Yugoslav Journal of Operations Research
33 (2023), Number 3, 389–407
DOI: https://doi.org/10.2298/YJOR220315001Z

A MATRIX GEOMETRIC SOLUTION OF A MULTI-SERVER QUEUE WITH WAITING SERVERS AND CUSTOMERS' IMPATIENCE UNDER VARIANT WORKING VACATION AND VACATION INTERRUPTION

Ines ZIAD
Laboratory of Mathematics of Sidi Bel Abbes, Kasdi Merbah University, Ouargla, Algeria
zines021@gmail.com

P. Vijaya LAXMI
Department of Applied mathematics, Andhra University, Visakhapatnam, India
vijayalaxmiau@gmail.com

E. Girija BHAVANI
Department of Applied mathematics, Andhra University, Visakhapatnam, India
girijabhavaniedadasari@gmail.com

Amina Angelika BOUCHENTOUF
Mathematics Laboratory, Djillali Liabes University of Sidi Bel Abbes, Algeria
Laboratory of Stochastic Models, Statistic and Applications, University of Saida-Dr. Moulay Tahar, Algeria
bouchentouf_amina@yahoo.fr

Shakir MAJID
Department of Mathematics, University of Ladakh, India
shakirku16754@gmail.com

Received: March 2022 / Accepted: February 2023

Abstract: This paper deals with a $M/M/c$ queueing system with waiting servers, balking, reneging, and K-variant working vacations subjected to Bernoulli schedule vacation interruption. Whenever the system is emptied, the servers wait for a while before synchronously going on vacation during which services are offered with a lower rate. We obtain the steady-state probabilities of the system using the matrix-geometric method. In addition, we derive important performance measures of the queueing model. Moreover, we construct a cost model and apply a direct search method to get the optimum service rates during both working vacation and regular working periods at lowest cost. Finally, numerical results are provided.

Keywords: Queueing models, matrix analytic method, performance measures, cost model, optimisation.

MSC: 60K25, 68M20, 90B22.

1. INTRODUCTION

Queueing theory addresses one of life's most infuriating experiences; waiting. Queueing is very common in diverse areas, such as industry, emergency services, military logistics, finance, telecommunication systems, computer systems, and so on. This subject has attracted many researchers' attention [1, 2, 3].

Vacation queues have gained a particular focus since Levy and Yechiali [4] because of their excellent applications in various real-life problems, including manufacturing/production, inventory systems, computers and communication systems, and so on. Eminent surveys on these models can be found in [5, 6, 7, 8] and the references therein.

The concept of working vacation (WV) policy at which the server continues providing service at a lower rate during the vacation period has been introduced by Servi and Finn [9]. Over the last years, a great variety of queueing models with working vacations in different context has been done, for a detailed overview on the theme, the readers may refer to [10, 11, 12, 13, 14, 15, 16, 17, 18].

Nevertheless, we often come across the cases where the vacation may be interrupted, like for instance when number of customers reaches a predetermined value. Here, the interruption of the vacation avoids significant waiting costs for customers. This concept was initiated by Li and Tian [19] and Li *et al.* [20]. Since then, many studies have been provided on the subject (cf. [21, 22, 23, 24, 25, 26, 27]).

Customers' impatience has a very bad impact on different real-life systems including telecommunication, manufacturing and production systems. Working vacation queues with impatient customers have been well studied. The behavior of customers' impatience in working vacation queueing models have been extensively analysed. Prominent research papers can be found in [28, 29, 30, 31, 32, 33].

Moreover, impatience behavior in queueing models with variant of multiple vacation where the server can take a determined number of sequential vacations if there is no customers present in the queue at the end of a vacation have been considerably investigated (e.g., [34, 35, 36, 37]).

In different practical contexts, the server waits for a while before taking a break once the system gets empty. This frequently happens while considering the human behavior as a server. This topic has been thoroughly investigated (e.g., [38, 39, 40, 41, 42, 43]).

In practice, multi-server queues with station vacation (the servers, all together, synchronously go on vacation) and server vacation (the servers individually take vacations) are more applicable than single server queueing models. However, the analysis of these systems appears to be limited due to their complexity [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54].

In recent decades, a large literature has been done on queueing models with impatience behavior of customers, nevertheless, no research work to date has investigated a variant working vacation queueing system with multiple servers, vacation interruption, waiting servers, balking and reneging. The present queueing model finds a powerful application in call centers (see Subsection 2.1).

The main goals of the current study are:

- To establish the stationary analysis of the suggested queueing system using the matrix geometric method;
- To derive different system characteristics.
- To develop a cost model in order to define the optimum service rates in both regular working period and working vacation period that optimize the total expected cost using the direct search method.

The remainder of this research article is structured as follows. In Section 2, we describe the queueing model. In Section 3, we use the matrix geometric method to obtain the stationary probabilities of the system. In Section 4, the system performance measures as well as the expected cost function per unit time are developed. Section 5 presents numerical examples for a sensitivity analysis and uses a direct search method to minimize the cost function, subject to the equilibrium condition. In 6 section, some conclusions are drawn.

2. THE MODEL

Consider a $M/M/c$ queueing system with K-kind of variant working vacations, vacation interruption, waiting servers, balking, and reneging:

- Customers arrive at the system in accordance to a Poisson process with rate λ.
- The service times during regular busy state are considered to be an i.i.d exponential random variables (r.v) with rate μ. The service discipline is FCFS.
- Whenever the system is emptied (the regular working period is ended), the servers stay idle before going on vacation (waiting servers), this period follows an exponential distribution with parameter ω.
- At a vacation completion, if no customers are present in the queue, the servers are allowed to take other vacations of shorter durations until the number of working vacations reached the maximum (defined by K-vacations), then the system returns to the regular working state, waiting for new customers. Type-j, $j = \overline{0, K-1}$ working vacation times are supposed to be i.i.d random variables that follow exponential distributions with parameter ϕ_j, where $\phi_j > \phi_{j-1}$.

- During the vacation time, incoming customers have the possibility for being serviced. Here, the service times are assumed to be an i.i.d exponential random variables with rate ν such that $\nu < \mu$. Within this period, when the service is completed, if there are some customers in the queue, the servers can stop (interrupt) the vacation under Bernoulli's rule and turn to the normal working state with probability β' or remain in the vacation state with probability $\beta = 1 - \beta'$. It should be noted that the service during the vacation can be offered only to the first arrival.
- If on arrival, the customer finds some servers idle, he will be directly served. Otherwise, the arrivals may join the queue with probability θ, or decide to balk with a complementary probability $\theta' = 1 - \theta$.
- During the working vacation time, the customers may get impatient and abandon the system (renege) if their services are not yet accomplished. Here, the impatience times are supposed to be i.i.d random variables that follow exponential distributions with rate ξ.
- The above variables (the inter-arrival times, service times, waiting server times, working vacation times, are impatience times) are mutually independent.

