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ANALYSIS OF BULK SERVICE QUEUING SYSTEM WITH REWORK, UNRELIABLE SERVER, RESUMING SERVICE AND TWO KINDS OF MULTIPLE VACATION

S. KARPAGAM
Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai, India
karpagammaths19@gmail.com

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Abstract: In this paper, we discuss a batch arrival bulk service queue with rework for the faculty item, possibility of breakdown, repair, and two kinds of multiple vacation with different threshold policy. The server serves the remaining service after repair completion. This type of queuing model has a lot of applications in the manufacturing/ production industries. Performance measures and stability conditions are obtained. Numerical illustrations are made to examine the validity of analytical results.

Keywords: Non-Markovian queue, rework, repair, multiple vacation.

MSC: 60K25, 90B22, 68M20.

1. INTRODUCTION

Quality has become one of the most significant competitive strategic instruments which many manufacturing industries have acknowledged as a key to the creation of products and services that will assure continuous success. Quality is a universal ideal and has become a worldwide problem. Manufacturing industries must guarantee that their processes are continually checked and product quality is improved in order to be effective and deliver quality products to consumers. Quality control is the process of comparing output to a standard and taking remedial action when the output does not match the standard. If the results are satisfactory, no further action is necessary; otherwise, remedial action is required. When a company fails to take quality control seriously, it invariably ends in scrapped, reworked, or returned items by consumers.

A manufacturing fault is a mistake done in the design or production stage of a product which causes malfunctioning of the product. It could be rectified after some rework. For example, in the automobile manufacturing company / Almira manufacturing company, if there is any defective component say lock is found during the production process or welding defect found during a quality control check-up, the defective needs to be repaired by rework. Reworking a faulty item might save the company from heavy loss. Also, during the manufacturing process, there is a probability of machine failure also besides the rework issue.

In a bulk manufacturing process, entry of defective component as rework in the production process or machine failure during the production cycle affects the queue length and delays the production process. Queuing theory has been used by many authors to solve rework problems in manufacturing industries to optimise the production process with minimum idle time. So, the most prevalent issues confronting the manufacturing sectors is the unintentional production of faulty products. A manufacturing fault is a mistake in the design or production of a product that causes it to malfunction. It could work after some rework.

Henga *e*t al. [1], analysed a multistage serial manufacturing system with rework loops and product polymorphism. Mangey Ram *e*t al. [2] discussed stochastic design exploration with a rework of flexible manufacturing system under copula-coverage approach. Karpagam, *e*t al. [3] studied a bulk service queuing system with rework by using supplementary variable technique.

Kuntal Bakuli *e*t al. [4] investigated fixed batch size bulk service queuing system with impatient Customers. Paranjothi *e*t al. [5] studied a discrete-time gated vacation queue with a general bulk service rule (L, K). He considered two queues separated by a gate. Shanthi *e*t al. [6] evaluated a transient behaviour of single server bulk service queuing system with working vacation. Sourav Pradhan [7] analysed a single server infinite-buffer batch-size-dependent service queue with Poisson arrival and versatile batch-service rule. AnyueChen *e*t al. [8] studied a modified Markovian bulk-arrival and bulk-service queue with general state-dependent control. Gupta *e*t al. [9] derived steady state joint probability distribution of the number of customers with the server as well as in the queue is obtained by using PGF method for the infinite buffer bulk service queue.

Krishnamoorthy *e*t al. [10] analysed k-Stages of bulk service queuing system with accessible batches for the service. Ayyappan *e*t al. [11] investigated an unreliable bulk queuing system with customer's impatience. Nithya *e*t al. [12] provided simulation modelling for bulk service queuing system involved in textile industry. Recently, the mathematicians, Binay Kumar [13], Zirem *e*t al. [14], Charan Jeet Singh *e*t al. [15], Sethi [16], Chakravarthy*e*t al. [17], Vijaya Laxmi Pikkala *e*t al. [18], Kerobyan *e*t al. [19] are discussed the unreliable queuing systems. Ayyappan *e*t al. [20], Bouchentouf, *e*t al. ([21, 22, 23, 24], Nithya *e*t al. [25], Kempa *e*t al. [26], Joshi *e*t al. [27], Sadhna Singh *e*t al. [28] have analysed a queuing system incorporated with multiple vacation.

In this paper, a service queuing system with rework of a faulty batch and different threshold policy multiple vacation with random machine /server failure is analysed. We utilised remaining service time as a supplementary variable in this study. The whole paper is partitioned into following sections: In section 1, Introduction, Section 2, Model formulations with some assumptions , notations related to the model are given. Section 3, describes the governing equations wherein queue size distribution is obtained. In section 4, we attempt to find out the performance indices of the system. Numerical results are provided in section 5. Section 6, covers the conclusion of the paper. The Flow chart for rework is given in Figure 1.

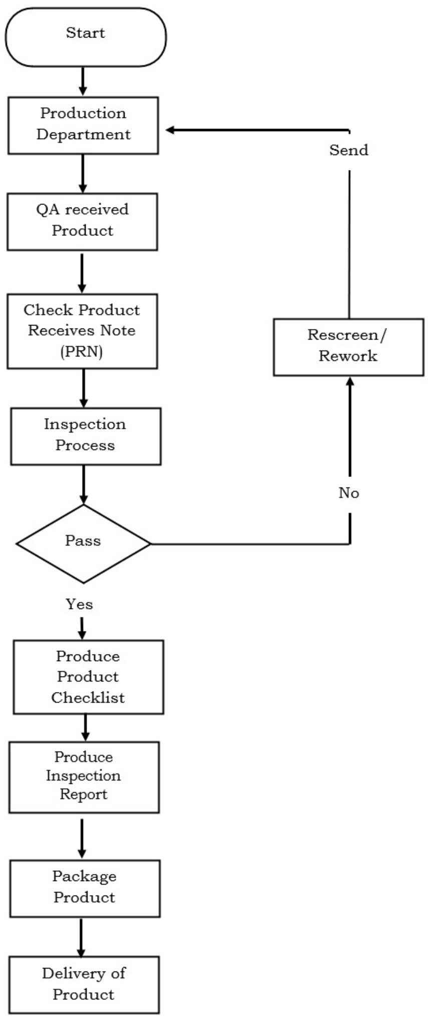

Figure 1: Flow chart for rework

2. MATHEMATICAL DESCRIPTION OF THE MODEL

We have developed a model with bulk service queue with the possibility of faculty item in the manufacturing industry with breakdown, repair and two types of vacation with different threshold policy. Breakdown follows an exponential distribution with rate 'α'. Service, checking, repair, type-I and type-II vacations are assumed to follow general distribution. The server starts production only if with a minimum of 'a' raw materials arrived. When a server breaks down, it is sent for repair and returns back after repair completion to complete the remaining service. On each batch service completion, it sent for quality checking if it found to faculty (with probability 'ε') then it sent for rework. Otherwise, if

i) The queue length (Q) is '$Q > a$' then server starts the service to the next batch by General Bulk Service Rule;

ii) The queue length is '$Q < a$' then the server starts the type-I vacation repeatedly until he finds minimum 'a' customers in the queue.

While getting back from vacation of type-I, if the queue has its length '$a \leq Q < N$' then the server performs the next vacation of type-II continuously, till it reaches the threshold value $N(a < b \leq N)$. After A practical case of injection moulding of plastic pipe is taken as an industry example is given in Figure 2. The results of the model could be used to reduce the production cost by reducing downtime and improve productivity.

