Yugoslav Journal of Operations Research
33 (2023), Number 1, 111–131
DOI: https://doi.org/10.2298/YJOR2109017032M

ANALYSIS OF SINGLE SERVER FINITE BUFFER QUEUE UNDER DISCOURAGED ARRIVAL AND RETENTION OF RENEGING

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Received: September 2021 / Accepted: December 2022

Abstract: Queuing models where arrival rates go down consequent to increase in the number of customers are called systems with discouraged arrivals. Discouraged arrivals are distinct from balking in the sense that balking implies that arriving customers do not join. In this paper, we model a single server finite buffer Markovian queuing model with discouraged arrival, balking, reneging and retention of reneged customers. The steady state probabilities are obtained using Markov process method. Closed form expression of traditional as well as some freshly designed performance measures are presented. We also perform sensitivity analysis to examine the variations in performance measures with the variations in system parameters. Our results are numerically illustrated through a field level problem with design connotations.

Keywords: Markovian-queue, balking, impatience, reneging, retention of reneging, discouraged arrival.

MSC: 60K25, 68M20, 90B22.

1. INTRODUCTION

Waiting for service has attained importance since the capacity of a service delivery mechanism is often far less than the demand for service. Queues are therefore an inevitable phenomenon of life. In today's competitive world, it is no secret that unless customers are satisfied with the quality of service, their loyalties would suffer. Waiting for service requires patience and all customers are not in a position or willing to wait long enough. In practice, the unwillingness to wait finds reflection through balking and reneging behavior of customers.

The phenomenon of customers leaving a queuing system without joining the queue is known as balking. Balking is not possible from an empty system. Haight [1] has provided a rationale which might influence a person to balk. It relates to the perception of the importance of being served which induces an opinion somewhere in between urgency, so that a queue of certain length will not be joined, to indifference where a non-zero queue is also joined. Based on the rate at which customers balk, balking can be divided into two types. If balking rate is constant across all states of the system, we call it state independent balking (SIB). If it varies with queue size, we call it state dependent balking (SDB) [2]. In this paper we shall assume SIB throughout.

Among the customers who join the queue, it is commonly observed that some get impatient because of waiting and leave the system without receiving service. In queuing parlance, joining the system and leaving without service is known as reneging. To a business manager, this implies loss both in terms of immediate revenues as well as reputation. Current business environment can hardly afford this. Therefore, it is very important to retain customers by applying retention strategies. A typical strategy involves providing an incentive to the customer not to renege. These incentives induce a desire on the impatient customer to remain in the system. Such customers are termed as retained customers. Under this framework, if a customer gets impatient (due to reneging), he may leave the queue with probability say p and may remain in the queue with probability q such that $p + q = 1$ [3, 4].

Continuing on with reneging behavior, it is not difficult to see that it can be classified into two types - reneging till beginning of service (henceforth denoted by R BOS) and reneging till end of service (henceforth denoted by R EOS). In case of first type, customers can renege while they are waiting in queue. If they begin receiving service, they cannot renege. A common example is the beauty parlor. A female customer can renege while she is waiting in queue. However once service begins i.e. haircut, facial or spa etc. begins, that customer cannot leave till service is over. On the other hand, in case of R EOS, a customer can renege from the queue as well as while service is going on. When an internet user requests a particular website, it takes a little time for server hosting the website to be accessed by the browser. During this time, the user may get impatient and disconnect. This is an example of second type of reneging. Both R BOS and R EOS are treated separately in this paper.

Moreover, usually customers hesitate to join a queue when a large number of customers are already present in the system. Essentially, this results in lowering of the arrival rate as the number of customers in the system goes up. This is known as discouraged arrival. The concept of discouraged arrival was first introduced by Natvig [5]. We can observe many queues with discouraged arrivals in our day to day life. For e.g. Computers with batch job processing where job submissions are discouraged when the system is used frequently and arrivals are modeled as a

Poisson process with state dependent arrival rate.

Even though one can observe reneging and balking in our day-to-day life, it is not very often that one can locate a paper analyzing these features simultaneously in a queuing system with the additional feature of discouraged arrivals and retention of reneging customers [6, 7]. Even if these have been analyzed, closed form expressions of important performance measures are not available [8, 9]. Kumar et al. [6] considered an M/M/1/N queuing system with balking. But they did not consider the concept of reneging and discouraged arrival. A two server Markovian queuing model with discouraged arrivals, reneging and retention of reneged customers was analyzed by Kumar and Sharma [9], but balking was not assumed. Kumar and Sharma [10] analyzed customers' impatience in a single server queuing model considering discouraged arrival and retention of reneged customers. Assumption of finite buffer restriction and balking was not considered in their paper. Closed form expressions of performance measures are also not available. In another relevant work, a M^X/M/c Bernoulli feedback queuing system with variant multiple working vacations and impatient timers was considered by Bouchentouf and Guendouzi [11] but their model did not assume balking, discouraged arrival or retention of reneged customers. El-Paoumy and Nabwey [12] analyzed an M/M/2/N queuing model with balking, reneging and heterogeneous server and obtained the steady state solution.

