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A NEW TYPE OF DIFFERENCE I-CONVERGENT SEQUENCE IN IFnNS

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Abstract: In this paper, we introduce the notion of a generalized difference I-convergent (i.e.Δ^m-I-convergent) and difference I-Cauchy (i.e.Δ^m-I-Cauchy) sequence in intuitionistic fuzzy n-normed spaces. Further, we prove some results related to this notion. Also, we study the concepts of a generalized difference I^*-convergent (i.e.Δ^m-I^*-convergent) sequence in intuitionistic fuzzy n-normed spaces and show the relation between them.

Keywords: Difference sequences, I-Convergence, I-Cauchy, I^*-Convergence, Intuitionistic fuzzy n-norm space.

MSC: 40A35, 40G99, 54E99.

1. INTRODUCTION

After Zadeh[1] introduced the idea of a fuzzy set, many authors have constantly put great efforts to establish fuzzy analogues of classical theories. Like other fields, the field of fuzzy topology is heading towards progressive developments by contributing beneficial applications in quantum particle physics. Gähler introduced 2-normed and n-normed linear spaces in [2, 3] and was further studied by Kim and Cho [4], Malceski [5], and Gunawan and Mashadi [6]. Vijayabalaji and Narayanan [7] defined fuzzy n-normed linear space. Atanassov [8] generalized fuzzy sets and proposed the notion of intuitionistic fuzzy sets (IFS). Furthermore, Deschrijver and Kerre [9] gave the properties of IFS. These notions helped Çoker [10] to define intuitionistic fuzzy topological spaces. Saadati and Park [11, 12] studied these spaces and their generalization, which helped them to obtain the concept of intuitionistic fuzzy normed space(IFNS). In 2007, Vijayabalaji et al.[13] presented the interesting notion of intuitionistic fuzzy n-normed space(IFnNS) as a generalization of fuzzy n-normed linear space and also introduced Cauchy and convergent sequence in IFnNS.

Fast [14] initiated the theory of statisitcal convergence which was further generalized by Kostyrko et al.[15] to the notion of I-convergence with the help of ideal I, which is the subset of a set of the natural number $\mathbb{N}$. Later on, I-convergence was studied in different types of spaces by several authors (see, [16, 17, 18, 19, 20, 21, 22, 23, 24]). Kizmaz [25] introduced the notion of difference sequence spaces, further Et and Çolak [26], Tripathy and Esi [27] analyzed this space. Gumus and Nuray [28] investigated the generalized difference ideal convergence of sequences in different spaces.

2. PRELIMINARIES

Throughout this article, we use $\mathbb{N}$ and $\mathbb{R}$ as the set of natural number and the set of real number, respectively.

Definition 1. *[15] A subset I of power set $P(\mathbb{N})$ is called ideal if it satisfies the following conditions:*

(i) $\emptyset \in I$,
(ii) $C \cup D \in I$ *for all* $C, D \in I$,
(iii) for all $C \in I$ *and* $D \subset C$ *then* $D \in I$.

Remark 2. *[15] An ideal I is called non-trivial if $I \neq P(\mathbb{N})$. If $\{\{n\} : n \in \mathbb{N}\} \subset I$, then I is called admissible ideal.*

Definition 3. *[15] A non-empty subset $\mathcal{F} \subseteq P(\mathbb{N})$ is known as filter in $\mathbb{N}$ if*

(i) $\emptyset \notin \mathcal{F}$,
(ii) for every $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,
(iii) for every $A \in \mathcal{F}$ *with* $A \subseteq B$, *one obtain* $B \in \mathcal{F}$.

Proposition 4. *[15] For every ideal I, there is a filter $\mathcal{F}(I)$ associated with I defined as follows:*

$$\mathcal{F}(I) = \{K \subseteq \mathbb{N} : K = \mathbb{N} \setminus A, \text{ for some } A \in I\}.$$

Definition 5. *[11] A t-norm $\bullet$ is a binary operation on $[0, 1]$, which satisfies the following conditions:*
(i) $\bullet$ *is associative and commutative*
(ii) $\bullet$ *is continuous*
(iii) $p \bullet 1 = p$, *for all* $p \in [0, 1]$
(iv) $\bullet$ *is non-decreasing i.e.,* $p \bullet q \leq r \bullet s$ *whenever* $p \leq r$ *and* $q \leq s$ *and* $p, q, r, s \in [0, 1]$.

Example 6. *$p_1 \bullet p_2 = p_1p_2$ and $p_1 \bullet p_2 = min(p_1, p_2)$ are two examples of continuous t-norm.*

Definition 7. *[16] Let $(V, \phi, \bullet)$ be generalized probabilistic n-norm space. A sequence $u = (u_j)$ in V is said to be Δ^m- I-convergent to $\zeta \in V$ with respect to the generalized probabilistic $n-$ norm if for each $s > 0$, $\epsilon \in (0,1)$ and $v_1, v_2, \ldots, v_{n-1} \in V$, the set*

$$\{j \in \mathbb{N} : \phi(v_1, v_2, \ldots, v_{n-1}, \Delta^m u_j - \zeta, s) \leq 1 - \epsilon\}$$

belongs to I and denoted by $I - \lim \Delta^m u_j = \zeta$.