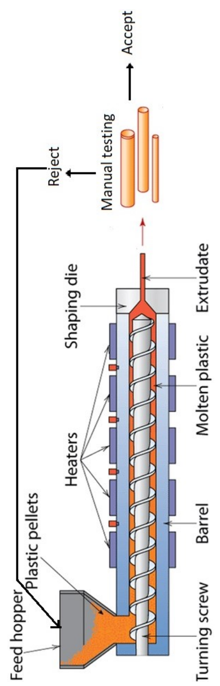

Figure 2: Industry application

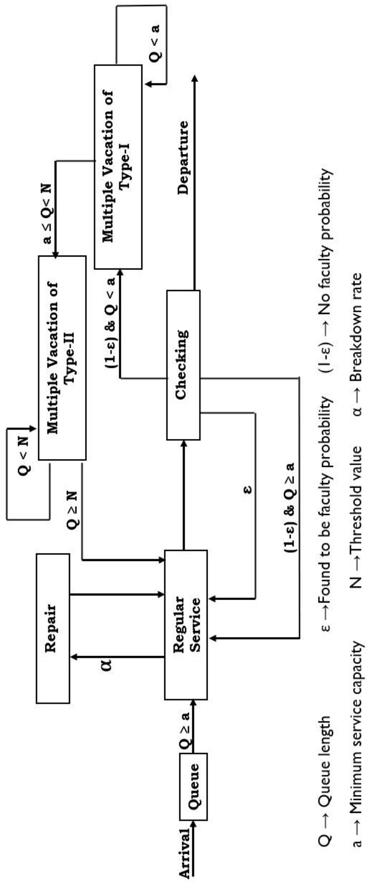

Figure 3: Pictorial representation of the model

2.1. Notations and probabilities

Notation	Description
λ	Group arrival rate
$X(z)$	Probability Generating Function (PGF) of X
$N_1(t)$	Number of customers in service station
$N_2(t)$	Number of customers in queue
$S^0(t)$	remaining service time
$C^0(t)$	remaining checking time
$R^{(0)}(t)$	remaining repair time
$V^{(0)}(t)$	remaining type-I vacation time
$W^{(0)}(t)$	remaining type-II vacation time

$g_k = Pr(X = k), k = 1, 2, 3, ...$

The notations of Cumulative Distribution Functions (CDF), Probability Density Function (PDF) and its Laplace-Stieltjes Transform (LST) are listed below:

	CDF	PDF	LST
Service time	S	$s(w)$	$\tilde{S}(\tau)$
Checking time	C	$c(w)$	$\tilde{C}(\tau)$
Repair time	R	$r(y)$	$R^*(\theta)$
Vacation Type-I	V	$v(w)$	$\tilde{V}(\tau)$
Vacation Type-II	W	$y(w)$	$\tilde{W}(\tau)$

$\psi(t) = (1), (2), (3), (4)$ and (5) denotes server is busy, checking, type-I, type-II vacation, and repair respectively.

$$\begin{aligned}
M_{r,j}(w,t)\Delta t &= Pr\{N_1(t) = r,\ N_2(t) = j,\ w \le S^0(t) \le w + \Delta t,\ \psi(t) = 1\}, \\
&\qquad\qquad a \le r \le b,\ j \ge 1, \\
C_{r,j}(w,t)\Delta t &= Pr\{N_1(t) = r,\ N_2(t) = j,\ w \le C^0(t) \le w + \Delta t,\ \psi(t) = 2\}, \\
&\qquad\qquad a \le r \le b,\ j \ge 1, \\
V_{l,j}(w,t)\Delta t &= Pr\{Z_1(t) = l, N_2(t) = j,\ w \le V^0(t) \le w + \Delta t,\ \psi(t) = 3\}, \\
&\qquad\qquad l \ge 1, j \ge 0. \\
W_{l,j}(w,t)\Delta t &= Pr\{Z_2(t) = l, N_2(t) = j,\ w \le W^0(t) \le w + \Delta t,\ \psi(t) = 4\}, \\
&\qquad\qquad l \ge 1, j \ge a. \\
R_n(w,y)\Delta t &= Pr\{N_2(t) = n, S^0(t) = w,\ y \le R^0(t) \le y + \Delta t\ \psi(t) = 5\},\ n \ge a.
\end{aligned}$$

3. GOVERNING EQUATIONS

The Kolmogorov backward equation governing the model as follows:

Server is in Service State

$$\begin{aligned}
-M'_{d,0}(w) &= -(\lambda + \alpha)M_{d,0}(w) + \varepsilon C_{d,0}(0)s(w) + R_{d,0}(w,0) \\
&\quad + (1-\varepsilon)\sum_{r=a}^{b} C_{r,d}(0)s(w),\ a \le d \le b, \qquad (1) \\
-M'_{d,j}(w) &= -(\lambda + \alpha)M_{d,j}(w) + \varepsilon C_{d,j}(0)s(w) + R_{d,j}(w,0) \\
&\quad + \sum_{k=1}^{j} M_{d,j-k}(w)\lambda g_k,\ j \ge 1,\ a \le d \le b-1, \qquad (2)
\end{aligned}$$

$$-M_{b,j}^{'}(w) = -(\lambda+\alpha)M_{b,j}(w) + \varepsilon C_{b,j}(0)s(w) + (1-\varepsilon)\sum_{r=a}^{b} C_{r,b+j}(0)s(w)$$
$$+ R_{b,j}(w,0) + \sum_{k=1}^{j} M_{b,j-k}(w)\lambda g_k,\ 1 \le j \le N-b-1, \tag{3}$$

$$-M_{b,j}^{'}(w) = -(\lambda+\alpha)M_{b,j}(w) + \varepsilon C_{b,j}(0)s(w) + (1-\varepsilon)\sum_{r=a}^{b} C_{r,b+j}(0)s(w)$$
$$+ \sum_{k=1}^{j} M_{b,j-k}(w)\lambda g_k + \sum_{l=1}^{\infty} V_{l,b+j}(0)s(w) + \sum_{l=1}^{\infty} W_{l,b+j}(0)s_(w)$$
$$+ R_{b,j}(w,0),\ j \ge N-b, \tag{4}$$

<u>Server is in Checking State</u>

$$-C_{d,0}^{'}(w) = -\lambda C_{d,0}(w) + M_{d,0}(0)c(w),\ a \le d \le b, \tag{5}$$

$$-C_{d,j}^{'}(w) = -\lambda C_{d,j}(w) + M_{d,j}(0)c(w) + \sum_{k=1}^{j} C_{d,j-k}(w)\lambda g_k,\ j \ge 1,$$
$$a \le d \le b, \tag{6}$$

<u>Server is in Repair</u>

$$-\frac{\partial}{\partial y}R_{d,0}(w,y) = -\lambda R_{d,0}(w,y) + \alpha M_{d,0}(w)r(y),\ a \le d \le b, \tag{7}$$

$$-\frac{\partial}{\partial y}R_{d,j}(w,y) = -\lambda R_{d,j}(w,y) + \alpha M_{d,j}(w)r(y) + \sum_{k=1}^{j} R_{d,j-k}(w,y)\lambda g_k,$$
$$j \ge 1,\ a \le d \le b, \tag{8}$$

<u>Server is in Type-I Vacation</u>

$$-{V_{1,0}}^{'}(w) = -\lambda V_{1,0}(w) + (1-\varepsilon)\sum_{r=a}^{b} C_{r,0}(0)v(w), \tag{9}$$

$$-{V_{1,j}}^{'}(w) = -\lambda V_{1,j}(w) + (1-\varepsilon)\sum_{r=a}^{b} C_{r,j}(0)v(w) + \sum_{k=1}^{j} V_{1,j-k}(w)\lambda g_k,$$
$$1 \le j \le a-1, \tag{10}$$

$$-{V_{1,j}}^{'}(w) = -\lambda V_{1,j}(w) + \sum_{k=1}^{j} V_{1,j-k}(w)\lambda g_k,\ j \ge a, \tag{11}$$

$$-{V_{l,0}}^{'}(w) = -\lambda V_{l,0}(w) + V_{l-1,0}(0)v(w),\ l \ge 2, \tag{12}$$

$$-{V_{l,j}}^{'}(w) = -\lambda V_{l,j}(w) + V_{l-1,j}(0)v(w) + \sum_{k=1}^{j} V_{l,j-k}(w)\lambda g_k,\ l \ge 2,$$
$$1 \le j \le a-1, \tag{13}$$

$$-V_{l,j}{}'(w) = -\lambda V_{l,j}(w) + \sum_{k=1}^{j} V_{l,j-k}(w)\lambda g_k,\ j \geq a,\ l \geq 2, \tag{14}$$