Awasthi [13] considered a single server finite buffer Markovian queuing system with reverse balking and reverse reneging. Steady state solutions of the model were obtained along with performance measures. Recently, the analysis of multi-server Markovian queue with reverse balking and position dependent reneging was presented by Saikia et al. [14]. They derived the generating function of the stationary system size distribution and also obtained the mean system size along with the other performance measures. However in [12, 13, 14], authors did not incorporate the concept of discouraged arrival and retention strategy. In recent times, analysis of single server as well as multi-server Markovian queues with impatient customers assuming Bernoulli feedback, balking, reneging, retention of reneged customers and waiting server under variant working vacation policy was addressed by Bouchentouf and Yahiaoui [15], Bouchentouf et al. [16, 23]. In a similar line retention of impatient customers with optimal service and working vacations has been analysed by VijayaLaxmi et al. [18] without considering the aspect of balking and buffer size. A single server Markovian queueing system with Bernoulli feedback, balking, server's states-dependent reneging, and retention of reneged customers under variant multiple vacation policy was analysed by Bouchentouf et al. [19]. A heterogeneous queuing system with reverse balking and reneging was discussed by Som and Kumar [20]. In another work Bouchentouf et al. [21] presented the mathematical modelling of a multi-server Markovian queue with Bernoulli feedback under variant of multiple vacation policy in the absence of retention strategy. In these papers [15-19] authors did not assume the concept of discouraged arrival. Rasheed and Manoharan [22] studied a Markovian queueing system in the absence of customers' impatience. Nevertheless, they assumed state dependent arrival as well as service rate. They have also derived the steady state probabilities and presented various measures of effectiveness. Some special cases have also been discussed. The concept of encouraged or discouraged arrival with modified reneging policy was used to analyze a single server Markovian queue by Kumar et al. [24] in the absence of retention strategy.

The stochastic modeling of a single server finite buffer Markovian queuing model incorporating the additional challenges of discouraged arrival, state independent balking and position independent reneging with retention strategy is the subject matter of this paper together with the derivation of performance measures. Importance of the queuing model stems from the fact that in the classical M/M/1 model, "it is assumed that the system can accommodate any number of units. In practice, this may seldom be the case. We have thus to consider the situation such that the system has limited waiting space and can hold a maximum number of k units (including the one being served)" [17]. Though this model has been analyzed by many researchers, however, to the best of our knowledge, the added restrictions of discouraged arrival, reneging, retention of reneged customers and balking has not been dealt with in the literature. Only a restricted version considering the single and multi-server queuing model assuming impatient customers is available in [3, 4, 10, 15-19]. In this paper, we try to address these gaps in the literature.

One of the real life applications where our model with the features of balking, reneging, discouraged arrival and retention of reneged customers can be seen in cash/payment counter in shopping malls. Because of space related constraint there is a physical limit in the size of the queue and hence the finite buffer queuing model is appropriate. Balking as well as discouraged arrival is very often observed specially during the busy hours. Customers reneging from the queue would mean loss of business and hence the management of shopping malls makes all possible efforts (e.g. offering discounts on certain products) to retain customers who are of reneging type.

The subsequent sections of this paper are structured as follows. In Section 2, we discuss the review of literature. Section 3 and Section 4 contain the derivation of steady state probabilities and performance measures. In Section 5, a numerical example is discussed. We perform sensitivity analysis in Section 6. In Section 7, concluding statements are presented. The necessary derivations are presented in Appendix.

2. MODEL ASSUMPTIONS

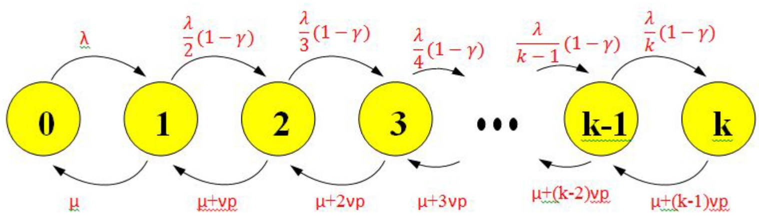
In this paper, we shall analyze the M/M/1/k model with discouraged arrivals, state independent balking, reneging and retention of reneged customers. The assumptions of the model are:

1. The arrival and service rates are assumed to follow exponential distribution with parameters λ and μ respectively.
2. Arrival rate is a function of the number of customers present in the system i.e. if there are n ($n > 1$) customers in the system, then a new customer enters the system with rate $\frac{\lambda}{n+1}$.

3. There is only one server.
4. The system capacity is restricted to k.
5. Each customer has a state independent balking probability. If a customer on arrival observes the existence of a queue but with system size below finite buffer restriction, the probability that he will balk is γ. It may be noted that our formulation requires that balking is possible only when the system is non-empty. There is no balking from an empty system.
6. Each customer joining the system is assumed to have a random patience time following exp (ν). It is also assumed that when a customer gets impatient (due to reneging), he may leave the queue with some probability, say p and may remain in the queue for service with probability q $(= 1 - p)$ as consequence of retention strategy. Reneging rules under R BOS and R EOS are considered separately.

3. THE SYSTEM STATE PROBABILITIES

Let n denote the state of the system and also let pn be the probability that there are n customers in the system under R BOS. The state transition rate diagram under R BOS is given by



Using rate in rate out principle and Markov process method, we have derived the steady state probabilities and are given by

$$\lambda p_0 = \mu p_1 \tag{3.1}$$

$$\lambda p_0 + (\mu + \nu p)p_2 = \frac{\lambda}{2}(1-\gamma)p_1 + \mu p_1 \tag{3.2}$$

$$\frac{\lambda(1-\gamma)}{n}p_{n-1} + (\mu + n\nu p)p_n + 1 = \frac{\lambda(1-\gamma)}{n+1}p_n + \{\mu + (n-1)\nu p\}p_n; n = 2, 3.., k-1 \tag{3.3}$$

$$\frac{\lambda}{k}(1-\gamma)p_{k-1} = \{\mu + (k-1)\nu p\}p_k; \quad n = k \tag{3.4}$$

