Yugoslav Journal of Operations Research
33 (2023), Number 2, 239–248
DOI: https://doi.org/10.2298/YJOR2111016031S

POLYHEDRAL COMPLEMENTARITY PROBLEM WITH QUASIMONOTONE DECREASING MAPPINGS

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Received: November 2021 / Accepted: November 2022

Abstract: The fixed point problem of piecewise constant mappings in $\mathbb{R}^n$ is investigated. This is a polyhedral complementarity problem, which is a generalization of the linear complementarity problem. Such mappings arose in the author's research on the problem of economic equilibrium in exchange models, where mappings were considered on the price simplex. The author proposed an original approach of polyhedral complementarity, which made it possible to obtain simple algorithms for solving the problem. The present study is a generalization of linear complementarity methods to related problems of a more general nature and reveals a close relationship between linear complementarity and polyhedral complementarity. The investigated method is an analogue of the well-known Lemke method for linear complementarity problems. A class of mappings is described for which the process is monotone, as it is for the linear complementarity problems with positive principal minors of the constraint matrix (class P). It is shown that such a mapping has always unique fixed point.

Keywords: Polyhedral complementarity, piecewise constant mappings, fixed point, duality, monotonicity, algorithm.

MSC: 90C33, 90-08.

1. INTRODUCTION

To find an equilibrium in the economic exchange model the author proposed an original approach of polyhedral complementarity [1], [2], [3]. The approach made it possible to develop finite algorithms both for the exchange model itself and for its variations and generalizations. The mathematical basis of the approach is the fixed points problem of special piecewise constant mappings on the price simplex. The presented paper investigates the mathematical fundamental principle of the approach. We consider piecewise constant mappings in $\mathbb{R}^n$, generated by a pair of mutually dual polyhedral complexes and study the fixed points problem of such mappings. Earlier this question was considered by the author under the assumption that the mappings are monotone [4]. For this case, the potentiality of the mappings was proved, and this made it possible to reduce the problem to an optimization one. Here the mappings without the assumption of monotonicity are considered. Local approximation of mappings leads to some generalization of the linear complementarity problem. The class of quasimonotone mappings is studied. For such a mapping a finite path-meeting algorithm is proposed, which can be regarded as an analogue of the well-known Lemke method [5], [6] for linear complementarity problems with positive principal minors of the constraint matrix (class P). For mappings on a simplex, the process was studied in [7] for the class of quasiregular mappings. In relation to the linear exchange model, the idea of such a process was used by the author in [3].

2. PIECEWISE CONSTANT MAPPINGS IN $\mathbb{R}^n$

The piecewise constant mappings under consideration are given by a pair of polyhedral complexes in duality. The fixed point problem for such mappings is formulated as a polyhedral complementarity problem. The description of this problem is as follows.

We consider polyhedrons in $\mathbb{R}^n$ as the sets, which are described by linear systems of equalities and inequalities. We get a face of the polyhedron when some of the inequalities turn into equalities.

Definition 1. *A vertex of a full-dimention polyhedron is called regular if exactly n inequalities from a system defining the polyhedron are performed at this vertex as equalities.*

We say that a family ω of polyhedrons is *polyhedral complex*, if each face of a polyhedron from ω is also in ω. We will use the terminology of combinatorial topology, referring to the elements of complexes as cells [8].

Definition 2 (Regularity condition of complex). *A complex ω is regular if all its vertexes are regular.*

For the cells we have a natural partial order: $\Omega_1 \prec \Omega_2$ if Ω_1 is a face of Ω_2.
Let two polyhedral complexes ω and ξ with the same number of cells r be given. Let $R \subset \omega \times \xi$ be a one-to-one correspondence: $R = \{(\Omega_i, \Xi_i)\}_{i=1}^r$ with $\Omega_i \in \omega$, $\Xi_i \in \xi$.