Server is in Type-II Vacation

$$-W_{1,a}{}'(w) = -\lambda W_{1,a}(w) + \sum_{l=1}^{\infty} V_{l,a}(0)y(w), \tag{15}$$

$$-W_{1,j}{}'(w) = -\lambda W_{1,j}(w) + \sum_{l=1}^{\infty} V_{l,j}(0)y(w) + \sum_{k=1}^{j} W_{1,j-k}(w)\lambda g_k, \quad a+1 \leq j \leq N-1 \tag{16}$$

$$-W_{1,j}{}'(w) = -\lambda W_{1,j}(w) + \sum_{k=1}^{j} W_{1,j-k}(w)\lambda g_k,\ j \geq N \tag{17}$$

$$-W_{l,a}{}'(w) = -\lambda W_{l,a}(w) + W_{l-1,a}(0)y(w),\ l \geq 2, \tag{18}$$

$$-W_{l,j}{}'(w) = -\lambda W_{l,j}(w) + W_{l-1,j}(0)y(w) + \sum_{k=1}^{j} W_{l,j-k}(w)\lambda g_k,\ l \geq 2, \quad a+1 \leq j \leq N-1 \tag{19}$$

$$-W_{l,j}{}'(w) = -\lambda W_{l,j}(w) + \sum_{k=1}^{j} W_{l,e-k}(w)\lambda g_k,\ l \geq 2,\ j \geq N. \tag{20}$$

While applying LST to the above equations (1) to (20), we get,

$$\tau\tilde{M}_{d,0}(\tau) - M_{d,0}(0) = (\lambda+\alpha)\tilde{M}_{d,0}(\tau) - \varepsilon C_{d,0}(0)\tilde{S}(\tau) - \tilde{R}_{d,0}(\tau,0) - (1-\varepsilon)\sum_{r=a}^{b} C_{r,d}(0)\tilde{S}(\tau),\ a \leq d \leq b, \tag{21}$$

$$\tau\tilde{M}_{d,j}(\tau) - M_{d,j}(0) = (\lambda+\alpha)\tilde{M}_{d,j}(\tau) - \varepsilon C_{d,j}(0)\tilde{S}(\tau) - \tilde{R}_{d,j}(\tau,0) - \sum_{k=1}^{j} \tilde{M}_{d,j-k}(\tau)\lambda g_k,\ a \leq d \leq b-1, j \geq 1, \tag{22}$$

$$\tau\tilde{M}_{b,j}(\tau) - M_{b,j}(0) = (\lambda+\alpha)\tilde{M}_{b,j}(\tau) - \tilde{R}_{b,j}(\tau,0) - \sum_{k=1} \tilde{M}_{b,j-k}(\tau)\lambda g_k - (1-\varepsilon)\sum_{d=a}^{b} C_{r,b+j}(0)\tilde{S}(\tau) - \varepsilon C_{b,j}(0)\tilde{S}(\tau), \quad 1 \leq j \leq N-b-1, \tag{23}$$

$$\tau\tilde{M}_{b,j}(\tau) - M_{b,j}(0) = (\lambda+\alpha)\tilde{M}_{b,j}(\tau) - \varepsilon C_{b,j}(0)\tilde{S}(\tau) + \sum_{k=1}^{j}\tilde{M}_{b,j-k}(\tau)\lambda g_k$$
$$- \sum_{l=1}^{\infty} V_{l,b+j}(0)\tilde{S}(\tau) - \sum_{l=1}^{\infty} W_{l,b+j}(0)\tilde{S}(\tau)$$
$$- (1-\varepsilon)\sum_{r=a}^{b} C_{r,b+j}(0)\tilde{S}(\tau) - \tilde{R}_{b,j}(\tau,0),\ j \geq N-b, \quad (24)$$

$$\tau\tilde{C}_{d,0}(\tau) - C_{d,0}(0) = \lambda\tilde{C}_{d,0}(\tau) - M_{d,0}(0)\tilde{C}(\tau),\ a \leq d \leq b, \quad (25)$$

$$\tau\tilde{C}_{d,j}(\tau) - C_{d,j}(0) = \lambda\tilde{C}_{d,j}(\tau) - M_{d,j}(0)\tilde{C}(\tau) + \sum_{k=1}^{j}\tilde{C}_{d,j-k}(\tau)\lambda g_k,$$
$$a \leq d \leq b, j \geq 1 \quad (26)$$

$$-\frac{\partial}{\partial y}\tilde{R}_{d,0}(\tau,y) = \lambda\tilde{R}_{d,0}(\tau,y) - \alpha\tilde{M}_{d,0}(\tau)r(y),\ a \leq d \leq b, \quad (27)$$

$$-\frac{\partial}{\partial y}\tilde{R}_{d,j}(\tau,y) = \lambda\tilde{R}_{d,j}(\tau,y) - \alpha\tilde{M}_{d,j}(\tau)r(y) - \sum_{k=1}^{j}\tilde{R}_{d,j-k}(\tau,y)\lambda g_k,$$
$$j \geq 1,\ a \leq d \leq b, \quad (28)$$

$$\tau\tilde{V}_{1,0}(\tau) - V_{1,0}(0) = \lambda\tilde{V}_{1,0}(\tau) - (1-\varepsilon)\sum_{r=a}^{b} C_{r,0}(0)\tilde{V}(\tau), \quad (29)$$

$$\tau\tilde{V}_{1,j}(\tau) - V_{1,j}(0) = \lambda\tilde{V}_{1,j}(\tau) - (1-\varepsilon)\sum_{r=a}^{b} C_{r,j}(0)\tilde{V}(\tau) - \sum_{k=1}^{j} V_{1,j-k}(\tau)\lambda g_k,$$
$$1 \leq j \leq a-1, \quad (30)$$

$$\tau\tilde{V}_{1,j}(\tau) - V_{1,j}(0) = \lambda\tilde{V}_{1,j}(\tau) - \sum_{k=1}^{j}\tilde{V}_{1,j-k}(\tau)\lambda g_k,\ j \geq a, \quad (31)$$

$$\tau\tilde{V}_{l,0}(\tau) - V_{l,0}(0) = \lambda\tilde{V}_{l,0}(\tau) - V_{l-1,0}(0)\tilde{V}(\tau),\ l \geq 2, \quad (32)$$

$$\tau\tilde{V}_{l,j}(\tau) - V_{l,j}(0) = \lambda\tilde{V}_{l,j}(\tau) - V_{l-1,j}(0)\tilde{V}(\tau) - \sum_{k=1}^{j}\tilde{V}_{l,j-k}(\tau)\lambda g_k,$$
$$1 \leq j \leq a-1,\ l \geq 2 \quad (33)$$

$$\tau\tilde{V}_{l,j}(\tau) - V_{l,j}(0) = \lambda\tilde{V}_{l,j}(\tau) - \sum_{k=1}^{j}\tilde{V}_{l,j-k}(\tau)\lambda g_k,\ j \geq a,\ l \geq 2, \quad (34)$$

$$\tau\tilde{W}_{1,a}(\tau) - W_{1,a}(0) = \lambda\tilde{W}_{1,a}(\tau) - \sum_{l=1}^{\infty} V_{l,a}(0)\tilde{W}(\tau), \quad (35)$$