The considered model is schematically depicted in the Figure 1.

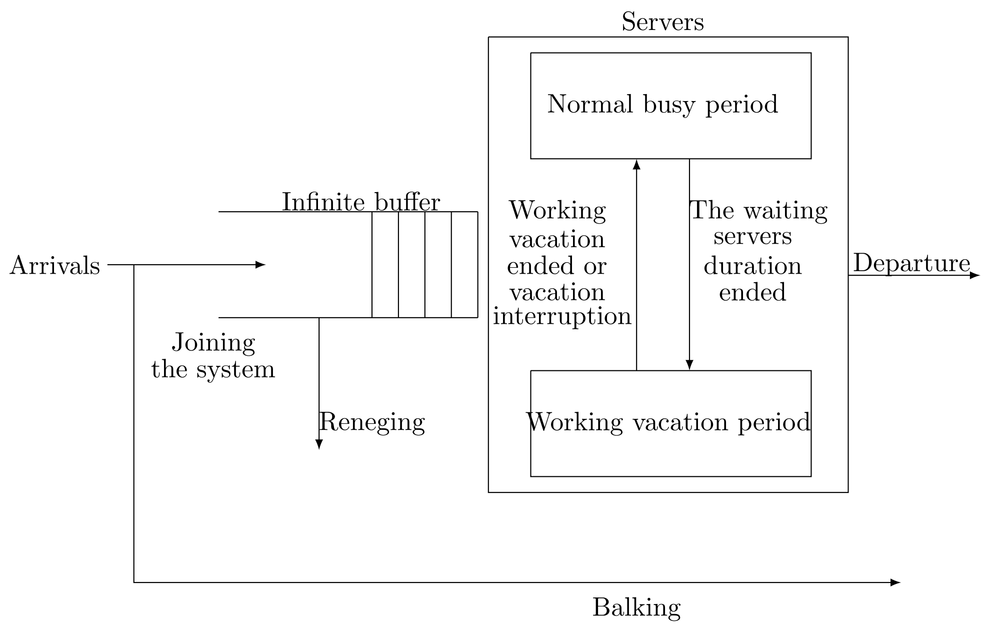

Figure 1: Schematic representation of the queueing system

2.1. Practical application

The proposed queueing model has a potential application in call centers, where the calls can reach the call center after traversing through different intermediate routers. If the agents are unoccupied, a call request is immediately processed. Otherwise, they will be waiting in a buffer to be serviced according to first-come-first-serve (FCFS) policy. When there is no calls in the system, before going synchronously 'as a group' on vacation, they will wait a random period of time (waiting servers duration). During the vacation period, the agents can deal with the new calls (if any) at the slower rate to economize the cost (working vacation period). During this period, at the end of each service, the agents check if there are new calls in the system and decide whether or not to return from their vacation, whether it is over or not (vacation interruption). In addition, if there is no calls request at a vacation completion, the agents starts a finite number of working vacations (variant working vacation). Otherwise, the agents begin a new regular busy period. During both periods, the call may balk based on the queue length. Further, during working vacation, a waiting call in the system may become impatient and quit the system after a long wait (reneging).

3. ANALYSIS OF THE MODEL

The behaviour of our queueing system is described by two-dimensional infinite state continuous-time Markov chain $\{(S(t);L(t));t \geq 0\}$ with state space $\Omega = \{(j,n) : n \geq 0, j = \overline{0,K}\}$, where $L(t)$ stands for the number of customers in the system and $S(t)$ specifies the state of the servers at time t, where

$$S(t) = \begin{cases} j, & \text{the system is on } (j+1)^{th} \text{ WV at time } t,\ j = \overline{0,K-1}; \\ K, & \text{the system is on regular working state at time } t. \end{cases}$$

Let $P_{j,n} = \lim\limits_{t\to\infty} P\{S(t) = j, L(t) = n\}$, $n \geq 0, j = \overline{0,K}$ be the system state probabilities of the process $\{(S(t);L(t)), t \geq 0\}$. Before proceeding with the analysis of the queueing system let us consider the following notations that are necessary for the rest of the article:

$$\chi_{0,n} = \begin{cases} \nu + \xi, & n = 1, \\ n(\beta\nu + \xi), & 2 \leq n \leq c-1, \\ c\beta\nu + n\xi, & n \geq c, \end{cases} \qquad \chi_{1,n} = \begin{cases} n\mu, & 1 \leq n \leq c-1, \\ \\ c\mu, & n \geq c, \end{cases}$$

and

$$\alpha_n = \begin{cases} 0, & n = 0,1 \\ n(1-\beta)\nu, & 2 \leq n \leq c-1, \\ c(1-\beta)\nu, & n \geq c. \end{cases}$$

The transition diagram is illustrated in Figure 2.

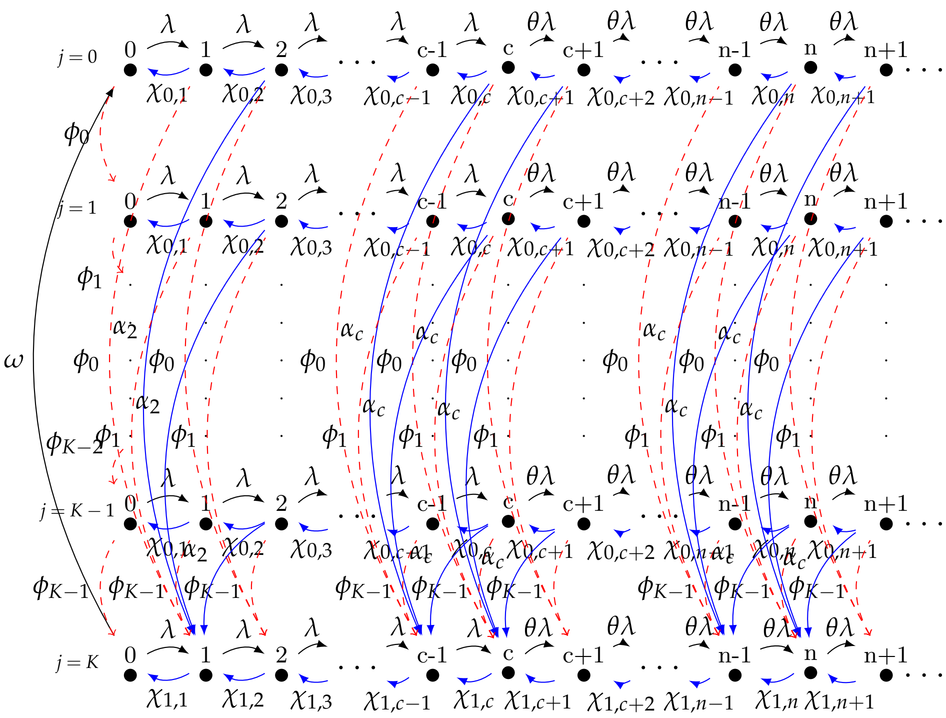

Figure 2: State-transition-rate diagram

3.1. Matrix-geometric method

In this section, we apply the matrix geometric method to obtain the stationary probabilities. According to [55], the infinitesimal generator $\mathbf{Q}$ for the process could be given as:

$$
\mathbf{Q} = \begin{pmatrix}
\mathbf{A}_0 & \mathbf{C}_0 & & & & & & & & & \\
\mathbf{B}_1 & \mathbf{A}_1 & \mathbf{C}_0 & & & & & & & & \\
 & \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{C}_0 & & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & & \\
 & & & \mathbf{B}_{c-1} & \mathbf{A}_{c-1} & \mathbf{C}_0 & & & & & \\
 & & & & \mathbf{B}_c & \mathbf{A}_c & \mathbf{C}_1 & & & & \\
 & & & & & \mathbf{B}_{c+1} & \mathbf{A}_{c+1} & \mathbf{C}_1 & & & \\
 & & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & & \mathbf{B}_N & \mathbf{A}_N & \mathbf{C}_1 & \\
 & & & & & & & & \mathbf{B}_N & \mathbf{A}_N & \mathbf{C}_1 \\
 & & & & & & & & & \ddots & \ddots
\end{pmatrix},
$$

where N is a large enough number so that when the number of customers $n \geq N$, we approximate the matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ by $\mathbf{A}_N$ and $\mathbf{B}_N$, respectively.

$$\mathbf{C}_0 = \begin{pmatrix} \lambda & & & & & & \\ & \lambda & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \lambda & \\ & & & & & & \lambda \end{pmatrix}_{K+1 \times K+1},$$

$$\mathbf{C}_1 = \begin{pmatrix} \theta\lambda & & & & & & \\ & \theta\lambda & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \theta\lambda & \\ & & & & & & \theta\lambda \end{pmatrix}_{K+1 \times K+1},$$

$$\mathbf{A}_0 = \begin{pmatrix} -(\lambda+\phi_0) & \phi_0 & & & & \\ & -(\lambda+\phi_1) & \phi_1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & -(\lambda+\phi_{K-1}) & \phi_{K-1} \\ \omega & & & & & -(\lambda+\omega) \end{pmatrix}_{K+1 \times K+1},$$

$$\mathbf{B}_1 = \begin{pmatrix} \chi_{0,1} & & & & & & \\ & \chi_{0,1} & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \chi_{0,1} & \\ & & & & & & \mu \end{pmatrix}_{K+1 \times K+1},$$

$$\mathbf{B}_n = \begin{pmatrix} \chi_{0,n} & & & & & n\beta'\nu \\ & \chi_{0,n} & & & & n\beta'\nu \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & \ddots & \vdots \\ & & & & \chi_{0,n} & n\beta'\nu \\ & & & & & n\mu \end{pmatrix}_{K+1 \times K+1}, \quad 2 \leq n \leq c-1,$$

$$\mathbf{B}_n = \begin{pmatrix} \chi_{0,n} & & & & & c\beta'\nu \\ & \chi_{0,n} & & & & c\beta'\nu \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & \ddots & \vdots \\ & & & & \chi_{0,n} & c\beta'\nu \\ & & & & & c\mu \end{pmatrix}_{K+1\times K+1}, \quad c \leq n \leq N-1,$$

$$\mathbf{A}_n = \begin{pmatrix} -\Theta_0 & & & & & \phi_0 \\ & -\Theta_1 & & & & \phi_1 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & -\Theta_{K-1} & \phi_{K-1} \\ & & & & & -(\lambda + n\mu) \end{pmatrix}_{K+1\times K+1}, \quad 1 \leq n \leq c-1,$$

where $\Theta_j = (\lambda + \phi_j + \alpha_n + \chi_{0,n})$, $0 \leq j \leq K-1$,

$$\mathbf{A}_n = \begin{pmatrix} -\delta_0 & & & & & \phi_0 \\ & -\delta_1 & & & & \phi_1 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & -\delta_{K-1} & \phi_{K-1} \\ & & & & & -(\theta\lambda + n\mu) \end{pmatrix}_{K+1\times K+1}, \quad c \leq n \leq N-1,$$

where $\delta_j = (\theta\lambda + \phi_j + \chi_{0,n} + c\beta'\nu)$, $0 \leq j \leq K-1$,

$$\mathbf{B}_n = \begin{pmatrix} \chi_{0,N} & & & & & c\beta'\nu \\ & \chi_{0,N} & & & & c\beta'\nu \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & \ddots & \vdots \\ & & & & \chi_{0,N} & c\beta'\nu \\ & & & & & c\mu \end{pmatrix}_{K+1\times K+1}, \quad n \geq N,$$

and $$\mathbf{A}_n = \begin{pmatrix} -\Gamma_0 & & & & & \phi_0 \\ & -\Gamma_1 & & & & \phi_1 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & -\Gamma_{K-1} & \phi_{K-1} \\ & & & & & -(\theta\lambda + c\mu) \end{pmatrix}_{K+1\times K+1}, \quad n \geq N,$$

where $\Gamma_j = (\theta\lambda + \phi_j + \chi_{0,N} + c\beta'\nu)$, $0 \leq j \leq K-1$.

Based on Neuts [55], the approximated system is stable and the steady-state probability vector exists iff $\mathbf{YC_1e} < \mathbf{YB_Ne}$, where $\mathbf{Y}$ is an invariant probability of the matrix $\psi = \mathbf{B_N} + \mathbf{A_N} + \mathbf{C_1}$. $\mathbf{Y}$ satisfies the equations $Y\psi = 0$ and $\mathbf{Ye_n} = \mathbf{1}$, where $\mathbf{e}_n$ is the column vector of appropriate dimension n with all elements equal to one.

