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# A MAXIMUM FLOW NETWORK INTERDICTION MODEL IN FUZZY STOCHASTIC HYBRID UNCERTAINTY ENVIRONMENTS

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**Abstract:** Uncertainty is an inherent characteristic of a decision-making process. Occasionally, historical data may be insufficient to accurately estimate the probability distribution suitable for an unknown variable. In these situations, we deal with fuzzy stochastic variables in solving a problem. As a result, decision-makers, particularly those in the military, are confronted with numerous issues. This article discusses the maximum network flow interdiction under fuzzy stochastic hybrid conditions. The capacity of arcs has been treated as a fuzzy stochastic variable in this problem. The primary objective of this paper is to propose a model to the decision-maker that can be used to manage unknown factors in the network. Since this topic is explored concurrently in a stochastic and fuzzy environment, it is impossible to solve it directly. Consequently, three probability-possibility, probability-necessity, and probability-credibility techniques are utilized to transform it into a deterministic state. Eventually, the proposed model's efficacy is demonstrated by presenting a numerical example.

**Keywords:** Network interdiction, fuzzy stochastic programming, probability-possibility approach, possibility measure, probability measure.

**MSC:** 90B15, 90B18, 90C15, 90C26, 90C90.

## 1. INTRODUCTION

Living in the information age, whether as a service provider or as a consumer, is all affected by the undeniable role of communication networks. These networks include computer networks, social networks, railways, airlines, and land networks. However, some of these services may threaten other countries and their residents, making it critical to identify these dangers and threats in the first stage and attempt to mitigate or prevent them in the subsequent steps. Undoubtedly, the most challenging aspect of the work is proper decision-making and adopting appropriate strategies in such scenarios, especially for military commanders who cannot evaluate their own decisions except during times of war. In these instances, formulation is one of the most crucial tools for analyzing the decision-making process, particularly in military domains, gaining experience, and even debugging errors without spending excessive money, which enables military decision-makers to assess their techniques, methods, and capabilities in a non-combat environment to discover prospective flaws. In recent decades, numerous scholars have considered the Network Interdiction Problem (NIP), which covers a wide range of networks. Moreover, the NIP is a combinatorial optimization problem, one of the most significant bi-level programming problems. In these types of problems, there is always a network in which the network user (attacker) in the first level attempts to optimize its target function. In contrast, the interdictor in the second level, with its limited budget, attempts to minimize this optimal value for the network user. As a result, we have a network in which each arc, in addition to the usual parameters, is assigned a parameter called the cost of interdiction. In its simplest form, this problem was introduced during the Vietnam War in the 1960s to disrupt the movement of Vietnamese soldiers and warfare equipment by U.S. military researchers [1]. Thus, the majority of early models concentrated on military applications, followed by applications such as counter-trafficking (drugs, weapons, humans, and nuclear materials [2], identifying the susceptibility of infrastructure networks to terrorist attacks [3], controlling infectious diseases such as swine flu, Ebola [4], and, more recently, coronavirus in emergencies, and monitoring computer networks [5], which increased the importance of this topic for researchers in the fields of military, network security, and healthcare. In the NIP, in general, traveling the shortest route [6], discovering the most reliable path [7], and lastly, delivering the maximum flow from the source node to the destination node [8] could be considered one of the attacker's most critical acts. Alternatively, the interdictor endeavors to inflict the greatest damage on the attacker's actions with its limited budget. These problems are classified into three broad categories: the shortest path interdiction problem, the most reliable path interdiction problem [9], and the maximum flow interdiction problem [10] in the literature. As novel research conducted in this field, a new interdiction problem called the minimum st-cut interdiction was proposed by Abdolahzadeh et al. [11], in which the attacker aims at selecting the least amount of st-cut, provided it intersects every possible path between the source and destination. Instead, the defender wishes to maximize the minimal cut amount by increasing the capacity of the arcs within a given budget. Xiao *e*t al. [12] investigated a two-objective model for the shortest network path interdiction problem with nodal interdiction. Forghani *e*t al. [13] established a two-objective model for the partial interdiction problem on capacitated hierarchical facilities. Their model has been recommended to anticipate and address the detrimental impacts of a malicious attack on multi-layered hierarchical capacitated facilities. Bigdeli *e*t al. [14] considered mathematically modeling the problem of identifying the enemy traversal path in ground defense. By solving their proposed model, the best solution is given to help the commanders of the ground unit to prevent the transfer of enemy forces and equipment. The majority of research in this field is focused on deterministic and stochastic interdictions as well as the study of the interdiction format via the game theory, also referred to as dynamic interdiction [15, 16, 17, 18, 19]. As the term "deterministic interdiction" suggests, the interdictor has comprehensive knowledge of a network's data and characteristics in this format. Conversely, some information in stochastic interdiction is unclear, e.g., the interdictor's edge capacities.