$$\tau\tilde{W}_{1,j}(\tau) - W_{1,j}(0) = \lambda\tilde{W}_{1,j}(\tau) - \sum_{l=1}^{\infty} V_{l,j}(0)\tilde{W}(\tau) - \sum_{k=1}^{j}\tilde{W}_{1,j-k}(\tau)\lambda g_k,$$
$$a+1 \leq j \leq N-1 \quad (36)$$

$$\tau\tilde{W}_{1,j}(\tau) - W_{1,j}(0) = \lambda\tilde{W}_{1,j}(\tau) - \sum_{k=1}^{j} W_{1,j-k}(\tau)\lambda g_k,\ j \geq N \tag{37}$$

$$\tau\tilde{W}_{l,a}(\tau) - W_{l,a}(0) = \lambda\tilde{W}_{l,a}(\tau) - W_{l-1,a}(0)\tilde{W}(\tau),\ l \geq 2, \tag{38}$$

$$\tau\tilde{W}_{l,j}(\tau) - W_{l,j}(0) = \lambda\tilde{W}_{l,j}(w) - W_{l-1,j}(0)\tilde{W}(\tau) - \sum_{k=1}^{j} \tilde{W}_{l,j-k}(\tau)\lambda g_k,$$
$$l \geq 2,\ a+1 \leq j \leq N-1 \tag{39}$$

$$\tau\tilde{W}_{l,j}(\tau) - W_{l,j}(0) = \lambda\tilde{W}_{l,j}(w) - \sum_{k=1}^{j} \tilde{W}_{l,j-k}(\tau)\lambda g_k,\ l \geq 2,\ j \geq N. \tag{40}$$

Again applying LST to the above equations (27) to (28), we get,

$$\theta\tilde{R}^*_{d,0}(\tau,\theta) - \tilde{R}_{d,0}(\tau,0) = \lambda\tilde{R}^*_{d,0}(\tau,\theta) - \alpha\tilde{M}_{d,0}(\tau)\tilde{R}(\theta),\ a \leq d \leq b, \tag{41}$$

$$\theta\tilde{R}^*_{d,j}(\tau,\theta) - \tilde{R}_{d,j}(\tau,0) = \lambda\tilde{R}^*_{d,j}(\tau,\theta) - \alpha\tilde{M}_{d,j}(\tau)\tilde{R}(\theta) - \sum_{k=1}^{j} \tilde{R}^*_{d,j-k}(\tau,\theta)\lambda g_k,$$
$$j \geq 1,\ a \leq d \leq b, \tag{42}$$
$$\tag{43}$$

The probability generating functions (PGFs) used for mathematical analysis are as follows:

$$\tilde{M}_d(z,\tau) = \sum_{j=0}^{\infty} \tilde{M}_{d,j}(\tau)z^j, \quad M_d(z,0) = \sum_{j=0}^{\infty} M_{d,j}(0)z^j, a \leq d \leq b,$$
$$\tilde{R}^*_d(z,\tau,\theta) = \sum_{j=0}^{\infty} \tilde{R}^*_{d,j}(\tau,\theta)z^j, \quad \tilde{R}_d(z,\tau,0) = \sum_{j=0}^{\infty} \tilde{R}_{d,j}(\tau,0)z^j, a \leq d \leq b,$$
$$\tilde{V}_l(z,\tau) = \sum_{j=0}^{\infty} \tilde{V}_{l,j}(\tau)z^j, \quad V_l(z,0) = \sum_{j=0}^{\infty} V_{l,j}(0)z^j,\ l \geq 1,$$
$$\tilde{W}_l(z,\tau) = \sum_{j=a}^{\infty} \tilde{W}_{l,j}(\tau)z^j, \quad W_l(z,0) = \sum_{j=a}^{\infty} W_{l,j}(0)z^j,\ l \geq 1. \tag{44}$$

4. PROBABILITY GENERATING FUNCTION OF THE QUEUE SIZE

Let P(z) be the probability generating function of the number of clients in the queue at an arbitrary time epoch of the proposed model. Then,

$$P(z) = \sum_{i=a}^{b} \tilde{M}_i(z,0) + \sum_{i=a}^{b} \tilde{C}_i(z,0) + \sum_{i=a}^{b} \tilde{R^*}_i(z,0,0) + \sum_{l=1}^{\infty} \tilde{V}(z,0) + \sum_{l=1}^{\infty} \tilde{W}_l(z,0). \tag{45}$$

The probability generating function is obtained as:

$$P(z) = \frac{\Big[L(z)\sum_{d=a}^{b-1}(z^b - z^d)m_d + (\tilde{V}(v(z)) - 1)A(z)\sum_{n=0}^{a-1}(m_n + v_n)z^n + (\tilde{W}(v(z)) - 1)A(z)\sum_{n=a}^{N-1}(v_n + w_n)z^n\Big]}{u(z)v(z)h(z)}. \tag{46}$$

where

$$\begin{aligned}
L(z) &= v(z)(\tilde{M}(u(z)) - 1) + u(z)\tilde{M}(u(z))(\tilde{C}(v(z)) - 1)\\
&\quad + \alpha(\tilde{M}(u(z)) - 1)(1 - R^*(v(z))),\\
A(z) &= v(z)(1 - \tilde{M}(u(z))) + u(z)\tilde{M}(u(z)) + \alpha(1 - \tilde{M}(u(z)))(1 - R^*(v(z)))\\
&\quad - z^b u(z) + (z^b - 1)\varepsilon u(z)\tilde{S}(u(z))\tilde{C}(v(z)),\\
h(z) &= z^b[1 - \varepsilon\tilde{S}(u(z))\tilde{C}(v(z))] - (1 - \varepsilon)\tilde{S}(u(z))\tilde{C}(v(z)),\\
u(z) &= \lambda + \alpha - \lambda X(z) - \alpha R^*(v(z)),\\
v(z) &= \lambda - \lambda X(z),\\
m_i &= (1 - \varepsilon)\sum_{i=a}^{b} C_{i,j}z^j,\\
v_l &= \sum_{l=1}^{\infty} V_{l,n}z^n,\\
w_l &= \sum_{l=1}^{\infty} W_{l,n}z^n.
\end{aligned}$$

4.1. Particular case

Case 1:
When there are no breakdown, faulty and second stage vacation then P(z) becomes,

$$P(z) = \frac{(\tilde{S}(v(z)) - 1)\sum_{n=a}^{b-1} m_n z^n + (\tilde{V}(v(z)) - 1)(z^b - 1)\sum_{n=0}^{a-1}(m_n + v_n)z^n}{(z^b - \tilde{S}(v(z)))v(z)} \tag{47}$$

which coincide with Jeyakumar et al. [29] without breakdown and closedown.

Case 2:
When there is no breakdown and rework then equation (46) becomes

$$P(z) = \frac{\left[(\tilde{S}(v(z)) - 1) \sum_{n=a}^{b-1} (z^b - z^n) m_n + (z^b - 1)(\tilde{W}(v(z)) - 1) \sum_{n=a}^{N-1} (v_n + w_n) z^n + (z^b - 1)(\tilde{V}(v(z)) - 1) \sum_{n=0}^{a-1} (m_n + v_n) z^n \right]}{(v(z))(z^b - \tilde{S}(v(z)))} \tag{48}$$

which coincides with Haridass et al. [30].

4.2. Computational Aspects

Equation (46) has the unknowns $m_0, ..., m_{b-1}, v_0, ..., v_{b-1}$ and $w_a, w_1, ..., w_{N-1}$. "We can express $w_i(i = a, a+1, ..., N-1)$ in terms of $v_i(i = a, a+1, ..., N-1)$ and $v_i(i = 0, 1, ..., N-1)$ in terms of $m_i(i = 0, 1, ..., a-1)$ such a way that numerator has only b constants. Now from equation (46) which is probability generating function of number of customers involves b unknowns. By Rouches's theorem h(z) has one zero on the boundary and b-1 inside the unit circle. Due to the analyticity of P(z), the numerator must vanish at these points and gives b equations with b unknowns, which can be solved by suitable numerical technique."