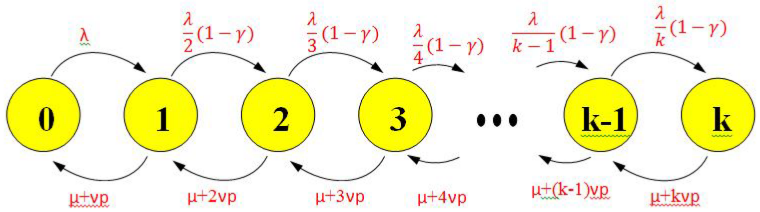
Solving recursively, we get

$$p_n = \left[\frac{\lambda^n(1-\gamma)^{n-1}}{n!\prod_{r=1}^{n}\{\mu+(r-1)\nu p\}}\right]p_0; \quad n=1,2,..,k \tag{3.5}$$

Using the normalizing condition $\sum_{n=0}^{k} p_n = 1$ we have obtained p_0 and is given as

$$p_0 = \left[1+\sum_{n=1}^{k}\frac{\lambda^n(1-\gamma)^{n-1}}{n!\prod_{r=1}^{n}\{\mu+(r-1)\nu p\}}\right]^{-1} \tag{3.6}$$

Under R EOS, customers may renege from queue as well as while being served. Let q_n denote the probability that there are n numbers of customers in the system. The state transition rate diagram under R EOS is given by



Applying the Markov process method, the steady state equations under R EOS are:

$$\lambda q_0 = (\mu+\nu p)q_1 \tag{3.7}$$

$$\lambda q_0 + (\mu+2\nu p)q_2 = \frac{\lambda}{2}(1-\gamma)q_1 + (\mu+\nu p)q_1 \tag{3.8}$$

$$\frac{\lambda(1-\gamma)}{n}q_{n-1}+\{\mu+(n+1)\nu p\}q_{n+1} = \frac{\lambda(1-\gamma)}{n+1}q_n+\{\mu+n\nu p\}q_n; n=2,3..,k-1 \tag{3.9}$$

$$\frac{\lambda}{k}(1-\gamma)q_{k-1} = \{\mu+(k\nu p\}q_k; \quad n=k \tag{3.10}$$

Solving recursively, we get

$$q_n = \left[\frac{\lambda^n(1-\gamma)^{n-1}}{n!\prod_{r=1}^{n}\{\mu+r\nu p\}}\right]q_0; \quad n=1,2,..,k \tag{3.11}$$

where q_0 is obtained from the normalizing condition $\sum_{n=0}^{\infty} q_n = 1$ and is given as

$$q_0 = \left[1+\sum_{n=1}^{k}\frac{\lambda^n(1-\gamma)^{n-1}}{n!\prod_{r=1}^{n}\{\mu+r\nu p\}}\right]^{-1} \tag{3.12}$$

4. PERFORMANCE MEASURES

Analysis of performance measures are important as these allow various queuing issues to be identified. Many of these are related to customers' dissatisfaction and managements perception of efficiency of system and are therefore of great interest. We provide below the closed form expression of some performance measures.

4.1. Mean system size (L) and Mean queue size (L_q)

In the analysis of queuing system one of the important measure is the average number of customers present in the system and it is usually denoted by 'L'. The derivation of mean system size (L) for two reneging rules are presented in the appendix section. Thus mean system size and mean queue size under R BOS are given by

$$L_{R\,BOS} = \frac{1}{\nu p}\left[(\lambda+\mu-\nu p)p_0 - (\mu-\nu p) + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}}{n}\right]$$

[*Derivation in Section A*.1]

$$\begin{aligned}
L_{q(R\,BOS)} &= \sum_{n=2}^{k}(n-1)p_n \\
&= L_{R\,BOS} + p_0 - 1 \\
&= \frac{1}{\nu p}\left[(\lambda+\mu)p_0 - \mu + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}}{n}\right].
\end{aligned}$$

Under R EOS mean system size and mean queue size are given by

$$L_{R\,EOS} = \frac{1}{\nu p}\left[\lambda q_0 - \mu(1-q_0) + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{q_{n-1}}{n}\right] \quad [Derivation\ in\ Section\ A.2]$$

$$\begin{aligned}
L_{q(R\,EOS)} &= L_{R\,EOS} - 1 + q_0 \\
&= \frac{1}{\nu p}\left[\lambda q_0 - (\mu+\nu p)(1-q_0) + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{q_{n-1}}{n}\right].
\end{aligned}$$

4.2. Effective arrival rate (λ^e)

Because of balking, discouraged arrival and finite buffer restriction, all the customers who arrive into the system may not be able to join. The rate at which customers actually enter the system is called effective arrival rate. The closed form expression of the effective arrival rate under the two rules of reneging are given by:

$$\begin{aligned}
\lambda^e_{R\,BOS} &= \lambda p_0 + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}}{n} \\
\lambda^e_{R\,EOS} &= \lambda q_0 + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{q_{n-1}}{n}.
\end{aligned}$$

4.3. Average reneging rate (*Avg rr*)

We assumed that a customer who is at state n has random patience time following $\exp(\nu)$.Thus, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rate ($Avg\,rr$) under both the reneging rules is given by

$$\begin{aligned} Avg\,rr_{R\,BOS} &= \sum_{n=2}^{k}(n-1)\nu p p_n \\ &= \nu p L_{R\,BOS} - \nu p + \nu p p_0 \\ &= \lambda p_0 - \mu(1-p_0) + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}}{n} \end{aligned}$$

$$\begin{aligned} Avg\,rr_{R\,EOS} &= \sum_{n=1}^{k} n\nu p q_n \\ &= \nu p L_{R\,EOS} \\ &= \lambda q_0 - \mu(1-q_0) + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{q_{n-1}}{n}. \end{aligned}$$

4.4. Mean rate of losing customer

Customers may be lost to the system in four ways viz: due to balking, reneging, discouraged arrival and due to finite buffer restriction. An analysis of these kinds of loss becomes important as it can provide a measure of total revenue lost. Consequently, management of any industry or department would like to know the proportion of total customers lost in order to have an idea of total business lost.