In this research, the focus will be on the maximum flow interdiction problem. In such cases, the goal of the interdictor is to prevent the flow of some undesired items (such as contraband, hostile weaponry, etc.) within a capacitated network. The interdictor accomplishes its objective by altering some of the network properties, such as the capacity of arcs. Numerous studies have been conducted on this subject. For instance, Ratliff *e*t al. [20] were among the first to address the cardinality problem or determine the most vital K of an arc. Janjarassuk *e*t al. [21] described interdiction variables using Bernoulli's random variable and presented a mixed-integer programming problem with several exponential constraints utilizing linearization methods. Chauhan [22] investigated a NIP based on the maximum flow, considering the arc capacity uncertainty and interdiction resource use. Afsharirad [23] proposed a novel interpretation of the network maximum flow interdiction problem, defining the concept of "optimal cut." Additionally, He offered a heuristic algorithm to acquire an approximate value for cut and its error bound. Mirzaei *e*t al. [24] examined the interdiction problem of a smuggling network, which involves organizing police efforts to prevent criminals from smuggling commodities successfully. This research is concerned with asymmetric information, unpredictable conditions, multi-commodity, and various sources and destinations in the literature about the network maximum flow interdiction problem. Bigdeli and Bavandi [25] studied a multi-period dynamic interdiction problem in fuzzy conditions investigated in order to help military decision-makers and commanders to choose an appropriate strategy.

Let us briefly review the network maximum flow interdiction problem. For $G = (N, A)$ as a directed graph, in which $N$ and $A$ are the sets of nodes and arcs, respectively, two specific nodes are considered the source and destination nodes, respectively shown by $s$ and $d$. Each arc $(i, j) \in A$ has a capacity of $u_{ij}$ and $x_{ij} \geq 0$ is the feasible flow on this arc, which cannot exceed the arc capacity. Let us assume that the interdictor has a total number of R units of sources used for the interdiction of arcs, and $r_{ij}$ units of sources are required for the interdiction of each arc. An artificial arc has been defined corresponding to each s and d node with $u_{ds} = \infty$ and $r_{ds} = \infty$. Assume $y_{ij}$ to be a binary decision variable defined as follows:

$$y_{ij} = \begin{cases} 1, & \text{if the arc(i, j) is interdicted} \\ 0, & o.w \end{cases}$$

Furthermore, a set of feasible decisions of the interdictor is demonstrated as follows:

$$Y = \left\{ y_{ij} \,\middle|\, \sum_{(i,j)\in A} y_{ij} r_{ij} \leq R, y_{ij} \in \{0,1\}, \forall (i,j) \in A \right\}$$

Accordingly, the network maximum flow interdiction problem can be formulated as below [26]:

**Problem 1:(DNIP)**

$$\begin{array}{ll} \min\limits_{y\in Y} \max\limits_{x} \; x_{ds} & \\ s.t. \quad \sum\limits_{j:(i,j)\in A'} x_{ij} - \sum\limits_{j:(j,i)\in A'} x_{ji} = 0 & \forall i \in N \\ \qquad 0 \leq x_{ij} \leq u_{ij} (1 - y_{ij}) & \forall (i,j) \in A, \end{array} \tag{1}$$

in which $A' = A \cup (d, s)$. The first constraint is the mass balance constraint, and the second is the arc capacity constraint.

In general, the capacity of arcs, the input and output flow values of vertices, and transmission costs may be dependent on elements such as weather conditions, network emergencies, maintenance, and traffic congestion, as well as certain unforeseen factors such as changes in petrol prices, complications related to the network components, and so on. As a result, it is possible that the flow in the network, and thus the maximum flow in the network, cannot be reliably measured in such scenarios. Indeed, despite all the desired features that the interdiction approaches have in the network, the measurement in them is carried out using traditional methods, and computations are made in a deterministic setting. However, in the real world, the most critical issue is the existence of uncertainty in the data. Liu [27] introduced the uncertain network problem to model a project scheduling problem with unknown durations. The inaccuracy or ambiguity of the data in fuzzy environments and the measurement errors and disruptions in the sample observed in stochastic environments are all significant causes of uncertainty. However, it should be emphasized that there is not sufficient data available to determine the probability distribution in many circumstances. In some instances, the data collected is inaccurate, and experts' judgments are imprecisely replaced. In such cases, we are confronted with a combination of stochastic and fuzzy phenomena [28, 29]. In other words, we are dealing with a random variable that acquires inaccurate or fuzzy values. To this end, this article introduces a novel model for the maximum network flow interdiction in a stochastic fuzzy environment. The concept of fuzzy random variables was introduced by Kwakernoak [30].

Mathematically, a fuzzy random variable can be regarded as a mapping from a probability space to a set of fuzzy variables[31]. Generally, calculations in a fuzzy stochastic uncertain environment would be more complicated because fuzzy and stochastic approaches are typically used differently to solve their respective models [32]. However, each approach its own optimization problems using the measure-based technique. The probability measure, necessity measure, credibility measure in a fuzzy environment, and the possibility measure in a stochastic environment are among the sizes typically applied in this area, usually in the form of Chance Constraint Programming (CCP) to solve probability and fuzzy models. This study will combine and use three measurement sets, namely the probability-possibility measure, probability-necessity measure, and probability-credibility measure. This research also aims to demonstrate the extent to which uncertainty of data overshadows a network's vulnerability.

## 2. FUZZY STOCHASTIC MAXIMUM FLOW INTERDICTION PROBLEM

In this section, we consider three fuzzy stochastic NIP models using three different approaches of possibility, necessity, and credibility in the form of CCP programming. The capacity of arcs $(i, j)$ are fuzzy parameters whose mean is a random variable with a normal distribution in these models.