Under the stability condition, the stationary probability vector $\mathbf{\Pi}$ of the generator $\mathbf{Q}$ exists, satisfying the balance equation $\mathbf{\Pi Q} = \mathbf{0}$ and $\mathbf{\Pi e}_n = 1$, where $\mathbf{0}$ is the row vector with all elements equal to zero. The vector $\mathbf{\Pi}$ partitioned as $\mathbf{\Pi} = [\mathbf{\Pi}_0, \mathbf{\Pi}_1, \mathbf{\Pi}_2, ...]$, where $\mathbf{\Pi}_n = [P_{0,n}, P_{1,n}, P_{2,n}, ..., P_{K,n}]$.

Clearly, when the stability condition is fulfilled, the sub-vectors of $\mathbf{\Pi}$, relating to various levels satisfy

$$\mathbf{\Pi}_n = \mathbf{\Pi}_N \mathbf{R}^{n-N},\ n \geq N, \tag{1}$$

where the matrix $\mathbf{R}$ is the minimal non-negative solution of the matrix quadratic equation

$$\mathbf{C}_1 + \mathbf{R}\mathbf{A}_N + \mathbf{R}^2\mathbf{B}_N \quad = \quad \mathbf{0}. \tag{2}$$

In fact, the QBD process is positive recurrent iff the spectral radius $Sp(\mathbf{R}) < 1$. However, it is quite complicated and tedious to define the explicit expression of the matrix $\mathbf{R}$ by resolving equation (2). Neuts [55] developed an iterative algorithm for numerically computing $\mathbf{R}$. Starting with initial iteration $\mathbf{R}_0 = 0$, we can compute the successive approximation

$$\mathbf{R}_{n+1} = -(\mathbf{C}_1 + \mathbf{R}_n^2\mathbf{B}_N)(\mathbf{A}_N)^{-1}, n \geq 0.$$

From equation $\mathbf{\Pi Q} = \mathbf{0}$, the governing system of difference equations can be given as

$$\mathbf{\Pi}_0\mathbf{A}_0 + \mathbf{\Pi}_1\mathbf{B}_1 = 0, \tag{3}$$

$$\mathbf{\Pi}_{n-1}\mathbf{C}_0 + \mathbf{\Pi}_n\mathbf{A}_n + \mathbf{\Pi}_{n+1}\mathbf{B}_{n+1} = 0,\ \ 1 \leq n \leq c \tag{4}$$

$$\mathbf{\Pi}_{n-1}\mathbf{C}_1 + \mathbf{\Pi}_n\mathbf{A}_n + \mathbf{\Pi}_{n+1}\mathbf{B}_{n+1} = 0,\ \ c+1 \leq n \leq N-1, \tag{5}$$

$$\mathbf{\Pi}_{n-1}\mathbf{C}_1 + \mathbf{\Pi}_n\mathbf{A}_N + \mathbf{\Pi}_{n+1}\mathbf{B}_N = 0,\ \ n \geq N, \tag{6}$$

and the normalizing condition

$$\sum_{n=0}^{\infty} \mathbf{\Pi}_n\mathbf{e} \quad = \quad 1. \tag{7}$$

From equations (3) to (6), after some mathematical manipulations, we obtain

$$\mathbf{\Pi}_{n-1} = \mathbf{\Pi}_n\varphi_n,\ \ 1 \leq n \leq N, \tag{8}$$

$$\mathbf{\Pi}_N[\varphi_N\mathbf{C}_1 + \mathbf{A}_N + \mathbf{R}\mathbf{B}_N] = \mathbf{0}, \tag{9}$$

where

$$\varphi_1 = -\mathbf{B}_1(\mathbf{A}_0^{-1}),\ \ \varphi_n = -\mathbf{B}_n(\mathbf{A}_{n-1} + \varphi_{n-1}\mathbf{C}_0)^{-1}, 2 \leq n \leq c+1,$$

$$\varphi_n = -\mathbf{B}_n(\mathbf{A}_{n-1} + \varphi_{n-1}\mathbf{C}_1)^{-1}, c+2 \leq n \leq N.$$

Using equations (7) and (8), we obtain

$$\mathbf{\Pi}_N \left[\sum_{n=1}^{N} \prod_{i=N}^{n} \varphi_i + (\mathbf{I} - \mathbf{R})^{-1} \right] \mathbf{e} \quad = \quad 1. \tag{10}$$

By solving equations (9) and (10), we find $\mathbf{\Pi}_N$. Then, We employ equations (1) and (8) to obtain $\mathbf{\Pi}_n$ for $n \geq 0$.

4. PERFORMANCE MEASURES AND COST MODEL

4.1. Performance measures

- The expected number of customers in the system during WV:

$$E[L_{wv}] = \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} n P_{j,n}.$$

- The expected number of customers in the system during regular working period:

$$E[L_K] = \sum_{n=1}^{\infty} n P_{K,n}.$$

- The expected number of customers in the system:

$$E[L] = E[L_{wv}] + E[L_K].$$

- The servers remain idle during regular working period with probability

$$P[I_b] = P_{K,0}.$$

- The servers remain idle during WV period with probability

$$P_{idle} = \sum_{j=0}^{K-1} P_{j,0}.$$

- The probability that the servers are in WV period is given as

$$P_{wv} = \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} P_{j,n}.$$

- The probability that the servers are busy during regular working state is as follows:

$$P_{busy} = 1 - P_{K,0} - \sum_{j=0}^{K-1} P_{j,0}.$$

- The average rate of reneging is

$$R_{ren} = \xi E[L_{wv}].$$

- Throughput is given as

$$T_P = \mu \sum_{n=1}^{c-1} n P_{K,n} + c\mu \sum_{n=c}^{\infty} P_{K,n} + \nu \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} P_{j,n}$$

4.2. Cost model

We develop a cost model to analyze the optimization of the cost function of the model. The total expected cost function per unit time is as:
$F[\mu,\nu] = c_l\, E[L] + c_b\,(\mu+\nu)P_{busy} + c_r\, R_{ren} + c_i\, P[I_b] + C_\mu\, \mu,$
where
$C_l \equiv$ Cost per unit time per customer present in the system ,
$C_b \equiv$ Cost per unit time when the servers are busy,
$C_r \equiv$ Cost per unit time when a customer reneges,
$C_i \equiv$ Cost per unit time when the servers are idle during busy period,
$C_\mu \equiv$ Fixed service purchase cost per unit during busy period.
The cost minimization problem is illustrated mathematically as:

$$F[\mu^*,\nu] = \text{minimize}_\mu F[\mu,\nu],$$

$$F[\mu^*,\nu^*] = \text{minimize}_\nu F[\mu^*,\nu].$$

5. SOME SPECIAL CASES

In this section, we present some important particular cases of our queueing model.