Definition 8. *[16] Let $(V, \phi, \bullet)$ be generalized probabilistic n-norm space. A sequence $u = (u_j)$ in V is said to be Δ^m- I-Cauchy with respect to the generalized probabilistic $n-$ norm if for each $s > 0$, $\epsilon \in (0,1)$ and $v_1, v_2, \ldots, v_{n-1} \in V$, there exists $k = k(\epsilon) \in \mathbb{N}$ in such a way that, the set*

$$\{j \in \mathbb{N} : \phi(v_1, v_2, \ldots, v_{n-1}, \Delta^m u_j - \Delta^m u_k, s) \leq 1 - \epsilon\}.$$

belongs to I.

Definition 9. *[11] A t-co-norm $\diamond$ is a binary operation on $[0,1]$, which satisfies the following conditions:*
(a) $\diamond$ is associative and commutative
(b) $\diamond$ is continuous
(c) $p \diamond 0 = p$, for all $p \in [0,1]$
(d) $\diamond$ is non-decreasing i.e., $p \diamond q \leq r \diamond s$ whenever $p \leq r$ and $q \leq s$ and $p, q, r, s \in [0,1]$

Example 10. *$p_1 \diamond p_2 = min(p_1 + p_2, 1)$ and $p_1 \diamond p_2 = max(p_1, p_2)$ are two examples of continuous t-co-norm.*

Remark 2.2.[11]
(1) For any two numbers $a, b \in (0,1)$ with $a > b$, there exist $c, d \in (0,1)$ such that $a \bullet c \geq b$ and $a \geq d \diamond b$.
(2) For any $e \in (0,1)$, there exist $f, g \in (0,1)$ such that $f \bullet f \geq e$ and $e \geq g \diamond g$.

Definition 11. *[13] The five-tuple $(V, \phi, \psi, \bullet, \diamond)$ is called an intuitionistic fuzzy n-normed space if V is a linear space over a field $\mathbb{F}$, $\bullet$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and ϕ & ψ denote the degree of membership & non-membership of $(u_1, u_2, \ldots, u_n, s) \in V^n \times (0, \infty)$ satisfy the following conditions $\forall$ $(u_1, u_2, \ldots, u_n) \in V^n$ and $s, t > 0$:*
(a) $\phi(u_1, u_2, \ldots, u_n, s) + \psi(u_1, u_2, \ldots, u_n, s) \leq 1$,
(b) $\phi(u_1, u_2, \ldots, u_n, s) > 0$,
(c) $\phi(u_1, u_2, \ldots, u_n, s) = 1$ iff $u_1, u_2, \ldots, u_n$ are linearly dependent,
(d) $\phi(u_1, u_2, \ldots, u_n, s)$ is invariant under any permutation of $u_1, u_2, \ldots, u_n$,
(e) $\phi(u_1, u_2, \ldots, \alpha u_n, s) = \phi\left(u_1, u_2, \ldots, u_n, \frac{s}{|\alpha|}\right)$, $\forall \alpha \neq 0$, $\alpha \in \mathbb{F}$,

(f) $\phi(u_1, u_2, ...u_{n-1}, u_n, t) \bullet \phi(u_1, u_2, ..., u_{n-1}, u'_n, s) \leq \phi(u_1, u_2, ..., u_{n-1}, u_n + u'_n, t+s)$,
(g) $\phi(u_1, u_2, \ldots, u_n, s) : (0, \infty) \to [0, 1]$ *is continuous in* s,
(h) $\lim_{s\to\infty} \phi(u_1, u_2, \ldots, u_n, s) = 1$ *and* $\lim_{s\to 0} \phi(u_1, u_2, \ldots, u_n, s) = 0$,
(i) $\psi(u_1, u_2, \ldots, u_n, s) < 1$,
(j) $\psi(u_1, u_2, \ldots, u_n, s) = 0$ *iff* $u_1, u_2, \ldots, u_n$ *are linearly dependent,*
(k) $\psi(u_1, u_2, \ldots, u_n, s)$ *is invariant under any permutation of* $u_1, u_2, \ldots, u_n$,
(l) $\psi(u_1, u_2, \ldots, \alpha u_n, s) = \psi\left(u_1, u_2, \ldots, u_n, \frac{s}{|\alpha|}\right)$, $\forall \alpha \neq 0,\ \alpha \in \mathbb{F}$,
(m) $\psi(u_1, u_2, \ldots, u_n, t) \diamond \psi(u_1, u_2, ..., u'_n, s) \geq \psi(u_1, u_2, ..., u_n + u'_n, t+s)$,
(n) $\psi(u_1, u_2, \ldots, u_n, s) : (0, \infty) \to [0, 1]$ *is continuous in* s,
(o) $\lim_{s\to\infty} \psi(u_1, u_2, \ldots, u_n, s) = 0$ *and* $\lim_{s\to 0} \psi(u_1, u_2, \ldots, u_n, s) = 1$.