### 2.1. Probability-possibility approach

Consider graph $G = (N, A)$. Assume $u_{ij}^s = (u_{ij}^s, u_{ij}^\alpha, u_{ij}^\beta)$ to indicate the fuzzy stochastic variables corresponding to the capacity of $u_{ij}$, in which $u_{ij}^s$ is a random variable with a normal distribution as $u_{ij}^s \sim N(u_{ij}, \sigma_{ij}^2)$, and $u_{ij}^\alpha$ and $u_{ij}^\beta$ are the order of left and right spread, respectively. Model(2) displays the NIP fuzzy stochastic model in the form of chance constraint programming using the probability measure of $Pr$ and the possibility measure of $Pos$.

**Problem 2:**

$$
\begin{array}{llll}
\min\limits_{y \in Y} \max\limits_{x} \ x_{ds} & & & \\
s.t. \quad \sum\limits_{j:(i,j)\in A'} x_{ij} - \sum\limits_{j:(j,i)\in A'} x_{ji} = 0 & \forall i \in N, & (c_1) & \\
\Pr\left[Pos\left(x_{ij} \le \tilde{u}_{ij}^s\left(1 - y_{ij}\right)\right) \ge \delta\right] \ge \gamma & \forall\,(i,j) \in A, & (c_2) & \quad (2)\\
x_{ij} \ge 0 & \forall\,(i,j) \in A. & (c_3) &
\end{array}
$$

where $\delta, \gamma \in [0, 1]$ are pre-determined confidence levels. The membership function of the random fuzzy variable $\tilde{u}_{ij}^s$ is defined as follows.

$$
\mu_{\tilde{u}_{ij}^s}(t) = \begin{cases} L\left(\frac{u_{ij}^s - t}{u_{ij}^\alpha}\right), & \text{if} \quad u_{ij}^s - u_{ij}^\alpha < t \le u_{ij}^s \\ R\left(\frac{t - u_{ij}^s}{u_{ij}^\beta}\right), & \text{if} \quad u_{ij}^s \le t < u_{ij}^s + u_{ij}^\beta \end{cases} \tag{3}
$$

The letters $L$ and $R$ are the abbreviations of non-increasing continuous functions $[0,1]$ to $[0,1]$, so that $L(0)=R(0)=1$ and $L(1)=R(1)=0$ are called left and right functions, respectively. Henceforth, we will assume:

$$L(x)=R(x)=\begin{cases} 1-x, & 0\le x\le 1 \\ 0, & o.w. \end{cases} \tag{4}$$

The constraints should be definite to solve model(2). For this purpose, first, the theorem [33] and then the lemma [34] can be expressed as below.

**Theorem 1.** *Let $\tilde{u}_{ij}^{s}$ is a fuzzy random vector and $g_j, j=1,\ldots,n$ are real-valued continuous functions, we have:*

1. *The possibility $Pos\left\{g_j\left(\tilde{u}_{ij}^{s}(\omega)\right)\le 0, j=1,...,n\right\}$ is a random variable.*
2. *The necessity $Nec\left\{g_j\left(\tilde{u}_{ij}^{s}(\omega)\right)\le 0, j=1,...,n\right\}$ is a random variable.*
3. *The credibility $Cr\left\{g_j\left(\tilde{u}_{ij}^{s}(\omega)\right)\le 0, j=1,...,n\right\}$ is a random variable.*

**Lemma 2.** *Let $\tilde{u}_1$ and $\tilde{u}_2$ to be two independent fuzzy numbers type $LR$.*

1. $Pos\left(\tilde{u}_1\ge\tilde{u}_2\right)\ge\alpha \quad\Leftrightarrow\quad \tilde{u}_{1,\alpha}^{R}\ge\tilde{u}_{2,\alpha}^{L}$
2. $Nec\left(\tilde{u}_1\ge\tilde{u}_2\right)\ge\alpha \quad\Leftrightarrow\quad \tilde{u}_{1,1-\alpha}^{L}\ge\tilde{u}_{2,\alpha}^{R}$

where $\tilde{u}_{1,\alpha}^{L}$, $\tilde{u}_{1,\alpha}^{R}$ and $\tilde{u}_{2,\alpha}^{L}$, $\tilde{u}_{2,\alpha}^{R}$ are the $\alpha$-cut corner points of fuzzy numbers $\tilde{u}_1$ and $\tilde{u}_2$.

The following theorem presents a deterministic substitute for the uncertain constraint $c_2$.