- For both $K \to \infty$, and $K = 1$, if $c = 1$, $\nu = 0$, $\phi_i = \phi,\ i = \overline{0, K-1}$, $\omega = 0$, $\beta = 1$, and $\theta = 1$, then our model reduces to the models investigated in Altman and Yechiali [56].

- If $c = 1$, $\nu = 0$, $\phi_i = \phi,\ i = \overline{0, K-1}$, $\omega = 0$, $\xi = 0$, $\beta = 1$, and $\theta = 1$, our system is reduced with that studied by Yue *et al.* [35].

- When $c = 1$, $\phi_i = \phi,\ i = \overline{0, K-1}$, $\omega = 0$, $\xi = 0$, and $\beta = 1$, our queueing model coincides with that examined by Vijaya Laxmi and Rajesh [36].

- When $\phi_i = \phi,\ i = \overline{0, K-1}$, $\omega = 0$, $\theta = 1$, and $\beta = 1$, and the customers may be impatient before the service begins, then the queueing model presented in the current paper match with that given by Vijaya Laxmi and Kassahun [49].

6. NUMERICAL ILLUSTRATIONS

Numerical computations were carried out in this section using Mathematica software, and the results are provided in the form of graphs given below. Unless their values are indicated in the appropriate places, the model parameters are assumed to be $\lambda = 1.0$, $\mu = 5.0$, $\beta = 0.6$, $\nu = 3.0$, $\theta = 0.5$, $\omega = 0.7$, $\xi = 0.8$, $\phi[0] = 0.5$, $\phi_i = \phi_{i-1} + 0.1,\ 1 \leq i \leq K-1$. Cost parameters are taken as $c_l = 25$, $c_b = 20$, $c_r = 10$, $c_i = 6$, and $c_\mu = 5$.

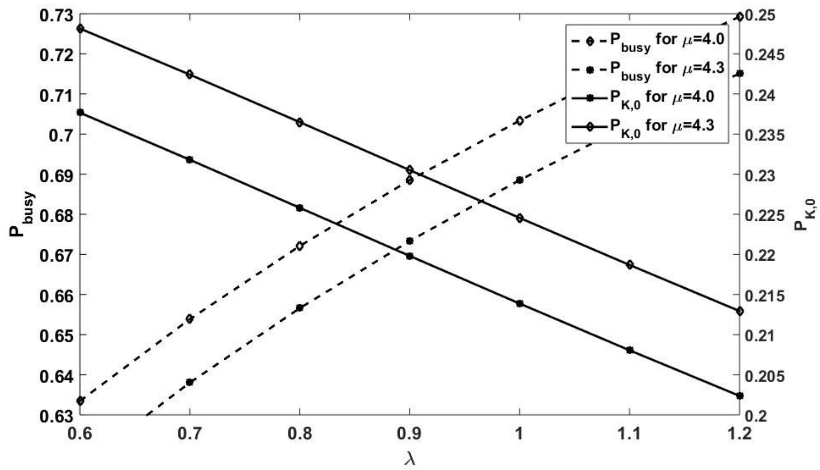

Figure 3: Effect of λ on P_{busy} and $P_{K,0}$ for different μ

Figure 3 shows the effect of arrival rate λ on P_{busy} and probability $P_{K,0}$, for different values of service rate in busy period μ. As λ increases, P_{busy} and $P_{K,0}$ increases and decreases, respectively. This is because, an increase in customers inflow into the system increases the probability of busy servers resulting in a decrease in server idleness. Further, for a fixed λ, contrary trend is observed as μ increases, which is true. The point of intersection of curves is indicated by the value of λ at which P_{busy} and $P_{K,0}$ are the maximum and minimum, respectively.

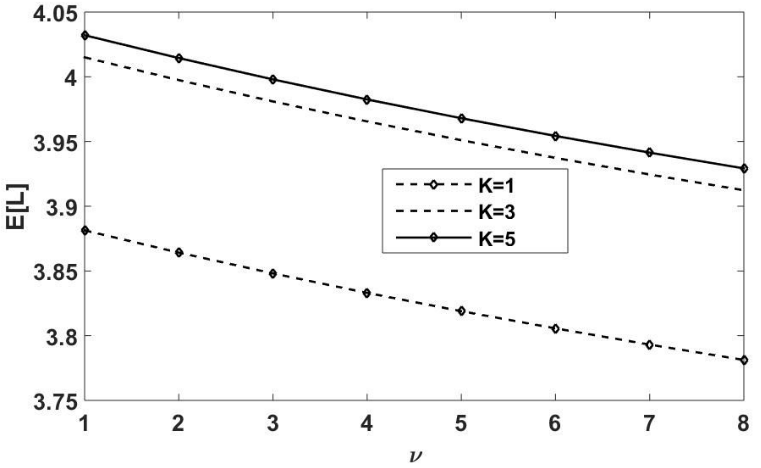

Figure 4: Effect of ν on $E[L]$ for different K

The impact of ν on $E[L]$, for different number of working vacations K, is shown in Figure 4. The figure shows that for a fixed K, the increase in ν decreases $E[L]$, which is obvious. Moreover, for a constant ν, $E[L]$ shows an opposite trend with the increase of K because of the slower service rates during the vacations. Also, $E[L]$ is observed smaller for $K = 1$.

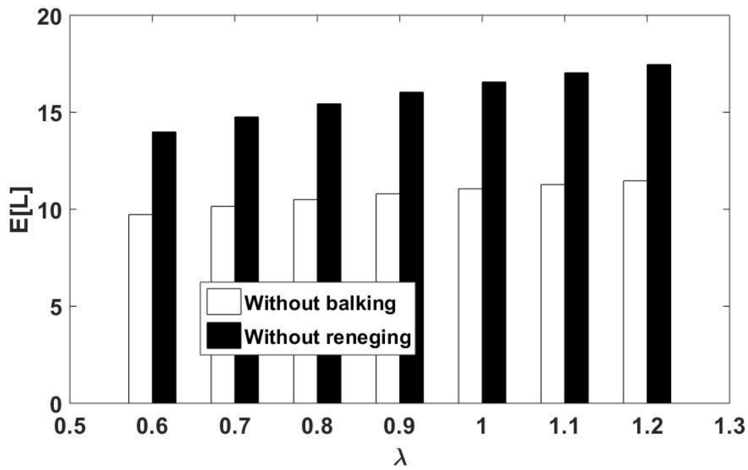

Figure 5: Effect of λ on $E[L]$

In Figure 5, we discuss the effect of λ on $E[L]$ for two scenarios; queueing model without balking and without reneging. We notice that when there is balking but no reneging, the system size is larger, and smaller when there is reneging but no balking. This demonstrates that for this system reneging constraint has a negative impact than balking factor.