We say that the complexes ω and ξ are *in duality by R* if the subordination of cells in ω and the subordination of the corresponding cells in ξ are opposite each other:

$$\Omega_i \prec \Omega_j \Longleftrightarrow \Xi_i \succ \Xi_j.$$

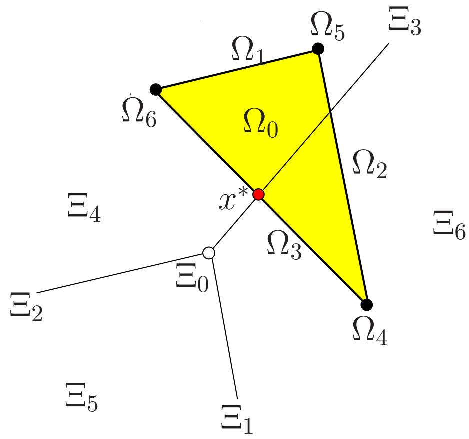

Figure 1: Polyhedral complemehtarity

The correspondence R defines the polyhedral complementarity problem: you need to find a point that belongs to both cells of some pair (Ω_i, Ξ_i).

$$x^* \text{ is the solution of the problem } \iff x^* \in \Omega_i \cap \Xi_i \text{ for some } i.$$

This is a natural generalization of linear complementarity, where (in the non-singular case) the complexes are formed by all faces of two simplex cones and one of them is R^n_+ .

In the future, it is assumed that in the complex ω there are vertexes, the cells Ω_i cover all space $\mathbb{R}^n$ and their relative interiors Ω_i° do not intersect. A piecewise constant point-to-set mapping F can be associated with such a complementarity problem. For each point $p \in \Omega_i^\circ$ the corresponding paired cell $\Xi_i \in \xi$ is taken as its image $F(p)$:

$$(\Omega_i, \Xi_i) \text{ is a pair, } p \in \Omega_i^\circ \Longrightarrow F(p) = \Xi_i.$$

It is easy to see that the fixed points of this mapping coincide with the solutions of the considered complementarity problem.

3. PATH-MEETING METHOD FOR THE POLYHEDRAL COMPLEMENTARITY PROBLEM

We will assume that the complexes generating the considered polyhedral complementarity problem are finite and regular.

Let (Ω, Ξ) be a pair of the considered problem. Let $L \supset \Xi$ and $M \supset \Omega$ be the affine hulls of the cells. We will assume that for any such pair of cells the following condition is satisfied:

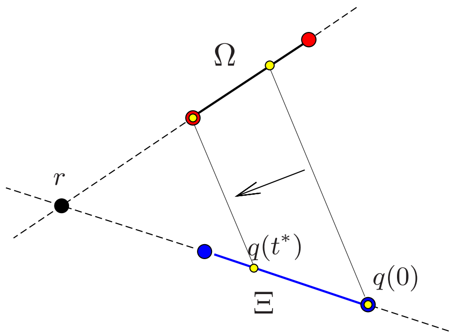

Figure 2: One step of path-meeting method

General situation condition: *The set $L \cap M$ is a singleton.*

The case when this condition is violated, we consider as a degenerate one. One step of the path-meeting method has following description.

At the beginning of the k step there are two cells Ω_k and Ξ_k, and two points $p^k \in \Omega_k$, $q^k \in \Xi_k$. Let L_k, M_k be the affine hulls of the cells.

We obtain the "ideal" point r^k as the intersection point of the affine hulls L_k and M_k : $\{r^k\} = L_k \cap M_k$. If $r^k \in \Omega_k$ and $r^k \in \Xi_k$, then r^k is a fixed point of the mapping.

Otherwise, we consider the moving points

$$p(t) = p^k + t(r^k - p^k) \quad q(t) = q^k + t(r^k - q^k), \quad t \in [0, 1).$$

Let be $t_k^* = \max t$ under the conditions $p(t) \in \Omega_k, q(t) \in \Xi_k$.

It is the case when $t_k^* < 1$. The two variants may occur:

$$t_k^* \text{ is limited by } p(t) \in \Omega_k \implies \text{the cell } \Omega_k \text{ is reducing,}$$

$$t_k^* \text{ is limited by } q(t) \in \Xi_k \implies \text{the cell } \Xi_k \text{ is reducing.}$$

We consider the situation when t_k^* is limited by both above conditions as degenerate.

Nondegeneracy condition 1. *Only one of the above two cases can occur.*

For the next step $k+1$ we take $p^{k+1} = p(t_k^*), q^{k+1} = q(t_k^*)$. Thus, it will be $p^{k+1} - q^{k+1} = (1 - t_k^*)(p^k - q^k)$. From this follows

$$p^{k+1} - q^{k+1} = (1 - t_k^*)(1 - t_{k-1}^*)...(1 - t_1^*)(p^1 - q^1). \tag{1}$$

<u>Geometric interpretation.</u>

From (1) follows, that all points $\theta^s = p^s - q^s$ are located on the ray

$$\Lambda_{\theta^1} = \{\theta = \tau\theta^1 | \tau \geq 0\}.$$

A Minkowski difference for each pair of cells (Ω, Ξ) can be considered:

$$\Omega\Xi = \{a - b \,|\, a \in \Omega, b \in \Xi\}.$$

The question is to find a pair (Ω, Ξ) for which $0 \in \Theta = \Omega\Xi$.