**Theorem 3.** *Let the fuzzy random variable $\tilde{u}_{ij}^{s}=(u_{ij}^{\alpha},u_{ij}^{s},u_{ij}^{\beta})$ to be the capacity on the $(i,j)$ arc. For the confidence levels $\delta,\gamma\in[0,1]$, the constraint $c_2$ is equivalent to the following constraint:*

$$x_{ij}\le\left(u_{ij}+R^{-1}(\delta)\,u_{ij}^{\beta}+\Phi^{-1}(1-\gamma)\,\sigma_{u_{ij}}\right)(1-y_{ij})$$

*Proof.* From lemma 2 and membership function 3, we have:

$$Pos\left(x_{ij}\le\tilde{u}_{ij}^{s}(1-y_{ij})\right)\ge\delta \quad\Leftrightarrow\quad x_{ij}\le\left(u_{ij}^{s}+R^{-1}(\delta)\,u_{ij}^{\beta}\right)(1-y_{ij})$$

Letting $\bar{u}_{ij}^{s}=u_{ij}^{s}+R^{-1}(\delta)\,u_{ij}^{\beta}$, since $u_{ij}^{s}\sim N\left(u_{ij},\sigma_{ij}^{2}\right)$, then $\bar{u}_{ij}^{s}$ is clearly a random variable with a mean of $m_{\bar{u}_{ij}}^{s}$ and variance of $\sigma_{\bar{u}_{ij}^{s}}^{2}$ as well. Thus, we have:

$$\begin{aligned}
&\Pr\left[\tfrac{x_{ij}}{(1-y_{ij})}\le\left(u_{ij}^{s}+R^{-1}(\delta)\,u_{ij}^{\beta}\right)\right]\ge\gamma\\
\Leftrightarrow\quad &\Pr\left[\frac{\frac{x_{ij}}{(1-y_{ij})}-m_{\bar{u}_{ij}^{s}}}{\sigma_{\bar{u}_{ij}^{s}}}\le\frac{\bar{u}_{ij}^{s}-m_{\bar{u}_{ij}^{s}}}{\sigma_{\bar{u}_{ij}^{s}}}\right]\ge\gamma\\
\Leftrightarrow\quad &1-\Phi\left[\frac{\frac{x_{ij}}{(1-y_{ij})}-m_{\bar{u}_{ij}^{s}}}{\sigma_{\bar{u}_{ij}^{s}}}\right]\ge\gamma\\
\Leftrightarrow\quad &x_{ij}\le\left(m_{\bar{u}_{ij}^{s}}+\Phi^{-1}(1-\gamma)\,\sigma_{\bar{u}_{ij}^{s}}\right)(1-y_{ij})\\
\Leftrightarrow\quad &x_{ij}\le\left(u_{ij}+R^{-1}(\delta)\,u_{ij}^{\beta}+\Phi^{-1}(1-\gamma)\,\sigma_{u_{ij}}\right)(1-y_{ij})
\end{aligned}$$

where $\Phi^{(-1)}$ is the inverse function of the distribution function $\Phi$ of the standard normal distribution $N(0,1)$. This concludes the proof of the theorem. $\square$

According to Theorems 1 and 3 and Lemma 2, the following model is the deterministic equivalent of Model(2).

**Problem 3:**

$$\begin{array}{llll}\min\limits_{y\in Y}\max\limits_{x} \; x_{ds} & & & \\ s.t. & \sum\limits_{j:(i,j)\in A'} x_{ij} - \sum\limits_{j:(j,i)\in A'} x_{ji} = 0 & \forall i \in N, & (c_1) \\ & x_{ij} \leq \bar{u}_{ij}\left(1-y_{ij}\right) & \forall\left(i,j\right)\in A, & (c'_2) \\ & x_{ij} \geq 0 & \forall\left(i,j\right)\in A. & (c_3)\end{array} \tag{5}$$

Where $\bar{u}_{ij} = u_{ij} + R^{-1}\left(\delta\right)u_{ij}^{\beta} + \Phi^{-1}\left(1-\gamma\right)\sigma_{u_{ij}}$.

Obviously, when the interdiction operation occurs on the $(i,j)$ edge, the constraint $c_2'$ causes zero flow. Moreover, in case there is no interdiction on this edge, the flow could increase up to $u_{ij} + R^{-1}\left(\delta\right)u_{ij}^{\beta} + \Phi^{-1}\left(1-\gamma\right)\sigma_{u_{ij}}$.

Given that $x=0$ is a feasible solution to the internal maximization problem in Model(5) and the feasible area corresponding to this problem is bounded, there is no duality gap according to the strong duality theorem. Additionally, the target function values of the primary and dual problems are identical in optimality. Thus, the internal maximization problem can be replaced by its dual problem with no influence on the optimal solution.

Assuming $\pi_i (i \in N)$ and $\beta_{ij}(i,j \in A)$ as dual variable vectors corresponding to the constraints $c_1$ and $c_2'$, respectively, Problem 3 can be rewritten as follows:

**Problem 4:**

$$\begin{array}{lll}\min & \sum\limits_{(i,j)\in A} \bar{u}_{ij}\beta_{ij}\left(1-y_{ij}\right) & \\ s.t. & \pi_i - \pi_j + \beta_{ij} \geq 0, & \forall\left(i,j\right)\in A \\ & \pi_d - \pi_s \geq 1, & \\ & \beta_{ij} \geq 0, & \forall\left(i,j\right)\in A \\ & y \in Y. & \end{array} \tag{6}$$

Due to the nonlinear expression in the target function, the earlier problem is also nonlinear. To commence the process of linearization, we define a non-zero variable, $\eta_{ij} = \beta_{ij}y_{ij} \geq 0$. In order to complete the procedure, the following constraints are then added to Problem4.