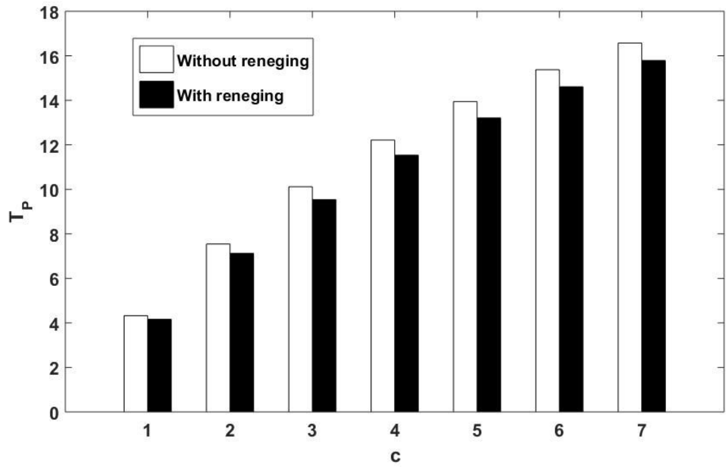

Figure 6: Effect of c on T_P

Figure 6 depicts the impact of number of servers c on throughput of the system T_p with and without reneging. As we see, the increase in c increases T_p. Also, for a fixed c, T_p is observed higher in the absence of reneging because of a longer queue.

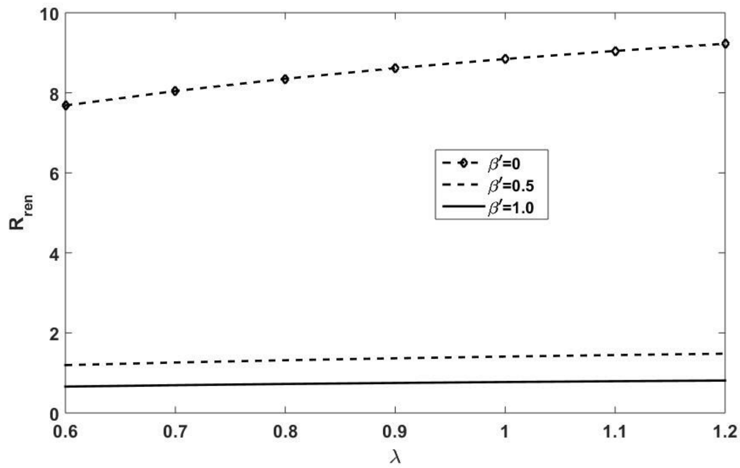

Figure 7: Effect of λ on R_{ren} for different β'

The impact of λ on R_{ren} is shown, for various vacation interruption probabilities β', in Figure 7. Initially, average reneging rate of the customer is high when there is no vacation interruption ($\beta' = 0$). Further, as β' increases, R_{ren} tends to decrease. This is due to as β' increases, the servers spend more in busy period and also service rates are higher during regular busy period.

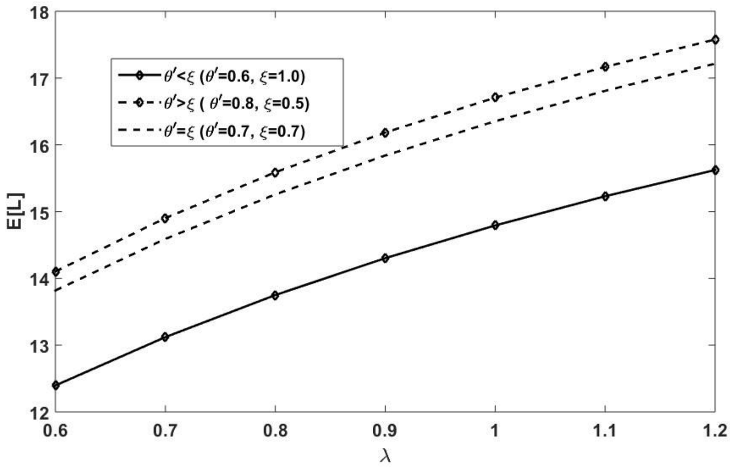

Figure 8: Effect of λ on $E[L]$ for different θ' and ξ

In Figure 8, we compare $E[L]$ in three cases $\theta' <=> \xi$ with respect to λ. This figure reveals that the system length is larger when $\theta' > \xi$ and smaller for $\theta' < \xi$. Hence, for the multi-server systems with balking and reneging of customers, in order to sustain the system properly, reneging rate should be maintained smaller than balking probability.

Table 1: Effect of λ on cost function

λ	(μ^*,ν)	$F[\mu^*,\nu]$	(μ^*,ν^*)	$F[\mu^*,\nu^*]$
0.8	(7.5,3.0)	471.331	(7.5,2.2)	463.783
1.0	(6.8,3.0)	509.298	(6.8,1.9)	501.366
1.2	(6.0,3.0)	539.570	(6.0,1.7)	528.748
1.4	(5.0,3.0)	563.339	(5.0,1.4)	546.111

Table 2: Effect of ξ on cost function

ξ	(μ^*,ν)	$F[\mu^*,\nu]$	(μ^*,ν^*)	$F[\mu^*,\nu^*]$
0.5	(7.2,3.0)	524.532	(7.2,2.4)	520.176
0.7	(6.9,3.0)	514.101	(6.9,2.0)	507.029
0.9	(6.7,3.0)	504.835	(6.7,1.9)	496.425
1.1	(6.5,3.0)	496.982	(6.5,1.7)	488.481

Table 3: Cost function vs. c

c	(μ^*,ν)	$F[\mu^*,\nu]$	(μ^*,ν^*)	$F[\mu^*,\nu^*]$
3	(5.1,3.0)	547.081	(5.1,1.3)	525.976
5	(7.7,3.0)	473.907	(7.7,2.6)	470.265
7	(8.4,3.0)	416.053	(8.4,3.7)	411.996
9	8.5,3.0	372.56	(8.5,4.3)	367.004

Using direct search method, the effect of λ, ξ, and c on optimum cost $F[\mu^*,\nu^*]$, optimum service rates, during busy period μ^*, and working vacation period ν^*, are shown in Tables 1-3, respectively.