The mean rate at which customers are lost under R BOS is given by
Restriction on joining the system $(\lambda - \lambda^e)$+Average reneging rate ($Avg\,rr$)

$$\lambda - \lambda^e_{(R\,BOS)} + Avgrr_{R\,BOS}$$
$$= \lambda - \mu(1-p_0)$$

and the mean rate at which customers are lost (under R EOS) is

$$\lambda - \lambda^e_{(R\,EOS)} + Avgrr_{R\,EOS}$$
$$= \lambda - \mu(1-q_0)$$

The proportion of customer lost and completing receipt of service can now be easily determined from the above results under R BOS and R EOS separately.

4.5. Actual load of the server

For this purpose, we need to consider only those customers who have received service as it constitutes the actual load of the server. From the server's point of view, this provides a measure of the amount of work he has to do. This can be referred to the rate at which customers reach the service station and let it be denoted as λ^s. Thus under R BOS

$$\begin{aligned}\lambda^s_{R\,BOS} = \lambda^e_{R\,BOS}(1 - &proportion\,of\,customers\,lost\,due\\ &to\,reneging\,out\,of\,those\,joining\,the\,system)\\ &= \lambda^e_{R\,BOS}\left[1 - \sum_{n=2}^{\infty}\frac{(n-1)\nu p p_n}{\lambda^e_{R\,BOS}}\right]\\ &= \lambda^e_{R\,BOS} - avg\,rr_{R\,BOS}\\ &= \mu(1-p_0).\end{aligned}$$

In case of R EOS, customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Thus,

$$\begin{aligned}\lambda^s_{R\,EOS} = \lambda^e_{R\,EOS}(1 - &proportion\,of\,customers\,lost\,due\\ &to\,reneging\,out\,of\,those\,joining\,the\,system)\\ &= \lambda^e_{R\,EOS}\left[1 - \sum_{n=c+1}^{\infty}\frac{(n-1)\nu p q_n}{\lambda^e_{R\,EOS}}\right]\\ &= \lambda^e_{R\,EOS} - \nu p(L_{R\,EOS} - q_1) + \nu p(1 - q_1 - q_0)\\ &= (\mu+\nu)(1-q_0).\end{aligned}$$

4.6. Average attrition rate (*AAR*)

Out of customers who join the queuing system, some would leave after service and others would become impatient and leave without service. Hence management has to arrange sufficient infrastructural facilities to account for customers leaving the system through these two streams. We now present the expression for the average attrition rate (AAR) under both the reneging rules.

$$\begin{aligned}AAR_{R\,BOS} &= \mu p_1 + (\mu+\nu p)p_2 + ...\{\mu + (k-1)\nu p\}p_k\\ &= (\mu - \nu p)(1-p_0) + \nu p L_{R\,BOS}\end{aligned}$$

$$\begin{aligned}AAR_{R\,EOS} &= (\mu+\nu p)q_1 + (\mu+2\nu p)q_2 + \cdots + (\mu + k\nu p)q_k\\ &= \mu(1-q_0) + \nu p L_{R\,EOS}.\end{aligned}$$

The percentage reduction in attrition rate due to retention strategy out of those who had joined the system is given by

$$\frac{AAR(p=1) - AAR(p<1)}{AAR(p=1)}$$

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4.7. Average retention rate (*ARR*)

We recall that out of those customers who join the queuing system, some would get impatient and renege and many others would be retained because of the retention strategy. It is therefore of interest to the management to have an idea of the rate at which customers are retained. We now present an expression for ARR under R BOS and R EOS

$$AAR_{R\,BOS} = \sum_{n=2}^{k}(n-1)\nu q p_n$$

$$= \frac{q}{p}\left[(\lambda+\mu-\nu p)-(\mu-\nu p)+\lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}}{n}-\nu p p_1-\nu q(1-p_0-p_1)\right]$$

$$AAR_{R\,EOS} = \sum_{n=2}^{k} n\nu q q_n$$

$$= \frac{q}{p}\left[\lambda q_0-\mu(1-p_0)+\lambda(1-\gamma)\sum_{n=2}^{k}\frac{q_{n-1}}{n}-\nu p q_1\right].$$

The proportion of customers retained due to retention strategy out of those who had joined the system under both the reneging rules is given by

$$\frac{Avg\ RR(p=1)-Avg\ RR(p<1)}{Avg\ RR(p=1)}$$

4.8. Impact of discouraged arrival (*DA*)

One of the features of our model is the assumption of discouraged arrival wherein the arrival rate of customers is a function of the state of the system. In order to give management a sense of business lost because of discouraged arrival we present the following performance measure.

Proportion of customer lost due to discouraged arrival under both the reneging rule is given by

$$\frac{\lambda^e(without\ DA)-\lambda^e(with\ DA)}{\lambda^e(without\ DA)}$$

5. NUMERICAL EXAMPLE

To illustrate the use of our results, we apply them to a queuing problem. We quote below an example from Allen (page 267 and 273) [25].