$$y_{ij} + \beta_{ij} - 1 \leq \eta_{ij}, \qquad \forall\left(i,j\right)\in A \tag{7}$$

$$\eta_{ij} \leq y_{ij}, \qquad \forall\left(i,j\right)\in A \tag{8}$$

$$\eta_{ij} \le \beta_{ij}, \qquad \forall (i,j) \in A \tag{9}$$

By considering the abovementioned replacements, Problem4 is converted into the following equivalent form:

$$\begin{array}{lrl}
\min & \sum\limits_{(i,j)\in A} \bar{u}_{ij}(\beta_{ij} - \eta_{ij}) & \\
s.t. & \pi_i - \pi_j + \beta_{ij} \ge 0, & \forall i \in N,\ \forall (i,j) \in A \\
& \pi_d - \pi_s \ge 1, & \\
& y_{ij} + \beta_{ij} - 1 \ge \eta_{ij}, & \forall (i,j) \in A \\
& \eta_{ij} \le y_{ij}, & \forall (i,j) \in A \\
& \eta_{ij} \le \beta_{ij}, & \forall (i,j) \in A \\
& \beta_{ij} \ge 0, & \forall (i,j) \in A \\
& \eta_{ij} \ge 0, & \forall (i,j) \in A \\
& y \in Y. &
\end{array} \tag{10}$$

The previous problem could be solved via common methods for mixed-integer linear programming methods.

### 2.2. Probability-necessity approach

In this section, the fuzzy necessity measure replaces the possibility measure in the previous section. To this end, Model(2) is transformed as follows:

**Problem 5:**

$$\begin{array}{lll}
\min\limits_{y\in Y} \max\limits_{x} \ x_{ds} & & \\
s.t. & \sum\limits_{j:(i,j)\in A'} x_{ij} - \sum\limits_{j:(j,i)\in A'} x_{ji} = 0 & \forall i \in N, \\
& \Pr\left[Nec\left(x_{ij} \le \tilde{u}^s_{ij}(1 - y_{ij})\right) \ge \delta\right] \ge \gamma & \forall (i,j) \in A, \\
& x_{ij} \ge 0 & \forall (i,j) \in A.
\end{array} \tag{11}$$

In order to transform Model(11) into a deterministic form, the procedure in the former section is followed, which has been presented in below:

**Problem 6:**

$$\begin{array}{lrl}
\min & \sum\limits_{(i,j)\in A} \bar{u}'_{ij}(\beta_{ij} - \eta_{ij}) & \\
s.t. & \pi_i - \pi_j + \beta_{ij} \ge 0, & \forall i \in N,\ \forall (i,j) \in A \\
& \pi_d - \pi_s \ge 1, & \\
& y_{ij} + \beta_{ij} - 1 \ge \eta_{ij}, & \forall (i,j) \in A \\
& \eta_{ij} \le y_{ij}, & \forall (i,j) \in A \\
& \eta_{ij} \le \beta_{ij}, & \forall (i,j) \in A \\
& \beta_{ij} \ge 0, & \forall (i,j) \in A \\
& \eta_{ij} \ge 0, & \forall (i,j) \in A \\
& y \in Y. &
\end{array} \tag{12}$$

where $\bar{u}'_{ij} = u_{ij} - L^{-1}(1-\delta)\, u^{\alpha}_{ij} + \Phi^{-1}(1-\gamma)\, \sigma_{u_{ij}}$.

The problem above can also be solved using standard mixed-integer linear programming techniques.

### 2.3. Probability-credibility approach

Similar to the previous section, another fuzzy measure will be developed here. Credibility measure is one of the essential fuzzy measures that is defined in terms of possibility and necessity measures. Model(13) is the probability-credibility form of Model(2):

**Problem 7:**

$$\begin{array}{lll} \min\limits_{y\in Y} \max\limits_{x} \; x_{ds} & & \\ s.t. \quad \sum\limits_{j:(i,j)\in A'} x_{ij} - \sum\limits_{j:(j,i)\in A'} x_{ji} = 0 & \forall i \in N, & \\ \quad \Pr\left[ Cr\left( x_{ij} \le \tilde{u}^{s}_{ij}(1-y_{ij}) \right) \ge \delta \right] \ge \gamma & \forall (i,j) \in A, & \\ \quad x_{ij} \ge 0 & \forall (i,j) \in A. & \end{array} \tag{13}$$

Theorem4 is presented to solve the probability-credibility programming of Model(13).

**Theorem 4.** *let $\tilde{u}_1 = (u^{\alpha}_1, u^{s}_1, u^{\beta}_1)$ and $\tilde{u}_2 = (u^{\alpha}_2, u^{s}_2, u^{\beta}_2)$ are two independent fuzzy numbers type LR. For $\delta \in [0,1]$, we have:*

1. *If $\delta \le 0.5$, that is:*

$$Cr\{\tilde{u}_1 \ge \tilde{u}_2\} \ge \delta \quad \Rightarrow u^{s}_1 + u^{\beta}_1 R^{-1}(2\delta) \ge u^{s}_2 - u^{\alpha}_2 R^{-1}(2\delta)$$