- As arrival rate λ increases, the optimum service rates μ^* and ν^* decrease and the minimum cost $F[\mu^*,\nu^*]$ increases, which is necessary in order to maintain the stability of the system.
- Further, the optimum service rates and the minimum cost decrease with the increase of reneging rate ξ. This agrees with the fact that to attract the reneged customers in the system there should be some decrease in the minimum cost.
- However, the minimum cost decreases and the optimum service rates grow with the increase of number of servers c.

7. CONCLUSION

In this paper, we investigated a multi-server queue with K- variant working vacations, vacation interruption, waiting servers and customers' impatience. The stationary solution of the queueing system is established. Different performance measures are derived. In addition, cost optimization along with numerical results are presented. Our results show that

1. An increase in the number of the servers increases the throughput of the system.
2. The average reneging rate lowers as the probability of the vacation interruption increases.

3. For the better maintenance of the system, reneging rate should be smaller than balking probability.
4. The minimum cost and optimal service rates are reduced when the reneging rate rises.

The method used in this paper can be applied to study different Markovian models, such as $Geo^X/G/c$ and $GI^X/Geo/1$ queues with variant working vacations, Bernoulli-schedule vacation interruption and impatient customers.

Funding. This research received no external funding.

REFERENCES

[1] A. Haghighi and D. Mishev, *Queuing models in industry and business.* Nova Publishers, 2008.
[2] ——, *Delayed and network queues*, 2016.
[3] ——, "Stochastic modeling in industry and management," in *Modeling and Simulation in Industrial Engineering.* Springer, 2018, pp. 131–178.
[4] Y. Levy and U. Yechiali, "Utilization of idle time in an M/G/1 queueing system," *Management Science*, vol. 22, no. 2, pp. 202–211, 1975.
[5] B. T. Doshi, "Queueing systems with vacations—a survey," *Queueing systems*, vol. 1, no. 1, pp. 29–66, 1986.
[6] H. Takagi, "Queueing analysis: a foundation of performance evaluation," *Vacation and priority systems*, vol. 1, no. 1, 1991.
[7] N. Tian and Z. G. Zhang, *Vacation queueing models: theory and applications.* Springer Science & Business Media, 2006, vol. 93.
[8] J.-C. Ke, C.-H. Wu, and Z. G. Zhang, "Recent developments in vacation queueing models: a short survey," *International Journal of Operations Research*, vol. 7, no. 4, pp. 3–8, 2010.
[9] L. D. Servi and S. G. Finn, "M/M/1 queues with working vacations (M/M/1/WV)," *Performance Evaluation*, vol. 50, no. 1, pp. 41–52, 2002.
[10] Y. Baba, "Analysis of a GI/M/1 queue with multiple working vacations," *Operations Research Letters*, vol. 33, no. 2, pp. 201–209, 2005.
[11] A. Banik, U. Gupta, and S. Pathak, "On the GI/M/1/N queue with multiple working vacations—analytic analysis and computation," *Applied Mathematical Modelling*, vol. 31, no. 9, pp. 1701–1710, 2007.
[12] M. Jain and P. K. Agrawal, "$M/E_k/1$ queueing system with working vacation," *Quality Technology & Quantitative Management*, vol. 4, no. 4, pp. 455–470, 2007.
[13] D.-A. Wu and H. Takagi, "M/G/1 queue with multiple working vacations," *Performance Evaluation*, vol. 63, no. 7, pp. 654–681, 2006.
[14] N.-S. Tian, J.-H. Li, and Z. G. Zhang, "Matrix analytic method and working vacation queues-a survey," *International Journal of Information and Management Sciences*, vol. 20, no. 4, pp. 603–633, 2009.
[15] A. Banik, "Analysis of single working vacation in GI/M/1/N and $GI/M/1/\infty$ queueing systems," *International Journal of Operational Research*, vol. 7, no. 3, pp. 314–333, 2010.
[16] M. Jain and A. Jain, "Working vacations queueing model with multiple types of server breakdowns," *Applied Mathematical Modelling*, vol. 34, no. 1, pp. 1–13, 2010.
[17] T. V. Do, "M/M/1 retrial queue with working vacations," *Acta Informatica*, vol. 47, no. 1, pp. 67–75, 2010.
[18] D.-Y. Yang and C.-H. Wu, "Cost-minimization analysis of a working vacation queue with n-policy and server breakdowns," *Computers & Industrial Engineering*, vol. 82, pp. 151–158, 2015.
[19] J. Li and N. Tian, "The M/M/1 queue with working vacations and vacation interruptions," *Journal of Systems Science and Systems Engineering*, vol. 16, no. 1, pp. 121–127, 2007.

[20] J.-H. Li, N.-S. Tian, and Z.-Y. Ma, "Performance analysis of GI/M/1 queue with working vacations and vacation interruption," *Applied Mathematical Modelling*, vol. 32, no. 12, pp. 2715–2730, 2008.

[21] Y. Baba, "The M/PH/1 queue with working vacations and vacation interruption," *Journal of Systems Science and Systems Engineering*, vol. 19, no. 4, pp. 496–503, 2010.

[22] M. Zhang and Z. Hou, "Performance analysis of MAP/G/1 queue with working vacations and vacation interruption," *Applied Mathematical Modelling*, vol. 35, no. 4, pp. 1551–1560, 2011.

[23] T. Li, Z. Wang, and Z. Liu, "Geo/Geo/1 retrial queue with working vacations and vacation interruption," *Journal of Applied Mathematics and Computing*, vol. 39, no. 1, pp. 131–143, 2012.

[24] S. Gao and Z. Liu, "An M/G/1 queue with single working vacation and vacation interruption under bernoulli schedule," *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 1564–1579, 2013.

[25] D. H. Lee and B. K. Kim, "A note on the sojourn time distribution of an M/G/1 queue with a single working vacation and vacation interruption," *Operations Research Perspectives*, vol. 2, pp. 57–61, 2015.

[26] R. Tian, L. Hu, and X. Wu, "Equilibrium and optimal strategies in M/M/1 queues with working vacations and vacation interruptions," *Mathematical Problems in Engineering*, vol. 2016, 2016.

[27] S. Majid and P. Manoharan, "Analysis of an M/M/1 queue with working vacation and vacation interruption," *Applications and Applied Mathematics: An International Journal (AAM)*, vol. 14, no. 1, pp. 19–33, 2019.

[28] D. Yue, W. Yue, and G. Xu, "Analysis of customers' impatience in an M/M/1 queue with working vacations," *Journal of Industrial & Management Optimization*, vol. 8, no. 4, pp. 895–908, 2012.