"Traffic to a message switching centre for Extraterrestrial Communications Corporation arrives in a random pattern (remember that 'random pattern' means exponential inter-arrival time) at an average rate of 240 messages per minute. The line has a transmission rate of 800 characters per second. The message length distribution (including control characters) is approximately exponential with an average length of 176 characters. Calculate the principal statistical measures of system performance assuming that a very large number message buffers is provided." Suppose, however, that it is desired to provide only the minimum number of messages buffers required to guarantee that

$$p^k < 0.005$$

How many buffers should be provided?

This is a design problem. Here λ= 4/sec and μ= 4.55/sec. As it is required by the switching centre, we examine the minimum number of message buffers with different choices of k. Though not explicitly mentioned, it is necessary to assume reneging and balking. Balking because in telecommunication systems, it is known that an incoming message that sees a workload may be admitted in to the system with a certain probability and rejected otherwise. Rejection implies balking. We shall assume that the balking probability is independent and is considered as 'γ'.

Further we assume R BOS because in telecommunication systems it is also known that messages usually have some real time constraints within which the message has to be processed. Messages received after the deadline is considered obsolete and discarded. This can be seen as reneging.

Let us assume alternative possible Markovian reneging rates of ν=0.1/sec, ν=.03333/sec. Put differently, these rates imply that a message joining the buffer will be alive if processing commences within 10sec and 30sec respectively on the average. Also, the probability that a customer will renege is considered as $p = 0.01$. We further assume that balking rate is independent of state and is taken as $\gamma = 0.001$ (one in 1000 message).

Various performance measures of interest computed under different reneging scenarios are given in Table 1 and 2. These measures were arrived at using a C++ program designed by the authors. Different choices of k were considered. Results relevant with regard to the requirement that the switching centre should provide only the minimum number of message buffers to guarantee should p^k <0.005 are presented in the tables. (All rates in the following tables are per second).

In case the reneging behavior follows exp (0.1) distribution, it is clear from the table 1 that an ideal choice of k could be 5 with p^k=0.0018. If the reneging distribution is exp (0.03333), then $k = 5$ appears to be close to the switching centre requirement with p^k=.00169 (table 2).

A few interesting observations can be made from the tables 1 and 2.

i) The reneging rate appears low possible we had to assume R_BOS and the fact that the average length of queue is very small, 0.29 for k=5.

ii) Only about 5% of the customers are lost due to reneging, balking and finite buffer possibly because the optimal finite buffer size of 5 is substantially higher than the average length of the system which stands at 0.87. Consequently, very few people are lost due to finite buffer restriction, balking and reneging in this example. In contrast, the impact of discouraged arrival is much more about 1/3rd of the customers are lost because of the same.

iii) The percentage reduction in attrition rate due to retention strategy also appears low possibly because the mean reneging rate itself is low as the average queue size is small.

iv) About 41% of time, the server appears to be idle. This is possibly because of two reasons. First the effective arrival rate into the server is only about 60% of the arrival rate. Secondly, the average length of queue is also low.

Table 1: Performance Measures assuming λ=4, μ=4.55, ν=0.1, p = 0.01 and γ=0.001

Performance Measure	Size of minimum number of message buffers	
	$k = 4$	$k = 5$
p^k	0.01031	0.0018
λ^s (i.e. arrival rate of customers reaching service station)	2.65637	2.65979
Effective arrival rate(λ^e)	2.65665	2.66008
Fraction of time server is idle (p^0)	0.41618	0.41543
Average length of queue	0.28559	0.29228
Average length of system	0.86941	0.876872
Mean reneging rate	0.00028	0.00029
Mean rate of customers lost	1.34363	1.34021
Proportion of customers lost due to reneging, balking and finite buffer	0.05179	0.05078
Average balking rate	0.00993	0.00999
Average Retention rate	0.75453	0.76958
Average attrition rate	2.65666	2.65996
Proportion of customers lost due to discouraged arrival	0.32609	0.33043
Proportion of customer retained due to retention strategy out of those who had joined the system	0.97597	0.98321
Percentage reduction in attrition rate due to retention strategy out of those who had joined the system	0.00546	0.00563

Table 2: Performance Measures assuming λ=4, μ=4.55, ν=0.03333, $p = 0.01$ and γ=0.001

Performance Measure	Size of minimum number of message buffers	
	$k = 4$	$k = 5$
p^k	0.01032	0.00169
λ^s (i.e. arrival rate of customers reaching service station)	2.65154	2.65989
Effective arrival rate(λ^e)	2.65658	2.65999
Fraction of time server is idle (p^0)	0.41616	0.41541
Average length of queue	0.28559	0.29236
Average length of system	0.86944	0.87691
Mean reneging rate	0.00095	0.00097
Mean rate of customers lost	1.34353	1.3401
Proportion of customers lost due to reneging, balking and finite buffer	0.05389	0.051778
Average balking rate	0.00573	0.00673
Average Retention rate	0.65454	0.66958
Average attrition rate	2.65656	2.65993
Proportion of customers lost due to discouraged arrival	0.32771	0.33239
Proportion of customer retained due to retention strategy out of those who had joined the system	0.98988	0.98999
Percentage reduction in attrition rate due to retention strategy out of those who had joined the system	0.00453	0.00289

6. SENSITIVITY ANALYSIS

It is important and also interesting to examine how server utilization varies in response to change in system parameters. The system parameters of interest are λ, μ, ν,k. We place below the effect of change in these system parameters on server utilization. For this purpose, we shall follow the following notational convention in the rest of this section. Let $p_n(\lambda, \mu, \nu, k)$ and $q_n(\lambda, \mu, \nu, , k)$ denote the probability that there are 'n' customers in a system with parameters in steady state under R BOS and R EOS respectively.