2. *If delta $> 0.5$, that is:*

$$Cr\{\tilde{u}_1 \ge \tilde{u}_2\} \ge \delta \quad \Rightarrow u^{s}_1 - u^{\alpha}_1 L^{-1}(2(1-\delta)) \ge u^{s}_2 + u^{\beta}_2 L^{-1}(2(1-\delta))$$

*Proof.* According to the fuzzy set theory, $\hat{u} = \tilde{u}_1 - \tilde{u}_2$ (in which there is also a type-$LR$ fuzzy number) is equal to $\left(\hat{u}^{\alpha} = u^{\alpha}_1 + u^{\beta}_2, \hat{u}^{s} = u^{s}_1 - u^{s}_2, \hat{u}^{\beta} = u^{\alpha}_2 + u^{\beta}_1\right)$, and hence the credibility of the fuzzy event $Cr\{\hat{u} \ge 0\}$, is expressed as follows:

$$Cr\{\hat{c} \ge 0\} = \begin{cases} 1, & 0 \le \hat{u}^{s} - \hat{u}^{\alpha}, \\ 1 - \frac{1}{2} L\left(\frac{\hat{u}^{s}}{\hat{u}^{\alpha}}\right), & \hat{u}^{s} - \hat{u}^{\alpha} \le 0 \le \hat{u}^{s}, \\ \frac{1}{2} R\left(\frac{-\hat{u}^{s}}{\hat{u}^{\beta}}\right), & \hat{u}^{s} \le 0 \le \hat{u}^{s} + \hat{u}^{\beta} \\ 0, & 0 > \hat{u}^{s} + \hat{u}^{\beta} \end{cases}$$

Now, let us consider $Cr\{\hat{u} \ge 0\} \ge \delta$. If $\delta \le 0.5$, then

$$\begin{aligned} \delta \le \tfrac{1}{2} R\left(\tfrac{-\hat{u}^{s}}{\hat{u}^{\beta}}\right) \quad &\Leftrightarrow \quad R^{-1}(2\delta) \ge \tfrac{-\hat{u}^{s}}{\hat{u}^{\beta}} \\ &\Leftrightarrow \quad \left(u^{\alpha}_2 + u^{\beta}_1\right) R^{-1}(2\delta) \ge -\left(\hat{u}^{s}_1 - u^{s}_2\right) \\ &\Leftrightarrow \quad u^{s}_1 + u^{\beta}_1 R^{-1}(2\delta) \ge u^{s}_2 - u^{\alpha}_2 R^{-1}(2\delta) \end{aligned}$$

and if $0.5 < \delta \leq 1$, that is

$$\begin{aligned}
\delta \leq 1 - \tfrac{1}{2} L\left(\tfrac{\hat{u}^s}{\hat{u}^\alpha}\right) \quad &\Leftrightarrow \quad 2(1-\delta) \geq L\left(\tfrac{\hat{u}^s}{\hat{u}^\alpha}\right) \\
&\Leftrightarrow \quad \left(u_1^\alpha + u_2^\beta\right) L^{-1}\left(2(1-\delta)\right) \geq \left(\hat{u}_1^s - u_2^s\right) \\
&\Leftrightarrow \quad u_1^s - u_1^\alpha L^{-1}\left(2(1-\delta)\right) \geq u_2^s + u_2^\beta L^{-1}\left(2(1-\delta)\right)
\end{aligned}$$

and the proof is concluded. □

According to Theorem4, the deterministic form of Problem7 in terms of the $\delta$ value is represented as the two following models:

**Problem 8:** $(\delta \leq 0.5)$

$$\begin{array}{lrl}
\min & \sum\limits_{(i,j)\in A} \bar{u}''_{ij}\left(\beta_{ij} - \eta_{ij}\right) & \\
s.t. & \pi_i - \pi_j + \beta_{ij} \geq 0, & \forall i \in N,\ \forall (i,j) \in A \\
& \pi_d - \pi_s \geq 1, & \\
& y_{ij} + \beta_{ij} - 1 \geq \eta_{ij}, & \forall (i,j) \in A \\
& \eta_{ij} \leq y_{ij}, & \forall (i,j) \in A \\
& \eta_{ij} \leq \beta_{ij}, & \forall (i,j) \in A \\
& \beta_{ij} \geq 0, & \forall (i,j) \in A \\
& \eta_{ij} \geq 0, & \forall (i,j) \in A \\
& y \in Y. &
\end{array} \tag{14}$$

and

**Problem 9:** $(\delta > 0.5)$

$$\begin{array}{lrl}
\min & \sum\limits_{(i,j)\in A} \bar{u}'''_{ij}\left(\beta_{ij} - \eta_{ij}\right) & \\
s.t. & \pi_i - \pi_j + \beta_{ij} \geq 0, & \forall i \in N,\ \forall (i,j) \in A \\
& \pi_d - \pi_s \geq 1, & \\
& y_{ij} + \beta_{ij} - 1 \geq \eta_{ij}, & \forall (i,j) \in A \\
& \eta_{ij} \leq y_{ij}, & \forall (i,j) \in A \\
& \eta_{ij} \leq \beta_{ij}, & \forall (i,j) \in A \\
& \beta_{ij} \geq 0, & \forall (i,j) \in A \\
& \eta_{ij} \geq 0, & \forall (i,j) \in A \\
& y \in Y. &
\end{array} \tag{15}$$

where $\bar{u}''_{ij} = u_{ij} + R^{-1}(2\delta)\, u_{ij}^\beta + \Phi^{-1}(1-\gamma)\,\sigma_{u_{ij}}$ and $\bar{u}'''_{ij} = u_{ij} - L^{-1}\left(2(1-\delta)\right) u_{ij}^\alpha + \Phi^{-1}(1-\gamma)\,\sigma_{u_{ij}}$.