Remark: The necessary and sufficient condition for the existence of steady state for the model under consideration. $\rho = \frac{(\lambda X_1 + \alpha R_1)E(S) + \lambda X_1 E(C)}{b(1-\varepsilon)}$

Result-1

$$w_n = \sum_{i=a}^{n} \varphi_{n-i} v_i, \ n = a, a+1, a+2..., N-1, \varphi_0 = \frac{\gamma_0}{1-\gamma_0} \tag{49}$$

where

$$\varphi_n = \frac{\gamma_n + \sum_{j=1}^{n} \gamma_j \varphi_{n-j}}{1-\gamma_0} \tag{50}$$

Result-2

$$v_n = \sum_{i=0}^{n} K_{n-i} m_i, \ n = 0, 1, 2..., a-1, \text{where } K_0 = \frac{\beta_0}{1-\beta_0}$$

$$K_n = \frac{\beta_n + \sum_{j=1}^{n} \beta_j K_{n-j}}{1-\beta_0}$$

Result-3

$$v_n = \sum_{i=0}^{a-1}(\beta_{n-i} + \sum_{j=0}^{a-1-i} K_j\beta_{n-j-i})m_i,\ n = a, a+1, a+2..., N-1, \tag{51}$$

where

$$K_n = \frac{\beta_n + \sum_{j=1}^{n}\beta_j\varphi_{n-j}}{1-\beta_0} \tag{52}$$

β_n, γ_n are the probabilities of 'n' customers arrive during main server's type-I and type-II vacation time respectively.

4.3. PGF of Queue Size at Various Completion Epochs

Server is busy:

$$P_b(z) = \frac{(1-\tilde{M}(u(z)))\Big[\sum_{d=a}^{b-1}(z^b - z^d)m_d + (\tilde{V}(v(z))-1)\sum_{n=0}^{a-1}(m_n+v_n)z^n + (\tilde{W}(v(z))-1)[\sum_{n=a}^{N-1}(v_n+w_n)z^n]\Big]}{u(z)h(z)}. \tag{53}$$

Server on repair:

$$P_r(z) = \frac{\alpha(1-\tilde{M}(u(z)))(1-R^*(v(z)))\Big[(\tilde{W}(v(z))-1)\sum_{n=a}^{N-1}(v_n+w_n)z^n + \sum_{d=a}^{b-1}(z^b-z^d)m_d + (\tilde{V}(v(z))-1)\sum_{n=0}^{a-1}(m_n+v_n)z^n\Big]}{u(z)v(z)h(z)}. \tag{54}$$

Server on vacation:

$$P_v(z) = \frac{(1-\tilde{V}(v(z)))\sum_{n=0}^{a-1}(m_n+v_n)z^n \quad (1-\tilde{W}(v(z)))\sum_{k=a}^{N-1}(v_k+w_k)z^k}{v(z)}. \tag{55}$$

4.4. Probability of Various Server States

Server on vacation:

$$P(V) = \frac{V_1\sum_{n=0}^{a-1}(m_n+v_n) + W_1\sum_{k=a}^{N-1}(v_k+w_k)}{\lambda X_1} \tag{56}$$

Server on repair:

$$P(R) = \frac{\alpha R_1 M_1 \Big[\sum_{d=a}^{b-1} (b-d) m_d + V_1 \sum_{n=0}^{a-1} (m_n + v_n) + W_1 [\sum_{n=a}^{N-1} (v_n + w_n)] \Big]}{u_1 v_1 h_1}. \tag{57}$$

Server is busy:

$$P(R) = \frac{M_1 \Big[\sum_{d=a}^{b-1} (b-d) m_d + V_1 \sum_{n=0}^{a-1} (m_n + v_n) + W_1 [\sum_{n=a}^{N-1} (v_n + w_n)] \Big]}{(\lambda X_1 + \alpha R_1) h_1} \tag{58}$$

4.5. Performance Measures

The expected length of busy period:

$$E(B) = \frac{E(S) + \varepsilon E(C) + \alpha E(R)}{\sum_{n=0}^{a-1} m_n}, \tag{59}$$

The expected length of idle period:

$$E(I) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \beta_{n-i} m_i} + \frac{E(W)}{1 - \sum_{n=a}^{N-1} \sum_{i=a}^{n} \gamma_{n-i} v_i}, \tag{60}$$

The mean queue length:

$$E(Q) = \frac{Nr^{(IV)} Dr^{(III)} - Nr^{(III)} Dr^{(IV)}}{4 (Dr^{(III)})^2}, \tag{61}$$

where

$$\begin{aligned}
Nr^{(III)} &= 3[L_2 \sum_{i=a}^{b-1} (b-i) m_i + V_1 A_2 \sum_{n=0}^{a-1} (m_n + v_n) + W_1 A_2 \sum_{n=a}^{N-1} (v_n + w_n)], \\
Nr^{(IV)} &= 4 L_3 \sum_{i=a}^{b-1} (b-i) m_i + 6 L_2 \sum_{i=a}^{b-1} (b(b-1) - i(i-1)) m_i \\
&\quad + [6 V_2 A_2 + 4 V_1 A_3] \sum_{n=0}^{a-1} (m_n + v_n) + 12 V_1 A_2 \sum_{n=0}^{a-1} n (m_n + v_n) \\
&\quad + [6 W_2 A_2 + 4 W_1 A_3] \sum_{n=0}^{a-1} (v_n + w_n) + 12 W_1 A_2 \sum_{n=0}^{N-1} n (v_n + w_n),
\end{aligned}$$

$$Dr^{(III)} = 6u_1v_1h_1,\ Dr^{(IV)} = 12[u_2v_1h_1 + u_1(v_2h_1 + v_1h_2)],$$
$$u_1 = -\lambda X_1 - \alpha R_1,\ v_1 = -\lambda X_1,$$
$$h_1 = b(1-\varepsilon) - (S_1 + C_1),\ u_2 = -\lambda X_2 - \alpha R_2,\ v_2 = -\lambda X_2,$$
$$h_2 = b(b-1)(1-\varepsilon) - 2b\varepsilon(S_1 + C_1) - (S_2 + 2S_1C_1 + C_2),$$
$$S_1 = (\lambda X_1 + \alpha R_1)E(S),\ C_1 = \lambda X_1 E(C),\ R_1 = \lambda X_1 E(R)$$
$$S_2 = (\lambda X_2 + \alpha R_2)E(S) + (-\lambda X_1 - \alpha R_1)^2 E(S^2),$$
$$C_2 = \lambda X_2 E(C) + (\lambda X_1)^2 E(C^2),\ R_2 = \lambda X_2 E(R) + (\lambda X_1)^2 E(R^2),$$
$$L_2 = 2[V_1S_1 + u_1C_1 - \alpha S_1R_1],$$
$$A_2 = u_2 - 2V_1S_1 + 2u_1S_1 + 2\alpha S_1R_1 + 2\varepsilon u_1(S_1 + C_1) - (1-\varepsilon)(2bu_1 + u_2)$$
$$- \varepsilon(u_2 + 2u_1(S_1 + C_1)),$$
$$L_3 = 3[v_2S_1 + v_1S_2 + u_2C_1 + u_1[2S_1C_1 + C_2] - S_2R_1 - S_1R_2],$$
$$A_3 = u_3 + 3u_2S_1 + 3u_1S_2 - 3[v_2S_1 + v_1S_2] - 3\alpha[S_2R_1 + S_1R_2]$$
$$+ 3\varepsilon u_1(S_2 + 2S_1C_1 + C_2) - (1-\varepsilon)(3b(b-1)u_1 + 3bu_2 + u_3)$$
$$+ 3\varepsilon(S_1 + C_1)(2bu_1 + u_2)$$
$$- \varepsilon[u_3 + 3u_2(S_1 + C_1) + 3u_1(S_2 + 2S_1C_1 + C_2)]$$

5. NUMERICAL EXAMPLES

The established analytical findings can be numerically derived by using an appropriate example. This section contains numerical findings for different performance indices. The computer program was created using the MATLAB software. For calculation purposes, we set the following system parameters:

1. Service time and Checking distribution is 2-Erlang. Let a=3, b=8 and N=10.
2. Batch size distribution of the arrival is geometric with mean 2.
3. Type-I and Type-II Vacation is exponential with parameters $\beta = 8$ and $\gamma = 10$ respectively.
4. Repair is exponential with parameters $\eta = 5$ and the breakdown rate is $\alpha = 1$.
5. Let μ_1 and μ_2 be the service and checking rate respectively.