i) Let $\lambda_1 > \lambda_0$, then

$$\frac{p_0(\lambda_1, \mu, \nu, k)}{p_0(\lambda_0, \mu, \nu, k)} < 1$$

$$\Rightarrow \frac{(\lambda_0 - \lambda_1)}{\mu} + \frac{(\lambda_0^2 - \lambda_1^2)(1-\gamma)}{2!\mu(\mu+\nu p)} + \cdots + \frac{(\lambda_0^k - \lambda_1^k)(1-\gamma)^{k-1}}{k!\mu(\mu+\nu p)\ldots\{\mu+(k-1)\nu p\}} < 0$$

which is true. Hence $p_0 \downarrow$ as $\lambda \uparrow$.

ii) Let $\mu_1 > \mu_0$, then

$$\frac{p_0(\lambda, \mu_1, \nu, k)}{p_0(\lambda_0, \mu_0, \nu, k)} > 1$$

$$\Rightarrow \lambda(1-\gamma)\left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right) + \frac{\lambda^2(1-\gamma)}{2!}\left[\frac{1}{\mu_0(\mu_0+\nu p)} - \frac{1}{\mu_1(\mu_1+\nu p)}\right] + \cdots + \frac{\lambda^k(1-\gamma)^{k-1}}{k!}\left[\frac{1}{\mu_0(\mu_0+\nu p)\ldots\{\mu_0+(k-1)\nu p\}} - \frac{1}{\mu_1(\mu_1+\nu p)\ldots\{\mu_1+(k-1)\nu p\}}\right] + \cdots > 0.$$

which is true. Hence $p_0 \uparrow$ as $\mu \uparrow$

iii) Let $\nu_1 > \nu_0$, then

$$\frac{p_0(\lambda, \mu, \nu_1, k)}{p_0(\lambda_0, \mu, \nu_1, k)} > 1$$

$$\Rightarrow \frac{\lambda^2(1-\gamma)}{2!}\left[\frac{1}{\mu(\mu+\nu_0 p)} - \frac{1}{\mu(\mu+\nu_1 p)}\right] + \ldots + \frac{\lambda^k(1-\gamma)^{k-1}}{k!}\left[\frac{1}{\mu_0(\mu+\nu_0 p)\ldots\{\mu+(k-1)\nu_0 p\}} - \frac{1}{\mu(\mu+\nu_1 p)\ldots\{\mu+(k-1)\nu_1 p\}}\right] + \cdots > 0$$

which is true. Hence $p_0 \uparrow$ as $\nu \uparrow$

iv) Let $k_1 > k_0$ then

$$\frac{p_0(\lambda,\mu,\nu,k_1)}{p_0(\lambda_0,\mu,\nu,k_0)} > 1$$

$$\Rightarrow \sum_{n=1}^{k_0} \frac{\lambda^n(1-\gamma)^{n-1}}{n!\prod_{r=1}^{n}\{\mu+(r-1)\nu p\}} - \sum_{n=1}^{k_1} \frac{\lambda^n(1-\gamma)^{n-1}}{n!\prod_{r=1}^{n}\{\mu+(r-1)\nu p\}} < 0$$

which is true . Hence $p_0 \downarrow$ as $k \uparrow$

Similar results are obtained under R EOS.

The managerial implications of the above results are obvious.

We can also study the variations in the other performance measures with response to change in system parameters numerically. We consider here the values of the different system parameters from the numerical example mentioned in Section 5 under R BOS. The results are computed using C++ program designed by authors.

From Table 3, it is clear that as the arrival rate increases, average system size, average reneging rate, ARR, AAR, proportion of total customer lost due to balking and reneging, rate at which customer reach the service system and effective arrival rate increases. The reasons are quite obvious. Increase in average arrival rate means more customers in the system and it leads to high levels of impatience. Loss of customers mean revenue lost. Since proportion of total customer lost increases with the increase in arrival rate, so the system manager must appoint some additional server or increase the service rate in order to reduce the revenue lost.

Table 3: Variations in L, $Avgrr$, p_0, ARR, proportion of total lost, λ^s and effl with respect to mean arrival rate (λ) Considering μ=4.55, ν=0.1, p = 0.01, γ=0.001 and k = 4

λ	L	Avg rr	ARR	p_0	AAR	Proportion of customer lost	λ^s	Effl
4	0.86941	0.01268	1.08622	0.41618	2.656657	0.505759	2.656372	2.65666
5	1.07569	0.03041	1.83487	0.33523	3.02512	0.652963	3.024709	3.02512
6	1.27265	0.05054	2.55787	0.27073	3.318715	0.808092	3.318172	3.37872
7	1.45851	0.08068	3.02642	0.21939	3.552464	0.970654	3.551785	3.55246
8	1.632	0.10081	3.95634	0.17851	3.738608	1.140051	3.737797	3.73861
9	1.97515	0.30111	4.56431	0.13856	3.920655	1.295819	3.919542	3.92066

From Table 4, it is evident that with increase in average rate of service, mean system size, average reneging rate, average balking rate, ARR and proportion of total customers lost due to impatience decreases. This means customers have to spend less time in the system and there will be less chances of get impatient which is the most idealistic situation for any firm. On the other hand, probability that the server is in idle condition, AAR and effective arrival rate to the system increases with the increase in average service rate which is quite obvious. Therefore, we can say that the theoretical development in this paper are really consistent with the proper functioning of the model.

