular form (Table 8).
Step 2: The crisp TP obtained using proposed ranking technique is shown in Table 9.

Table 8: Tabular form: Example 2

Factories↓ Warehouses→	**P**	**Q**	**R**	**S**	***Supply***
A	$(2,4,6,8,9)$	$(3,5,7,8,9)$	$(2,4,5,6,7)$	$(3,4,6,7,12)$	30
B	$(0,2,5,6,8)$	$(4,5,6,8,11)$	$(2,3,5,7,11)$	$(1,5,6,9,11)$	27
C	$(1,2,3,4,5)$	$(2,3,4,6,8)$	$(4,5,6,8,9)$	$(6,7,8,9,13)$	40
D	$(3,5,6,7,8)$	$(1,5,6,7,8)$	$(2,7,8,9,10)$	$(3,3,4,5,9)$	50
Demand	20	40	34	53	

Table 9: Defuzzified form: Example 2

Factories↓ Warehouses→	**P**	**Q**	**R**	**S**	**Supply**
A	5.77	6.33	4.72	6.75	30
B	4.16	6.95	5.8	6.27	27
C	3	4.66	6.41	8.9	40
D	5.72	5.1	6.77	5.16	50
Demand	20	40	34	53	

Step 3: The problem is balanced. So, we go to step 4.
The IBFS obtained on application of **steps 4, 5, 6 and 7** is:
$\mathbf{x_{13} = 30}$, $\mathbf{x_{21} = 20}$, $\mathbf{x_{23} = 4}$, $\mathbf{x_{24} = 3}$, $\mathbf{x_{32} = 40}$, $\mathbf{x_{44} = 50}$.

We now apply MODI method to find optimal transportation problem.

Step 8: The solution obtained (Table 10) is checked for optimality. It comes out to be optimal.

Thus, the optimal transportation cost is
$\mathbf{Z = 4.72 \text{ x } 30 + 4.16 \text{ x } 20 + 5.8 \text{ x } 4 + 6.27 \text{ x } 3 + 4.66 \text{ x } 40 + 5.16 \text{ x } 50}$
$\mathbf{= 711.21}$

Table 10: Optimal Solution: Example 2

Factories↓ Warehouses→	**P**	**Q**	**R**	**S**	**Supply**
A	– / 5.77	– / 6.33	**30** / 4.72	– / 6.75	30
B	**20** / 4.16	– / 6.95	**4** / 5.8	**3** / 6.27	27
C	– / 3	**40** / 4.66	– / 6.41	– / 8.9	40
D	– / 5.72	– / 5.1	– / 6.77	**50** / 5.16	50
Demand	20	40	34	53	

7. COMPARISON OF THE RESULT

7.1. Comparison with existing methods of finding IBFS

Table 11 and Table 12 present the comparison of the solutions obtained by proposed method with some existing methods.

The algorithm put forward by us first uses the proposed method to find IBFS followed by MODI method. The advantage of using this combination is that the product method gives IBFS closer to the optimal solution (in most of the problems), which reduces the number of iterations to obtain optimal solution and MODI method ensures the optimality of the solution. These methods when applied successively, eventually leads us to optimal solution of the TP in lesser time and involving lesser computations.

The method proposed by us gives optimal solution in crisp form. Different authors have expressed contrasting point of views in this matter. Although, it has some limitations, but obtaining a crisp optimal solution makes its comparison with the solutions obtained using different methods, easier. Also, due to this, the solution can be interpreted easily as it is free of uncertainty. As a result, decision making process becomes less complicated.

Table 11: Comparison Table: Example 1

Methods	***IBFS***	***Optimum Solution***	***No. of iterations***
North-West corner method [41]	5378.19	3625.67	5
CoR Method [42]	3771.47	3625.67	2
LCM [1]	3771.47	3625.67	2
Russell's Approximation Method (RAM) [43]	3895.09	3625.67	2
VAM [44]	3625.67	3625.67	1
Gorhe and Ghadle Method (GGM)[45]	3625.67	3625.67	1
Proposed Method	**3625.67**	**3625.67**	**1**

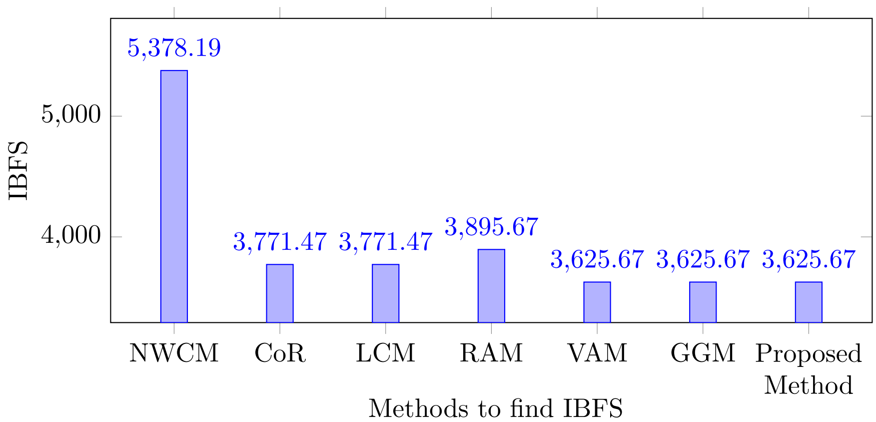

Figure 7: Comparison of optimum solution obtained for example 1 using different methods

From Figures 7 and 8, we can conclude that the proposed algorithm is more reliable and time-saving in comparison to some of the existing methods.