The moving points $\theta(t) = p(t) - q(t)$ move straight ahead to the point 0, crossing different sets Θ. The process starts at the point $\theta^1 = p^1 - q^1$. The point θ^{k+1} will be closer to point 0 than θ^k if $t_k^* > 0$.

Nondegeneracy condition 2. *The start point $\theta^1 = p^1 - q^1$ belongs to the interior of the start cell $\Theta_1 = \Omega_1\Xi_1$. The dimension of the reducing cell decreases only by one.*

This method was proposed in [3] for searching the equilibrium prises in the linear exchange model. In [7] it was considered for decreasing quasi-regular mappings on simplex.

4. LOCAL APPROXIMATION AND GENERALIZED LINEAR COMPLEMENTARITY PROBLEM

Consider a full-dimension cell $\Omega \in \omega$ and its vertex $\hat{p}$. By the assumption of regularity, to this vertex corresponds exactly n inequalities from the system defining the cell, which turn into equalities at this vertex. These inequalities form a system of the form:

$$(a_i, x) \geq b_i, \quad i = 1, ..., n. \tag{2}$$

The system describes a simplex cone W with vertex $\hat{p}$. This is the tangent cone for the cell Ω in the vertex $\hat{p}$.

In the dual complex ξ, the cell Ω and the vertex $\hat{p}$ correspond to the vertex $\hat{q}$ and the full-dimensional cell Ξ adjoining to $\hat{q}$ Thus, $(\Omega, \hat{q})$ and $(\hat{p}, \Xi)$ are the pairs in the correspondence of the complexes.

Each inequality of system (2) corresponds some one-dimensional cell with vertex $\hat{q}$. Let some $h^i \neq 0$ be a direction-vector from the vertex $\hat{q}$ inside this cell. Similarly to the cone W, we obtain a simplex cone V, which is the tangent cone for the cell Ξ in the vertex $\hat{q}$. This cone can be specified as the conical hull of the vectors h^i, which are shifted to the vertex $\hat{q}$:

$$q \in V \Longrightarrow q = \hat{q} + \sum_{i=1}^{n} h^i \lambda_i, \lambda_i \geq 0, i = 1, ..., n.$$

These two simplex cones define two polyhedral complexes in duality and form the complementarity problem with the complementarity condition of the form:

$$((a_i, x) - b_i)\lambda_i = 0, \quad i = 1, ..., n.$$

This is a generalization of the usual linear complementarity problem. This complementarity problem can be considered as a *local approximation* of initial problem. The usual form of the problem can be obtained as follows.

We introduce a vector λ with components λ_j, a vector b with components b_i, a matrix A with the rows a_i and a matrix H with the columns h^j. Let be $G = AH$ and $d = b - A\hat{q}$. The resulting complementarity problem now can be rewritten as:

$$G\lambda \geq d, \quad \lambda \geq 0,$$

$$(G\lambda - d, \lambda) = 0$$

It is easy to verify that the application of the Lemke method to this problem will be accompanied by the movement of the generated points along the ray of the form:

$$\Lambda_d = \{\vartheta = d - \tau e | \tau \geq 0\},$$

where $e = (1, ..., 1)$.

It is known that if the matrix G has positive principal minors (class P), then in the Lemke process, the τ parameter strictly decreases at each step. From the further presentation it will follow that in this case there will be a strict decreasing of the parameter From the further presentation it will follow that in this case there will be a strict decreasing of the parameter in the paths-meeting method.

Remark 3. It should be noted that if in any local approximation the vectors h^i can be taken equal a_i, then the considered mapping is the monotone decreasing one. For such a mapping in [4] its potentiality was proved.

5. QUASIMONOTONE DECREASING MAPPINGS

Definition 4. *A polyhedral complementarity mapping is called quasimonotone decreasing, if in every local approximation of the mapping all principal minors of the resulting matrix G are positive.*

Theorem 5. *For a quasimonotone decreasing mapping under the nondegeneracy assumptions the paths-meeting method gives a fixed point in a finite number of steps.*

Proof. To prove the theorem, we only need to show that at each step there will be a positive shift t^*. Then none of the passed pairs (Ω, Ξ) can be repeated : if a point moving on the ray leaves a polyhedron , then it cannot return to the polyhedron again. Thus the finiteness of the process will follow from the finiteness of the cells in the complexes.