For different service rate, arrival rate, breakdown rate, faulty probability and repair rate, the performance measures $E(Q)$, $E(W)$, $E(B)$ and $E(I)$ are calculated (Tables 1-5 and Figures 4-11).

Table 1: Service rate vs Performance measures ($\lambda = 5,\ \mu_2 = 25,\ \varepsilon = 0.2, \alpha = 1$ and $\eta = 5$)

μ_1	ρ	$E(Q)$	$E(W)$	$E(B)$
11.0	0.232955	4.68418	0.468418	0.370506
11.5	0.225543	4.35571	0.435571	0.364994
12.0	0.218750	4.07075	0.407075	0.359968
12.5	0.212500	3.82181	0.382181	0.355366
13.0	0.206731	3.60297	0.360297	0.351137
13.5	0.201389	3.40948	0.340948	0.347236
14.0	0.196429	3.23749	0.323749	0.343627
14.5	0.191810	3.08388	0.308388	0.340278
15.0	0.187500	2.94606	0.294606	0.337161
15.5	0.183468	2.82190	0.282190	0.334254
16.0	0.179687	2.70961	0.270961	0.331535
16.5	0.176136	2.60770	0.260770	0.328987
17.0	0.172794	2.51489	0.251489	0.326594
17.5	0.169643	2.43010	0.243010	0.324343
18.0	0.166667	2.35242	0.235242	0.322221
18.5	0.163851	2.28106	0.228106	0.320217
19.0	0.161184	2.21532	0.221532	0.318321
19.5	0.158654	2.15462	0.215462	0.316525
20.0	0.156250	2.09844	0.209844	0.314822

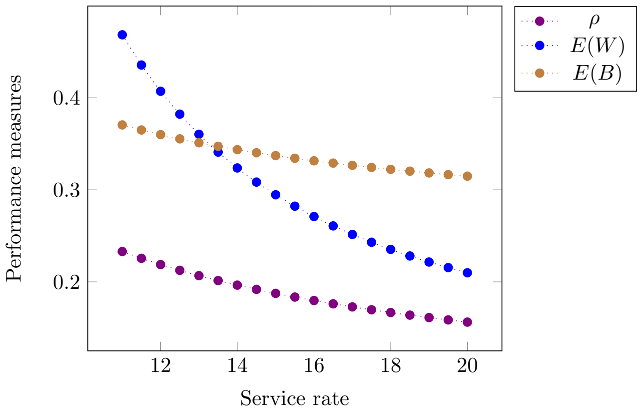

Figure 4: Service rate (vs) Performance measures

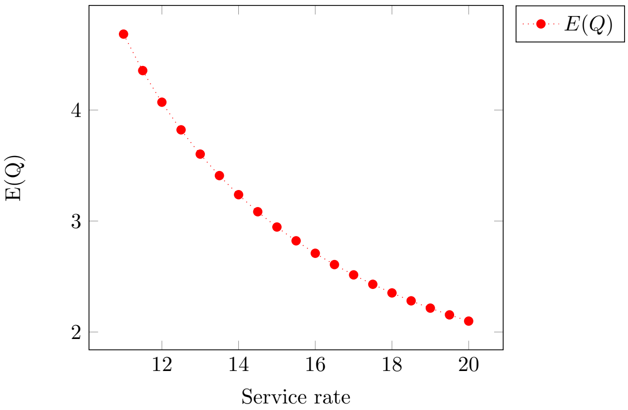

Figure 5: Service rate (vs) E(Q)

Table 2: Arrival rate vs Performance measures ($\mu_1 = 10,\ \mu_2 = 20,\ \varepsilon = 0.2, \alpha = 1$ and $\eta = 5$)

λ	Vacation of Type-I $(V)>$ Type-II (W)		Both vacations are of same length $(V = W)$	
	$E(Q)$	$E(W)$	$E(Q)$	$E(W)$
3	2.78522	0.46420	2.80437	0.46739
4	3.92603	0.49075	3.94341	0.49293
5	5.57869	0.55787	5.59006	0.55901
6	7.86895	0.65575	7.86869	0.65573
7	10.9632	0.78309	10.9444	0.78175
8	15.0869	0.94293	15.0412	0.94007
9	20.5531	1.14184	20.4707	1.13726
10	27.8124	1.39062	27.6819	1.38410
11	37.5384	1.70629	37.3469	1.69759
12	50.7908	2.11628	50.5235	2.10515

Table 3: Breakdown rate vs Performance measures ($\lambda = 10,\ \mu_1 = 10,\ \mu_2 = 15,\ \varepsilon = 0.2,\ \eta = 5$)

α	ρ	$E(Q)$	$E(W)$	$E(B)$
1.0	0.583333	30.4902	1.52451	0.331096
1.5	0.614583	30.4909	1.52455	0.480051
2.0	0.645833	31.3283	1.56642	0.659695
2.5	0.677083	31.3581	1.56791	0.879108
3.0	0.708333	33.1976	1.65988	1.151260
3.5	0.739583	36.2061	1.81030	1.495340
4.0	0.770833	40.7257	2.03629	1.941020
4.5	0.802083	47.3468	2.36734	2.536690
5.0	0.833333	57.1304	2.85652	3.367070

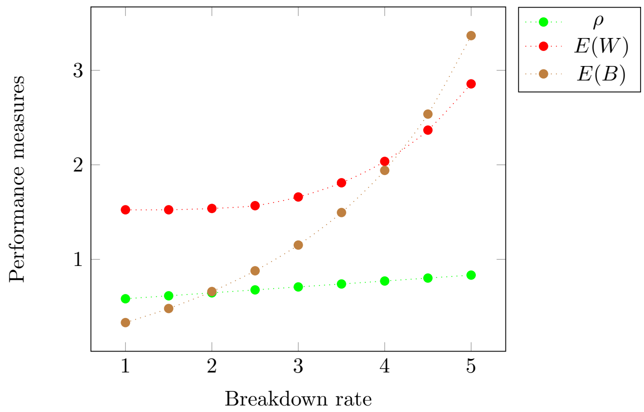

Figure 6: Breakdown rate (vs) Performance measures

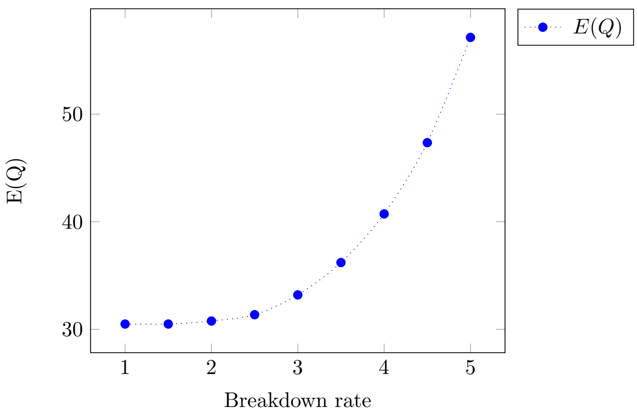

Figure 7: Breakdown rate (vs) E(Q)