Table 12: Comparison table: Example 2

Methods	***IBFS***	***Optimum Solution***	***No. of iterations***
North-West corner method [41]	882.97	711.21	5
CoR Method [42]	719.01	711.21	2
Russell's Approximation Method (RAM) [43]	719.01	711.21	2
LCM [1]	719.01	711.21	2
VAM [44]	719.01	711.21	2
Gorhe and Ghadle's Method (GGM)[45]	711.21	711.21	1
Geetha and Selvakumari's Method (GSM) [40]	838	838	1
Proposed Method	**711.21**	**711.21**	**1**

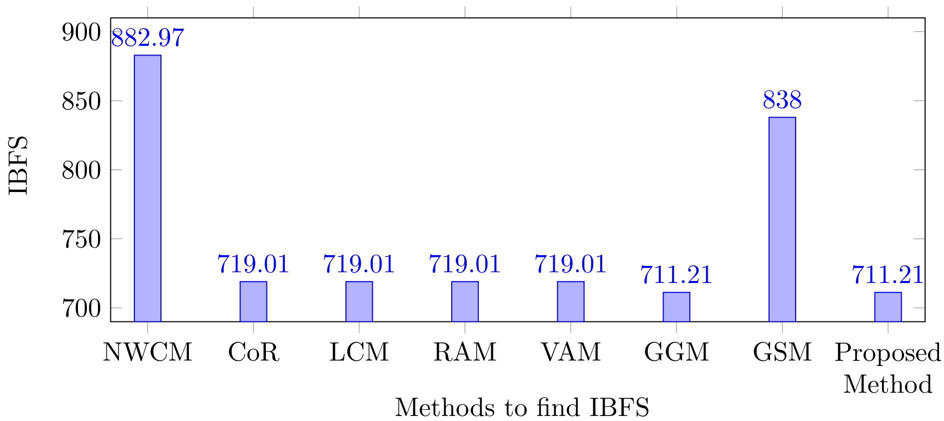

Figure 8: Comparison of optimum solution obtained for example 2 using different methods

7.2. Comparison of the solutions obtained on converting the pentagonal transportation problem to trapezoidal and triangular transportation problem

As discussed in the Definition 2, PFN is converted to TrFN and TFN by taking $w_1 = w_2 = 1$ and $w_1 = w_2 = 0$ respectively.

We converted the examples considered above to trapezoidal and triangular TPs and then compared the results attained by the methods put forward by different authors.

It can be concluded from table 13 that the solutions vary for the same problem, when converted to triangular and trapezoidal TPs. However, this difference is very slight (negligible in some case). Also, when the trapezoidal problem corresponding to example 2 is solved using the method proposed by Kaur and Kumar [16], the optimal solution obtained is very close to solution obtained by our method but the procedure of our method is comparatively more efficient and less tedious.

Table 13: Comparison table

	Pentagonal FTP		**Trapezoidal FTP**		**Triangular FTP**	
	Proposed Method		**(1) Poonam et al. Proposed Method [46]**		**(1) Poonam et al. Proposed Method [47]**	
	IBFS	Optimal Solution	IBFS	Optimal Solution	IBFS	Optimal Solution
Example 1	3625.67	3625.67	3618.06	3618.06	3623.18	3623.18
Example 2	711.21	711.21	725	705	647.25	647.25
			(2)Mathur et al. Proposed Method [27]		**Kumar and Subramanian Proposed Method [48]**	
			IBFS	Optimal Solution	IBFS	Optimal Solution
Example 1			3618.06	3618.06	3776.5	3630.75
Example 2			748	705	655.39	647.09

8. CONCLUSION

This research article proposes an algorithm to solve PFTP in which first the proposed ranking technique and then the proposed new method to find IBFS of crisp valued transportation problem is applied. The merits of the method proposed in this paper are as follows:

(i) The proposed ranking technique easily converts the pentagonal fuzzy numbers to crisp numbers.

(ii) The solution is obtained as a crisp number which makes its comparison, with existing methods, easier.

(iii) In case, hexa-section fuzzification approach [49] is used to convert interval data to pentagonal fuzzy numbers, this ranking technique converts it into a crisp number which is just the mid-point of the interval. Hence, solving interval data based transportation problem using this technique along with hexa-section fuzzification approach becomes very easy.

(iv) The solution obtained by this method is very close to the optimal solution. Hence, number of iterations to obtain optimal solution is comparatively less.

Also, it can be deduced from the comparison of the solution with other methods that this method is more effective and less tedious than the existing methods, since the IBFS obtained by our method is found to be very close to the optimal solution. Thus, this method is of great importance in industrial field. Moreover, this ranking technique can also be applied to problems of some other fields in which data is in the form of pentagonal fuzzy number.

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