The proof uses the logic of the Lemke method [6],[5]. Recall the main property characterizing the class P matrices . Let G^l be the l-th column of the matrix $G \in P$. Then the decomposition coefficient of the l-th unit vector e_l on the vector G^l is positive :

$$Gg = e_l, \quad g = (g_1, ..., g_n) \quad \Longrightarrow \quad g_l > 0.$$

It is clear that a similar statement will be true if the vectors G^l and e_l are interchanged: e_l will be placed in the matrix as its l-th column, and the G^l will be the right side of the linear system.

Now let us return to the consideration of the kth step of the process. There is a pair of cells (Ω_k, Ξ_k). One of these cells will expand, and the other will narrow. The consideration is symmetric for the cells. It is enough to consider the case of the cell expansion for one of the cells. Consider the case when the cell Ξ_k is obtained by expanding of the cell Ξ_{k-1} and, consequently, the cell Ω_k is obtained by narrowing the cell Ω_{k-1}.

Consider a local approximation of the problem, choosing the full-dimension cell $\Omega \supset \Omega_k$ with its vertex $\hat{p} \in \Omega_k$. They are associated in the complex ξ with the vertex $\hat{q} \in \Xi_k$ and the full-dimensional cell $\Xi \supset \Xi_k$.

Let us follow the description of the local approximation. Part of system (2) forms a system of equations describing the affine hull M_k of the cell Ω_k:

$$(a_i, x) = b_i, \quad i \in I_k. \tag{3}$$

Let be $I_k = \{1, ..., s\}$. Accordingly, the affine hull L_k of the Ξ_k cell will have the following description:

$$q \in L_k \Longrightarrow q = \hat{q} + \sum_{i \in I_k} h^i \lambda_i, \tag{4}$$

Thus, to find the "ideal" point r^k, you need to solve for corresponding $\bar{\lambda}_i^k, i \in I_k$, a system of the form:

$$\sum_{j \in I_k} (a_i, h^j) \bar{\lambda}_j^k = b_i - (a_i, \hat{q}), \quad i \in I_k. \tag{5}$$

It is necessary to show that a positive shift in the direction to the point r^k is possible without violating the conditions $p(t) \in \Omega_k, q(t) \in \Xi_k$.

Consider the case when the inequality $(a_s, x) \geq b_s$ from the description of the Ω_{k-1} cell is limiting at the previous step. For the point r^{k-1} we have $(a_s, r^{k-1}) < b_s$, and

$$(a_s, r^{k-1}) + \delta = b_s \tag{6}$$

with some $\delta > 0$.

It follows from the Nondegeneracy condition 2 that the condition $p(t) \in \Omega_k$ will not prevent small shifts in the direction to the "ideal point" r^k . Thus it must be considered the condition $q(t) \in \Xi_k$. We have $q(t) = q^k + t(r^k - q^k)$, and we must show, that a positive shift is possible from $t = 0$.

We have the simplest case when $I_k = \{s\}$. In this case $q(t) = \hat{q} + \lambda_s h^s$. It must be shown, that the condition $q(t) \in \Xi_k$ will be performed at small positive values λ_s. It will be so, if r^k as $q(1)$ is obtained by some $\bar{\lambda}_s > 0$. The value of such $\bar{\lambda}_s$ we obtain from $(a_s, r^k) = b_s$:

$$(a_s, \hat{q} + \bar{\lambda}_s h^s) = b_s.$$

In considered case we have $r^{k-1} = \hat{q}$. Using (6) for b_s , we obtain

$$(a_s, \hat{q}) + \bar{\lambda}_s (a_s, h^s) = (a_s, \hat{q}) + \delta,$$

and finaly

$$\bar{\lambda}_s(a_s, h^s) = \delta. \tag{7}$$

Here $\delta > 0$ and the factor (a_s, h^s) is positive as a diagonal element of the matrix G in the linear complementarity problem, obtained by the local approximation of the original mapping, which is assumed quasimonotone one. Thus $\bar{\lambda}_s$ is positive.

Now consider the case, when $|I_k| \geq 2$.