Table 4: Faculty probability vs Performance measures ($\lambda = 10,\ \mu_1 = 10,\ \mu_2 = 15,\ \alpha = 1,\ \eta = 5$)

ε	ρ	$E(Q)$	$E(W)$	$E(B)$	$E(I)$
0.00	0.466667	19.2720	0.96360	0.264526	0.150569
0.02	0.476190	20.0475	1.00238	0.269087	0.148653
0.04	0.486111	20.8876	1.04438	0.273960	0.146633
0.06	0.496454	21.8003	1.09001	0.279188	0.144499
0.08	0.507246	22.7953	1.13977	0.284816	0.142242
0.10	0.518519	23.8842	1.19421	0.290902	0.139852
0.12	0.530303	25.0806	1.25403	0.297511	0.137317
0.14	0.542636	26.4009	1.32004	0.304725	0.134624
0.16	0.555556	27.8650	1.39325	0.312642	0.131757
0.18	0.569106	29.4974	1.47487	0.321383	0.128701
0.20	0.583333	31.3283	1.56642	0.331096	0.125436
0.22	0.598291	33.3957	1.66979	0.341971	0.121941
0.24	0.614035	35.7480	1.78740	0.354246	0.118191
0.26	0.630631	38.4474	1.92237	0.368229	0.114160
0.28	0.648148	41.5760	2.07880	0.384329	0.109814
0.30	0.666667	45.2437	2.26219	0.403092	0.105118
0.32	0.686275	49.6016	2.48008	0.425272	0.100029
0.34	0.707071	54.8629	2.74314	0.451931	0.094498
0.36	0.729167	61.3392	3.06696	0.484627	0.088466
0.38	0.752688	69.5040	3.47520	0.525725	0.081866
0.40	0.777778	80.1140	4.00570	0.579018	0.074617

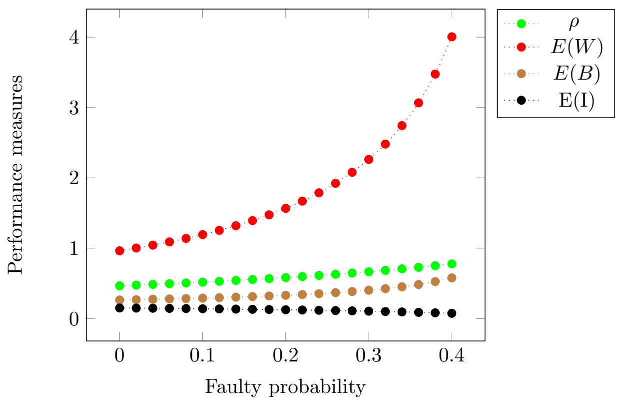

Figure 8: Faulty probability (vs) Performance measures

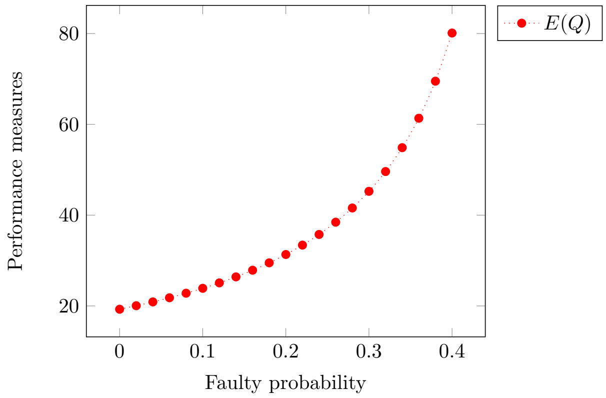

Figure 9: Faulty probability (vs) E(Q)

Table 5: Repair rate vs Performance measures ($\lambda = 10,\ \mu_1 = 10,\ \mu_2 = 15,\ \varepsilon = 0.2,\ \alpha = 5$)

η	ρ	$E(Q)$	$E(W)$	$E(B)$
3.0	0.833333	55.9064	2.79532	3.34295
3.2	0.813802	49.6391	2.48196	2.79813
3.4	0.796569	45.2933	2.26466	2.40664
3.6	0.781250	42.1452	2.10726	2.11217
3.8	0.767544	39.7895	1.98947	1.88289
4.0	0.755208	37.9823	1.89911	1.69949
4.2	0.744048	36.5680	1.82840	1.54956
4.4	0.733902	35.4439	1.77219	1.42479
4.6	0.724638	34.5387	1.72693	1.31940
4.8	0.716146	33.8021	1.69010	1.22924
5.0	0.708333	33.1976	1.65988	1.15126
5.2	0.701122	32.6980	1.63490	1.08317
5.4	0.694444	32.2828	1.61414	1.02322
5.6	0.688244	31.9361	1.59680	0.97005
5.8	0.682471	30.9910	1.54955	0.94244

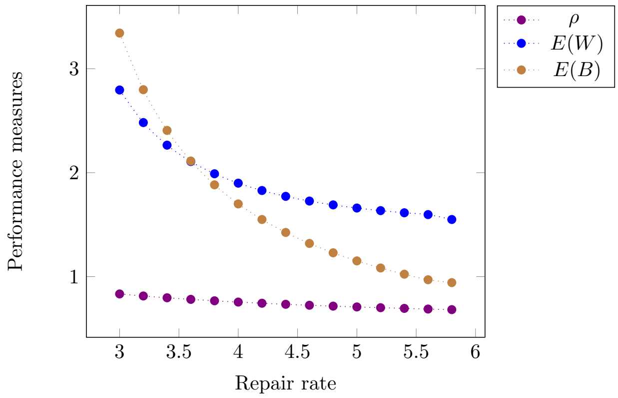

Figure 10: Repair rate (vs) Performance measures

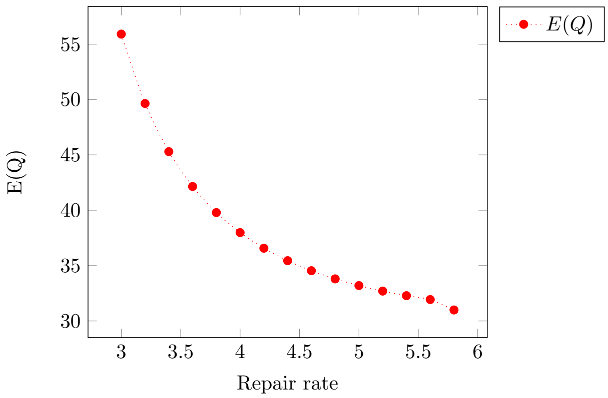

Figure 11: Repair rate (vs) E(Q)

The numerical results have been shown graphically in Figure 4 to 7. From the above results it may be noted that with increasing faulty probability and breakdown rate, the queue length is increasing it affects the production schedule badly. The production Engineer has to find out a solution by changing the failing parts or change of material quality or change the machine itself if the failure rate is very high to get satisfactory products with reduced the down time.

6. CONCLUSIONS

In this paper, we have discussed bulk service queuing system with active breakdown and repair, faulty probability, rework and variant threshold policy for vacations. To the best of knowledge of authors, there is no work has been done in the queuing literature with these combinations. Manufacturing industries must guarantee that their processes are continually checked and product quality is improved in order to be effective and deliver quality products to consumers. When a company fails to take quality control seriously, it invariably ends in scrapped, reworked, or returned items by consumers. The significant contribution in the proposed model is rework of the faulty item will reduce the heavy loss for the industry. This work can be further extended by incorporating the concepts of priority service and single vacation. The performance measures like, the mean number of consumers in the queue, the average waiting time of consumers in the queue, mean busy period, expected idle period and probability of various server state are obtained. The established analytical findings can be numerically derived.