Table 4: Variations in L, $Avgrr$, p_0, ARR, AAR, proportion of total lost, λ^s and effl with respect to mean service rate (μ) Considering λ=4, ν=0.1, p = 0.01, γ=0.001 and k = 4

μ	L	Avg rr	ARR	p_0	AAR	Proportion of customer lost	λ^s	Effl
4.55	0.86941	0.00029	1.08622	0.41618	2.65666	0.505759	2.65637	2.65666
5.55	0.71633	0.0002	0.90564	0.48696	2.84759	0.404759	2.84739	2.84759
6.55	0.60844	0.00015	0.73456	0.54331	2.99149	0.337178	2.99134	2.99149
7.55	0.52856	0.0001	0.60234	0.58892	3.10363	0.288814	3.10363	3.10306
8.55	0.46705	0.00009	0.51678	0.6265	3.1935	0.252573	3.19341	3.1935
9.55	0.41832	7.6E-05	0.40239	0.65791	3.26703	0.224376	3.26696	3.99527

Table 5: Variations in L, $Avgrr$, p_0, ARR, proportion of total lost, λ^s and effl with respect to average reneging rate (ν) Considering λ=4, μ=4.55, p = 0.01, γ=0.001 and k = 4

ν	L	Avg rr	ARR	p_0	AAR	Proportion of customer lost	λ^s	Effl
0.1	0.86941	0.00029	1.08622	0.41618	2.65666	0.505759	2.65637	2.65666
0.5	0.86882	0.00143	1.58478	0.41632	2.65719	0.5058884	2.65577	2.65719
1	0.86806	0.00285	1.98252	0.41648	2.65786	0.506041	2.65502	2.65786
1.5	0.86733	0.00426	2.36744	0.41665	2.65853	0.506196	2.65427	2.65853
2	0.86659	0.00567	2.89316	0.41681	2.65919	0.506351	2.65352	2.65919
2.5	0.86587	0.00707	3.29683	0.41697	2.65985	0.506506	2.65277	2.65985

It can be observed from table 5 that average reneging rate, proportion of total customers lost and the probability that the system is in empty state increases with the increase in average reneging rate. This is because as the average reneging rate of the system increases, more and more customers are leaving from the queue without receiving service due to impatience. At the same time this results in decrease in average system size and the rate at which customers reach the server.

Table 6: Variations in L, $Avgrr$, p_0, ARR, proportion of total lost, λ^s and effl with respect to system capacity (k) Considering λ=4, μ=4.55, ν=0.1, p = 0.01, γ=0.001 and k = 4

k	L	Avgrr	ARR	p_0	AAR	Proportion of customer lost	λ^s	Effl
4	0.86941	0.01268	1.05223	0.41618	2.656657	0.50576	2.65637	2.65666
5	0.89734	0.03142	1.33481	0.31522	3.16512	0.66296	3.1247	3.01519
6	1.23619	0.05825	2.56715	0.25073	3.338711	0.81808	3.31813	3.27871
7	1.43842	0.08157	2.92641	0.21036	3.652169	0.96066	3.46924	3.53279
8	1.63101	0.11023	3.25135	0.18362	3.83588	1.11006	3.64782	3.4556
9	1.91595	0.30156	3.86492	0.11656	3.911671	1.39512	3.81954	3.63571

From Table 6, it is clear that as the system capacity increases, average system size, average reneging rate, ARR, AAR, proportion of total customer lost due to balking and reneging, rate at which customer reach the service system and effective arrival rate increases. If we increase the capacity of the system, that means we can accommodate more number of customers and it causes high level of impatience among the customers. Loss of customer means the over all loss in the business. Since proportion of total customer lost increases and the idle time for the server decreases with the increase in system capacity so the system manager must not increase the system capacity or must appoint some additional server or have to increase the service rate in order to reduce the revenue loss.

7. CONCLUSION

The analysis of a single server finite buffer Markovian queuing system with discouraged arrival, state- independent balking and retention of reneging has been presented. Even though balking, discouraged arrival and retention of reneging have been discussed by others, explicit expression are not available under both the reneging rules. This paper makes a contribution here. Closed form expressions of number of traditional as well as some newly designed performance measures have been derived. To study the change in various performance measures corresponding to change in system parameters, sensitivity analysis has also been presented. A numerical example has been discussed to demonstrate results derived. The numerical example is of indicative nature meant to illustrate the benefits of our theoretical results in a design context.This paper can be extended in numerous ways. One of the pointer to future research is inclusion of cost consideration into the modeling. We have spoken of retention strategies which involves cost. The cost of implementing these strategies vis-a-vis additional business generated has to be examined. The derived results can also be extended considering general distribution.

Funding. This research received no external funding.