As can be observed, all of the models provided thus far have in common the usage of probability measure and triple measures of possibility, necessity, and credibility in solving the uncertainty anticipated. Possibility and necessity measures are used to evaluate the possibility of occurrence and the necessity of its fulfillment. Furthermore, the credibility measure guarantees the decision-maker that a fuzzy event will undoubtedly occur.

## 3. NUMERICAL EXAMPLE

Consider a battlefield where alliance forces are engaged in combat with the enemy in two distinct locations, with the enemy forces sandwiched between the two. Allied troops in one area have demanded equipment from the other area. The routes connect the two regions, either via adversaries or unknown routes. A number of obscured pathways with unknown costs and capacities have been identified, each of which is considered a fuzzy random variable as $\tilde{u}_1^s = (u_{ij}^{\alpha}, u_{ij}^{s}, u_{ij}^{\beta})$, in which the objective is to maximize the flow of equipment to allied forces. We will formulate it as a maximum flow interdiction problem. To this end, each combat zone is regarded as a network node, whereas direct routes between the two zones are regarded as network arcs. In Figure 1, the aforementioned network is depicted, in which $s$ and $t$ serve as the source and destination nodes, respectively. Table 2 contains information on indeterministic fuzzy stochastic variables and the resources required to interdict each edge. The total amount of resources necessary for interdicted edges is equal to $R = 9$.

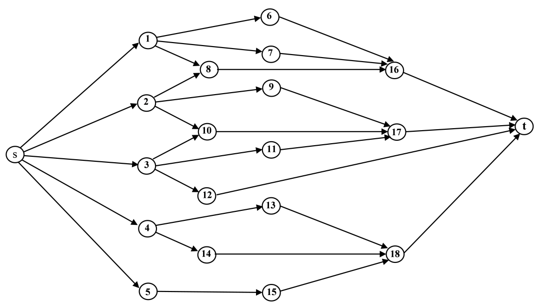


Figure 1: Network $G$ for example

Table 1: Example data

| Arc | Capacity | Resource | Arc | Capacity | Resource |
|---|---|---|---|---|---|
| (s,1) | $(N \sim (9, 3^2), 1, 3)$ | 8 | (4,14) | $(N \sim (13, 3^2), 1, 5)$ | 2 |
| (s,2) | $(N \sim (6, 0.5^2), 2, 4)$ | 9 | (5,15) | $(N \sim (6, 4^2), 2, 4)$ | 6 |
| (s,3) | $(N \sim (9, 0.25^2), 1, 3)$ | 7 | (6,16) | $(N \sim (8, 5^2), 2, 5)$ | 3 |
| (s,4) | $(N \sim (11, 4^2), 2, 5)$ | 10 | (7,16) | $(N \sim (2, 1^2), 3, 6)$ | 4 |
| (s,5) | $(N \sim (4, 3^2), 1, 4)$ | 12 | (8,16) | $(N \sim (5, 2^2), 4, 7)$ | 3 |
| (1,6) | $(N \sim (6, 1^2), 3, 5)$ | 4 | (9,17) | $(N \sim (8, 5^2), 2, 4)$ | 3 |
| (1,7) | $(N \sim (2, 0.5^2), 2, 4)$ | 4 | (10,17) | $(N \sim (9, 4^2), 2, 3)$ | 3 |
| (1,8) | $(N \sim (7, 1^2), 1, 3)$ | 3 | (11,17) | $(N \sim (11, 1^2), 1, 2)$ | 2 |
| (2.8) | $(N \sim (11, 2^2), 1, 3)$ | 5 | (12,d) | $(N \sim (11, 1^2), 1, 2)$ | 6 |
| (2,9) | $(N \sim (5, 0.5^2), 3, 5)$ | 4 | (13,18) | $(N \sim (11, 2^2), 2, 4)$ | 3 |
| (2,10) | $(N \sim (4, 1^2), 1, 2)$ | 5 | (14,18) | $(N \sim (2, 3^2), 4, 6)$ | 4 |
| (3,10) | $(N \sim (3, 0.25^2), 2, 4)$ | 5 | (15,18) | $(N \sim (7, 1^2), 2, 4)$ | 4 |
| (3,11) | $(N \sim (10, 5^2), 3, 5)$ | 5 | (16,d) | $(N \sim (13, 5^2), 3, 5)$ | 8 |
| (3,12) | $(N \sim (5, 2^2), 1, 2)$ | 5 | (17,d) | $(N \sim (9, 4^2), 1, 4)$ | 6 |
| (4,13) | $(N \sim (6, 4^2), 2, 4)$ | 3 | (18,d) | $(N \sim (18, 2^2), 2, 6)$ | 7 |