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REFERENCES

[1] H. Zhang, L. Aipinga, L. Xuemeia, X. Liyuna, and G. Moronia, "Modeling and Performance Evaluation of Multistage Serial Manufacturing Systems with Rework Loops and Product Polymorphism", *Procedia CIRP*, vol. 63, pp. 471-476, 2017.

[2] R. Mangey, and N. Goyal, "Stochastic Design Exploration with Rework of Flexible Manufacturing System Under Copula-Coverage Approach", *International Journal of Reliability, Quality and Safety Engineering*, vol. 25, no. 2, 1850007, pp. 1-20, 2018.

[3] S.Karpagam, G. Ayyappan, and B. Somasundaram, "A Bulk queuing System with Rework in Manufacturing Industry with Starting Failure and Single Vacation", *International Journal of Applied and Computational Mathematics*, vol. 6, no. 6, pp. 1–22, 2020.

[4] B. Kuntal, and P. Manisha, "Bulk Service Queuing System with Impatient Customers: A computational approach", *Thailand Statistician*, vol. 15, no. 1, pp. 1–10, 2017.

[5] N. Paranjothi, G. Paulraj, and R. Sivasamy, "A Discrete-Time Gated Vacation Queue with a Bulk Service", *AIP Conference Proceedings 2177*, vol. 2177, no. 1, 020054, 2019.

[6] S. Shanthi, M. G. Subramanian, and A. G. Sekar, "Computational approach for transient behaviour of $M/M(a,b)/1$ bulk service queuing system with working vacation", *Journal of Mathematical and Computational Science*, vol. 10, no. 6, pp. 2557-2578, 2020.

[7] S. Pradhan, "On the distribution of an infinite-buffer queuing system with versatile bulk-service rule under batch-size-dependent service policy: $M/G_n^{(a,Y)}/1$", *International Journal of Mathematics in Operational Research*, vol. 16, no. 3, pp. 331-378, 2020.

[8] A. Chen, X. Wu, and J. Zhang, "Markovian bulk-arrival and bulk-service queues with general state-dependent control", *queuing Systems*, vol. 95, no. 3, pp. 331–378, 2020.

[9] G.K. Gupta, and A. Banerjee, "Analysis of infinite buffer general bulk service queue with state dependent balking", *International Journal of Operational Research*, vol. 40, no. 2, pp. 137–161, 2021.

[10] A. Krishnamoorthy, J. Anu Nuthan, and V. Vishnevsky, "Analysis of a k-Stage Bulk Service Queuing System with Accessible Batches for Service", *Mathematics*, vol. 9, no. 5, pp. 1–16, 2021.

[11] G. Ayyappan, and M. Nirmala, "Analysis of customer's impatience on bulk service queuing system with unreliable server, setup time and two types of multiple vacations", *International Journal of Industrial and Systems Engineering*, vol. 38, no. 2, pp. 198–222, 2021.

[12] R.P. Nithya, and M. Haridass, "Modelling and simulation analysis of a bulk queuing system", *Kybernetes*, vol. 50, no. 2, pp. 263-283, 2021.

[13] B. Kumar, "Unreliable bulk queuing model with optional services, Bernoulli vacation schedule and balking", *International Journal of Mathematics in Operational Research*, vol. 12, no. 3, pp. 293–316, 2018.

[14] D. Zirem, M. Boualem, K. Adel-Aissanou, and D. Aissani, "Analysis of a single server batch arrival unreliable queue with balking and general retrial time", *Quality Technology & Quantitative Management*, vol. 16, no. 6, pp. 1-24, 2018.

[15] C. J. Singh, S. Kaur, and M. Jain, "Analysis of bulk queue with additional optional service, vacation and unreliable server", *International Journal of Mathematics in Operational Research*, vol. 14, no. 4, pp. 517–540, 2019.

[16] R. Sethi, M. Jain, R. K. Meena, and D. Garg, "Cost Optimization and ANFIS Computing of an Unreliable $M/M/1$ queuing System with Customers' Impatience Under N-Policy", *International Journal of Applied and Computational Mathematics*, vol. 6, no. 2, pp. 1-14, 2020.

[17] S. R. Chakravarthy, Shruti, and R. Kulshrestha, "A queuing model with server breakdowns, repairs, vacations, and backup server", *Operations Research Perspectives*, vol. 7, pp. 1-13, 2020.

[18] V. L. Pikkala, and G. B. Edadasari, "Variant working vacation Markovian queue with second optional service, unreliable server and retention of reneged customers", *International Journal of Mathematics in Operational Research*, vol. 19, no. 1, pp. 45-64, 2021.

[19] Kerobyan, R., and Kerobyan K., "Virtual Waiting Time in Single-Server queuing Model $M/G/1$ with Unreliable Server and Catastrophes", in *Dudin A., Nazarov A., Moiseev A. (eds) Information Technologies and Mathematical Modelling. queuing Theory and Applications. ITMM 2020. Communications in Computer and Information Science*, 1391.

[20] G. Ayyappan, and M. Nirmala, "An $M^{[X]}/G(a,b)/1$ queue with breakdown and delay time to two phase repair under multiple vacation", *Applications and Applied Mathematics: An International Journal*, vol. 13, no. 2, pp. 639–663, 2018.

[21] A. A. Bouchentouf, and A. Guendouzi, "Cost optimization analysis for an $M^{[X]}/M/c$ vacation queuing system with waiting servers and impatient customers", *SeMA Journal*, vol. 76, no. 2, pp. 309-341, 2019.

[22] A. A. Bouchentouf, and L. Medjahri, "Performance and economic evaluation of differentiated multiple vacation queuing system with feedback and balked customers", *Applications and Applied Mathematics: An International Journal*, vol. 14, no. 1, pp. 46–62, 2019a.

[23] A. A. Bouchentouf, and A. Guendouzi, "The $M^{[X]}/M/c$ Bernoulli feedback queue with variant multiple working vacations and impatient customers: performance and economic analysis", *Arabian Journal of Mathematics*, vol. 9, no. 4, pp. 309–327, 2020.

[24] A. A. Bouchentouf, M. Cherfaoui, M. Boualem, and L. Medjahri, "Variant vacation queuing system with Bernoulli feedback, balking and server's states-dependent reneging", *Yugoslav Journal of Operations Research*, vol. 31, no. 4, pp. 557–575, 2021.

[25] R. P. Nithya, and M. Haridass, "Cost optimisation and maximum entropy analysis of a bulk queuing system with breakdown, controlled arrival and multiple vacations", *International Journal of Operational Research*, vol. 39, no. 3, pp. 279-305, 2020.

[26] W. M. Kempa, and R. Marjasz, "Distribution of the time to buffer overflow in the $M/G/1/N-$type queuing model with batch arrivals and multiple vacation policy", *Journal of the Operational Research Society*, vol. 71, no. 3, pp. 447-455, 2020.

[27] P. K. Joshi, S. Gupta, and K. N. Rajeshwari, "A study of steady state analysis of $D/M/1$ model and $M/G/1$ model with multiple vacation queuing systems", *South East Asian Journal of Mathematics and Mathematical Sciences*, vol. 16, no. 1, pp. 37-50, 2020.

[28] S. Singh, and R.K. Srivastava, "Markovian queuing System for Bulk Arrival and Retrial Attempts with Multiple Vacation Policy", *International Journal of Mathematics Trends and Technology*, vol. 67, no. 1, pp. 21-28, 2021.

[29] S. Jeyakumar, and B. Senthilnathan, "A study on the behaviour of the server breakdown without interruption in a $M^{[x]}/G(a,b)/1$ queuing system with multiple vacations and closedown time", *Applied Mathematics and Computation*, vol. 219, no. 5, pp. 2618–2633, 2012.

[30] M. Haridass, and R. Arumuganathan, "Analysis of a batch arrival general bulk service queuing system with variant threshold policy for secondary jobs", *International Journal of Mathematics in Operational Research*, vol. 3, no. 1, pp. 56–77, 2011.