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A. APPENDIX

A.1. Derivation of $P'(1)$ under R BOS

We know that the probability generating function is given by

$$P(s) = \sum_{n=0}^{\infty} p_n s^n$$

Multiplying both sides of the equation (3.2) by s^1, (3.3) by s^n and (3.4) by s^k and summing over the respective range of n we get the equation as follows:

$$\lambda p_0 s^0 + \frac{1}{s}(\mu + \nu p)p_2 s^2 = \frac{\lambda}{2}(1-\gamma)p_1 s^1 + \mu p_1 s^1 \tag{A.1.1}$$

$$\sum_{n=2}^{k-1} \frac{\lambda(1-\gamma)}{n} s p_{n-1} s^{n-1} - \sum_{n=2}^{k} \frac{\lambda(1-\gamma)}{n} p_n s^n =$$

$$\sum_{n=2}^{k-1} \{\mu + (n-1)\nu p\} p_n s^n - \frac{1}{s}\sum_{n=2}^{k-1} (\mu + n\nu p) p_{n+1} s^{n+1} \tag{A.1.2}$$

$$\frac{\lambda}{k}(1-\gamma)p_{k-1}s^{k-1} = \{\mu + (k-1)\nu p\} p_k s^k \tag{A.1.3}$$

Now adding (A.1.1), (A.1.2) and (A.1.3)

$$\Rightarrow \lambda s p_0 s^0 + \lambda(1-\gamma)s \sum_{n=2}^{k-1} \frac{p_{n-1}s^{n-1}}{n} + \frac{\lambda(1-\gamma)s}{k} p_{k-1}s^{k-1} + \frac{1}{s}(\mu + \nu p)$$

$$p_2 s^2 + \frac{1}{s}\sum_{n=2}^{k-1} (\mu + n\nu p) p_{n+1} s^{n+1} =$$

$$\frac{\lambda(1-\gamma)}{2} p_1 s^1 + \lambda(1-\gamma) \sum_{n=2}^{k-1} \frac{p_n s^n}{n+1} + \mu$$

$$p_1 s^1 + \sum_{n=2}^{k-1} \{\mu + (n-1)\nu p\} p_n s^n + \{\mu + (k-1)\nu p\} p_k s^k$$

$$\Rightarrow \lambda s p_0 s^0 + \lambda(1-\gamma)s \sum_{n=2}^{k} \frac{p_{n-1}s^{n-1}}{n} +$$

$$\frac{\mu}{s}[P(s)-p_1s^1-p_0s^0]+\frac{\nu p}{s}[2p_2s^2+\cdots+kp_ks^k-(p_2s^2+\cdots+p_ks^k)]=$$

$$\lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}s^{n-1}}{n}+\mu[P(s)-p_0]+\nu p[2p_2s^2+\cdots+kp_ks^k-(p_2s^2+\cdots+p_ks^k)]$$

$$\Rightarrow \lambda sp_0s^0+\lambda(1-\gamma)s\sum_{n=2}^{k}\frac{p_{n-1}s^{n-1}}{n}+\frac{\mu}{s}[P(s)-p_1s^1-p_0s^0]+\nu p[P'(s)-p_1]-$$

$$\frac{\nu p}{s}[P(s)-p_1s-p_0]=\lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}s^{n-1}}{n}+\mu[P(s)-p_0]+$$

$$\nu ps[P'(s)-p_1]-\nu p[P(s)-p_1s-p_0]$$

$$\Rightarrow P'(s)\nu p=\lambda p_0+\lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}s^{n-1}}{n}-\frac{\mu}{s}P(s)+\frac{\nu pP(s)}{s}-\frac{\nu pp_0}{s}+\frac{\mu p_0}{s}.$$

Now,

$$\lim_{s\to 1-}P'(s) = \lim_{s\to 1-}\frac{1}{\nu p}\left[\lambda p_0+\lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}s^{n-1}}{n}-\frac{\mu}{s}P(s)+\frac{\nu pP(s)}{s}-\frac{\nu pp_0}{s}+\frac{\mu p_0}{s}\right]$$

$$\Rightarrow P'(1) = \frac{1}{\nu p}\left[\lambda p_0+\lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}s^{n-1}}{n}-\mu+\nu p-\nu pp_0+\mu p_0\right]$$

$$\Rightarrow P'(1) = \frac{1}{\nu p}\left[(\lambda+\mu-\nu p)p_0-(\mu-\nu p)+\lambda(1-\gamma)\sum_{n=2}^{k}\frac{p_{n-1}s^{n-1}}{n}\right].$$

A.2. Derivation of $Q'(1)$ under R_EOS

Multiplying both sides of the equation (3.8) by s^1, (3.9) by s^n and (3.10) by s^k and summing over the respective range of n we get the equation as follows

$$\lambda q_0s^0+\frac{1}{s}(\mu+2\nu p)q_2s^2=\frac{\lambda}{2}(1-\gamma)q_1s^1+(\mu+\nu p)q_1s^1 \tag{A.2.1}$$

$$\sum_{n=2}^{k-1}\frac{\lambda(1-\gamma)}{n}sq_{n-1}s^{n-1}-\sum_{n=2}^{k}\frac{\lambda(1-\gamma)}{n}q_ns^n=$$

$$\sum_{n=2}^{k-1}\{\mu+n\nu p\}q_ns^n-\frac{1}{s}\sum_{n=2}^{k-1}\{\mu+(n+1)\nu p)q_{n+1}s^{n+1} \tag{A.2.2}$$

$$\frac{\lambda}{k}(1-\gamma)q_{k-1}s^{k-1}=(\mu+k\nu p)q_ks^k \tag{A.2.3}$$

Now continuing in a similar way to the derivation of $L_{R\,BOS}$ after summing the equation (A.2.1), (A.2.2) and (A.2.3), the mean system size under R_EOS is calculated and is obtained as,

$$Q'(1) = \frac{1}{\nu p}\left[\lambda q_0 - \mu(1-q_0) + \lambda(1-\gamma)\sum_{n=2}^{k}\frac{q_{n-1}}{n}\right]$$

Abbreviations:
Avgrr: Average reneging rate
ARR: Average retention rate
AAR: Average attrition rate
p_0: probability that the server is in idle condition
λ^s: Rate at which customer reach the service station
effl: Effective arrival rate