Table 2: The optimal value for $\delta = \gamma = 0.1 - 0.9$

| $(\delta, \gamma)$ | Prob-Pos | | Prob-Nec | | Prob-Cr | |
|---|---|---|---|---|---|---|
| | OV | Interdicted Arcs | OV | Interdicted Arcs | OV | Interdicted Arcs |
| (0.1,0.1) | 447.82 | (4,14),(6,16),(8,16),(11,17) | 302.82 | (4,14),(8,16),(9,17),(11,17) | 436.02 | (4,14),(6,16),(8,16),(11,17) |
| (0.2,0.2) | 407.30 | (4,14),(6,16),(8,16),(11,17) | 267.30 | (4,14),(8,16),(9,17),(11,17) | 377.30 | (4,14),(6,16),(8,16),(11,17) |
| (0.3,0.3) | 373.74 | (4,14),(8,16),(11,17),(13,18) | 212.34 | (4,14),(8,16),(10,17),(11,17) | 283.54 | (4,14),(8,16),(10,17),(11,17) |
| (0.4,0.4) | 341.98 | (4,14),(8,16),(11,17),(13,18) | 212.34 | (4,14),(8,16),(10,17),(11,17) | 283.54 | (4,14),(8,16),(10,17),(11,17) |
| (0.5,0.5) | 310.50 | (4,14),(8,16),(11,17),(13,18) | 187.00 | (4,14),(8,16),(10,17), (11,17) | 237.00 | (4,14),(8,16),(10,17),(11,17) |
| (0.6,0.6) | 279.02 | (4,14),(8,16),(11,17),(13,18) | 161.66 | (4,14),(8,16),(10,17),(11,17) | 201.66 | (4,14),(8,16),(10,17),(11,17) |
| (0.7,0.7) | 246.91 | (4,14),(8,16),(11,17),(13,18) | 131.61 | (4,14),(8,16),(11,17),(13,18) | 165.73 | (4,14),(8,16),(10,17),(11,17) |
| (0.8,0.8) | 211.65 | (4,14),(8,16),(11,17),(13,18) | 101.45 | (4,14),(8,16),(11,17),(13,18) | 121.45 | (4,14),(8,16),(11,17),(13,18) |
| (0.9,0.9) | 168.83 | (4,14),(8,16),(11,17),(13,18) | 62.55 | (4,14),(11,17),(13,18) | 72.95 | (4,14),(11,17),(13,18) |
| CPU sec | 0:00:00.771 | | 0:00:00.135 | | 0:00:00.225 | |

In order to create a better depiction of the spectrum of changes, $\delta$ and $\gamma$ were selected within the $[0.1, 0.9]$ interval. The calculations were performed using the GAMS 24.1.2 program. Table 3 shows the objective function variations and the interdicted edges for different values of $\delta$ and $\gamma$. Based on the acquired values, it is concluded that the best solution corresponds to $\delta = \gamma = 0.9$. If the decision-maker is risk-averse, the best response will favor necessity. On the other hand, a risk-taking decision-maker may select the best possibility solution. A decision-maker who intends to monitor the risk can benefit from an optimal credibility solution. The schematic representation of the optimal solution to the problem is presented in Figure 2.

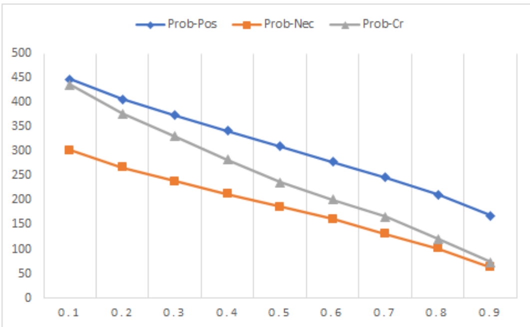


Figure 2: Probability-Possibility, Probability-necessarily and Probability-Credibility Optimal Solution

## 4. CONCLUSIONS AND SUGGESTIONS

This study aimed to analyze the network maximum flow interdiction problem under fuzzy stochastic conditions to assist decision-makers in selecting an effective strategy. In this case, the capacity of arcs was assumed to be a fuzzy stochastic variable. The fuzzy stochastic maximum flow interdiction problem was initially converted into the deterministic ultimate flow interdiction problem utilizing probability-possibility, probability-necessity, and probability-credibility approaches to address the proposed model. Subsequently, by using the duality technique, the resulting deterministic bi-level problem was simplified into a single-level problem. Finally, a standard linearization approach was proposed for generating a mixed-integer linear optimization problem. The presented models are applicable for both risk-taking and risk-averse decision-makers. In other words, risk-averse decision-makers typically prioritize the optimal necessity solutions, whereas risk-takers may choose the optimal possibility solutions. Decision-makers pursuing risk monitoring and control are more interested in the optimal credibility solutions. These are used to help decision-makers, particularly military decision-makers, establish appropriate policies. Time complexity can be seen as one of the challenges of solving the interdiction problem in large dimensions. However, meta-heuristic algorithms can be used to solve large-scale problems. To do this, the problem must first be transformed into its deterministic equivalent form. Recommendations for future studies could encompass the dynamic model of this problem in indeterministic fuzzy random environments. Additionally, the fuzzy randomness can be examined on other parameters such as edge time instead of edge capacities.

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