Let $\bar{\lambda}_i^{k-1}, i \in I_k \setminus \{s\}$ correspond the point r^{k-1}. We obtain for these quantities a system of the form:

$$\sum_{j \in I_k \setminus s} (a_i, h^j)\bar{\lambda}_j^{k-1} = b_i - (a_i, \hat{q}), \quad i \in I_k \setminus \{s\}, \tag{8}$$

$$\sum_{j \in I_k \setminus s} (a_s, h^j)\bar{\lambda}_j^{k-1} + \delta = b_s - (a_s, \hat{q}) \tag{9}$$

It is easy to see that system (5) is obtained from system (8-9) by replacing the last column $e_s = (0, ..., 0, 1)$ in the matrix of the system with the column $G^s = ((a_1, h^s), ..., (a_s, h^s))$. The point q^k is characterized by $\bar{\lambda}_s^{k-1} = 0$. In view of the Nondegeneracy condition 2 other values $\bar{\lambda}_i^{k-1}$ are positive. To show that the condition $q(t) \in \Xi_k$ does not prevent small shifts in the direction to the point r^k, it is necessary to make sure that $\bar{\lambda}_s^k$ is positive too.

It is known from the theory of the simplex-method of linear programming that the solution of system (5) can be obtained from the solution of system (8-9) using the decomposition coefficients of the column G^s in the columns of system (8-9). If these coefficients of the decomposition form the vector $g = (g_1, ..., g_s)$, then for $\bar{\lambda}_s^k$ we have the following formula

$$\bar{\lambda}_s^k = \frac{\delta}{g_s}$$

We have $\delta > 0$. It follows from the mentioned property of the class P that g_s is positive too . Thus, $\bar{\lambda}_s^k > 0$.

This completes the consideration of the case when at the previous step the magnitude of the shift was limited by the condition $p(t) \in \Omega_{k-1}$
The case when condition $q(t) \in \Xi_{k-1}$ turned out to be the limiting one at the previous step is considered similarly.

The Theorem is proved.

6. UNIQUENESS OF A FIXED POINT

Theorem 6. *A quasimonotone decreasing mapping has a unique fixed point.*

Proof. Under Nondegeneracy condition of the process the sequence of intersected sets Θ_i in the path-meeting method is defined uniquely by the starting point $\vartheta^1 \in \Theta^1$. Therefore, the end point $\vartheta^\nu = 0$ will belong to only one of the sets

Θ_i, if it is true for the starting point. Otherwise, moving along the ray Λ_0 in the opposite direction from the point $\vartheta = 0$, we would obtain a different set Θ for starting point ϑ^1 other than Θ^1. Thus, to proof the uniqueness of the fixed point, it is enough to make sure that there is a point ϑ, which possess this special property of uniqueness. By assumption there are a finite number of the cells $\Omega \in \omega$ and they cover all the space. Consider a unlimited cell of full dimension $\hat{\Omega}$. By assumption for complex ω this cell has vertexes. Therefore, $\hat{\Omega} = \hat{M} + \hat{Q}$, were $\hat{M}$ is the convex hull of all vertexes of the cell $\hat{\Omega}$ and $\hat{Q}$ is a finitely generated cone with the vertex $q = 0$. It is clear that as a $\hat{\Omega}$ cell can be chosen a such one for which the corresponding cone $\hat{Q}$ has the nonempty interior $\hat{Q}^\circ$. Take an arbitrarily vector $h \in \hat{Q}^\circ$ and consider the ray

$$\Lambda_h = \{p = \tau h | \tau \geq 0\}.$$

We have $\Lambda_h \subset \hat{Q}^\circ$ and the infinitely far part of this ray will be in the interior of the cell $\hat{\Omega}$, but not in the others unlimited cells Ω, because, as it can be easy shown, the corresponding cones Q° do not have common points. The mentioned infinitely far part of the ray Λ_h will be also in the interior of the corresponding cell $\hat{\Theta}$, but not in the others unlimited cells Θ, because they are shifts of the corresponding cells Ω. Thus the points of the infinitely far part of the ray Λ_h will have the mentioned special property of uniqueness.

The Theorem is proved.

7. CONCLUSION

The fixed point problem of a piecewise constant mapping in $\mathbb{R}^n$ is considered under condition that the mapping is quasimonotone decreasing one. It is shown that such a mapping has always a unique fixed point, which can be obtained by the path-meeting method of polyhedral complementarity in a finite number of steps.

These studies continue the earlier works of the author as part of an original approach of polyhedral complementarity to the equilibrium problem in exchange models, where the mappings are considered on the price simplex.

Funding. The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. 0324-2019-0018). The work was supported in part by the Russian Foundation for Basic Research, project 19-010-0091.

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