[29] N. Selvaraju and C. Goswami, "Impatient customers in an M/M/1 queue with single and multiple working vacations," *Computers & Industrial Engineering*, vol. 65, no. 2, pp. 207–215, 2013.

[30] V. Goswami, "Analysis of impatient customers in queues with bernoulli schedule working vacations and vacation interruption," *Journal of Stochastics*, vol. 2014, 2014.

[31] A. A. Bouchentouf, A. Guendouzi, and A. Kandouci, "Performance and economic study of heterogeneous M/M/2/N feedback queue with working vacation and impatient customers," *ProbStat Forum*, vol. 12, no. 1, pp. 15–35, 2019.

[32] P. Vijaya Laxmi, P. Rajesh, and T. Kassahun, "Analysis of a variant working vacation queue with customer impatience and server breakdowns," *International Journal of Operational Research*, vol. 40, no. 4, pp. 437–459, 2021.

[33] M. M. N. GnanaSekar and I. Kandaiyan, "Analysis of an M/G/1 retrial queue with delayed repair and feedback under working vacation policy with impatient customers," *Symmetry*, vol. 14, no. 10, p. 2024, 2022.

[34] J.-C. Ke, "Operating characteristic analysis on the $M^{[x]}/G/1$ system with a variant vacation policy and balking," *Applied Mathematical Modelling*, vol. 31, no. 7, pp. 1321–1337, 2007.

[35] D. Yue, W. Yue, Z. Saffer, and X. Chen, "Analysis of an M/M/1 queueing system with impatient customers and a variant of multiple vacation policy," *Journal of Industrial & Management Optimization*, vol. 10, no. 1, pp. 89–112, 2014.

[36] P. Vijaya Laxmi and P. Rajesh, "Analysis of variant working vacations queue with customer impatience," *International Journal of Management Science and Engineering Management*, vol. 12, no. 3, pp. 186–195, 2017.

[37] ——, "Performance measures of variant working vacation on batch arrival queue with reneging," *Int. J. Math. Arch*, vol. 8, no. 8, pp. 85–96, 2017.

[38] R. Padmavathy, K. Kalidass, and K. Ramanath, "Vacation queues with impatient customers and a waiting server," *Int. J. Latest Trends Softw. Eng*, vol. 1, no. 1, pp. 10–19, 2011.

[39] S. I. Ammar, "Transient solution of an M/M/1 vacation queue with a waiting server and impatient customers," *Journal of the Egyptian Mathematical Society*, vol. 25, no. 3, pp. 337–342, 2017.

[40] B. Deepa and K. Kalidass, "The markovian vacation queues with a waiting server and

geometric abandonments," *Int. J. Pure Appl. Math*, vol. 118, pp. 1903–1910, 2018. [Online]. Available: http://www.ijpam.eu

[41] A. A. Bouchentouf, A. Guendouzi, and A. Kandouci, "Performance and economic analysis of markovian bernoulli feedback queueing system with vacations, waiting server and impatient customers," *Acta Universitatis Sapientiae, Mathematica*, vol. 10, no. 2, pp. 218–241, 2018.

[42] M. Suranga Sampath and J. Liu, "Impact of customers' impatience on an M/M/1 queueing system subject to differentiated vacations with a waiting server," *Quality Technology & Quantitative Management*, vol. 17, no. 2, pp. 125–148, 2020.

[43] A. A. Bouchentouf, A. Guendouzi, and S. Majid, "On a finite-buffer markovian queue with differentiated working vacation policy, bernoulli schedule vacation interruption, balking and reneging," *Croatian Operational Research Review*, pp. 21–37, 2020.

[44] C.-H. Lin and J.-C. Ke, "Multi-server system with single working vacation," *Applied Mathematical Modelling*, vol. 33, no. 7, pp. 2967–2977, 2009.

[45] M. Jain and S. Upadhyaya, "Synchronous working vacation policy for finite-buffer multiserver queueing system," *Applied Mathematics and Computation*, vol. 217, no. 24, pp. 9912–9916, 2011.

[46] S. M. Ganie and P. Manoharan, "Impatient customers in an M/M/c queue with single and multiple synchronous working vacations," *Pakistan Journal of Statistics and Operation Research*, pp. 571–594, 2018.

[47] A. A. Bouchentouf and A. Guendouzi, "Cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers," *SeMA Journal*, vol. 76, no. 2, pp. 309–341, 2019.

[48] L. Yahiaoui, A. A. Bouchentouf, and M. Kadi, "Optimum cost analysis for an Geo/Geo/c/N feedback queue under synchronous working vacations and impatient customers," *Croatian Operational Research Review*, pp. 211–226, 2019.

[49] P. Vijaya Laxmi and T. Kassahun, "Analysis of variant working vacation queue with reneging under a multi-server environment," *International Journal of Management Science and Engineering Management*, vol. 15, no. 2, pp. 130–137, 2020.

[50] A. A. Bouchentouf and A. Guendouzi, "The $M^X/M/c$ bernoulli feedback queue with variant multiple working vacations and impatient customers: performance and economic analysis," *Arabian Journal of Mathematics*, vol. 9, no. 2, pp. 309–327, 2020.

[51] A. A. Bouchentouf, M. Cherfaoui, and M. Boualem, "Analysis and performance evaluation of markovian feedback multi-server queueing model with vacation and impatience," *American Journal of Mathematical and Management Sciences*, vol. 40, no. 3, pp. 261–282, 2021.

[52] A. A. Bouchentouf, M. Boualem, M. Cherfaoui, and L. Medjahri, "Variant vacation queueing system with bernoulli feedback, balking and server's states-dependent reneging," *Yugoslav Journal of Operations Research*, vol. 31, no. 4, pp. 557–575, 2021.

[53] A. A. Bouchentouf, M. Boualem, L. Yahiaoui, and H. Ahmad, "A multi-station unreliable machine model with working vacation policy and customers' impatience," *Quality Technology & Quantitative Management*, pp. 766–796, 2022.

[54] M. Houalef, A. A. Bouchentouf, and L. Yahiaoui, "A multi-server queue in a multi-phase random environment with waiting servers and customers' impatience under synchronous working vacation policy," *Journal of the Operations Research Society of China*, pp. 1–29, 2022.

[55] M. Neuts, "Matrix geometric solutions in stochastic models: An algorithmic approach, johns hopkins university press," *Baltimore, MD, USA*, 1981.

[56] E. Altman and U. Yechiali, "Analysis of customers' impatience in queues with server vacations," *Queueing systems*, vol. 52, no. 4, pp. 261–279, 2006.