Example 12. *If* $(V, |.|)$ *forms an* n*-normed linear space, let for all* $a, b \in [0, 1]$, t*-norm is defined as* $a \bullet b = ab$ *and* t*-co-norm is defined as* $a \diamond b = min\{a+b, 1\}$,

$$\phi(u_1, u_2, \ldots, u_n, s) = \frac{s}{s + |u_1, u_2, \ldots, u_n,|} \ \text{ and } \ \psi(u_1, u_2, \ldots, u_n, s) = \frac{|u_1, u_2, \ldots, u_n,|}{s + |u_1, u_2, \ldots, u_n|}$$

Then $(V, \phi, \psi, \bullet, \diamond)$ *forms an intuitionistic fuzzy* n*-normed space.*

Definition 13. *[13] In an IFnNS* $(V, \phi, \psi, \bullet, \diamond)$, *convergence of a sequence* (u_j) *to* ζ *is defined as for a given* $\epsilon > 0$, $s > 0$ *and* $v_1, v_2, \ldots, v_{n-1} \in V$, *there exists* $k_0 \in \mathbb{N}$ *such that*

$$\phi(v_1, v_2, \ldots, v_{n-1}, u_j - \zeta, s) > 1 - \epsilon \ \text{ and } \ \psi(v_1, v_2, \ldots, v_{n-1}, u_j - \zeta, s) < \epsilon,\ \forall j \geq k_0.$$

Definition 14. *[13] In an IFnNS* $(V, \phi, \psi, \bullet, \diamond)$, *a Cauchy sequence* (u_j) *is defined as for a given* $\epsilon > 0$, $s > 0$ *and* $v_1, v_2, \ldots, v_{n-1} \in V$, *there exists* $k_0 \in \mathbb{N}$ *such that for all* $j, k \geq k_0$

$$\phi(v_1, v_2, \ldots, v_{n-1}, u_j - u_k, s) > 1 - \epsilon \ \text{ and } \ \psi(v_1, v_2, \ldots, v_{n-1}, u_j - u_k, s) < \epsilon.$$

3. MAIN RESULTS

In this section, we define $\Delta^m - I$-convergent and $\Delta^m - I$-Cauchy sequences on IFnNS, we also establish some results related to the properties of these sequences.

Definition 15. *Let* $(V, \phi, \psi, \bullet, \diamond)$ *is an IFnNS and* $I \subset P(\mathbb{N})$ *is a non trivial ideal. Let* $(u_j) \in V$ *is said to be generalized difference* I*-convergent (i.e.* $\Delta^m - I$*-convergent) to* $\zeta \in V$ *with respect to intuitionistic fuzzy* n*-norm* (ϕ, ψ), *if for every* $\epsilon > 0$, $s > 0$ *and* $v_1, v_2, \ldots, v_{n-1} \in V$, *the set*

$$\Big\{ j \in \mathbb{N} : \phi(v_1, v_2, \ldots, v_{n-1}, \Delta^m u_j - \zeta, s) \leq 1 - \epsilon \ \text{ or } \ \psi(v_1, v_2, \ldots, v_{n-1}, \Delta^m u_j - \zeta, s) \geq \epsilon \Big\} \in I.$$

Hence, We write it as $I_{(\phi,\psi)} - \lim\ \Delta^m u_j = \zeta$.

Definition 16. *Let $(V,\phi,\psi,\bullet,\diamond)$ is an IFnNS and $I\subset P(\mathbb{N})$ is a non trivial ideal. Let $(u_j)\in V$ is said to be generalized difference I-Cauchy (i.e. $\Delta^m - I$-Cauchy) with respect to intuitionistic fuzzy n-norm (ϕ,ψ), if for every $\epsilon>0$, $s>0$ and $v_1,v_2,\dots,v_{n-1}\in V$, there exists a natural number $N=N(\epsilon)$ such that the set*

$$\Big\{j\in\mathbb{N}:\phi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\Delta^m u_N,s)\le 1-\epsilon \text{ or } \psi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\Delta^m u_N,s)\ge\epsilon\Big\}\in I.$$

Lemma 17. *Let $(V,\phi,\psi,\bullet,\diamond)$ is an IFnNS and $I\subset P(\mathbb{N})$ is a non trivial ideal. $u=(u_j)$ is a sequence in V. Then for every $\epsilon>0$, $s>0$ and $v_1,v_2,\dots,v_{n-1}\in V$, the following are equivalent:*

(1) $I_{(\phi,\psi)}-\lim_{j\to\infty}\Delta^m u_j=\zeta$;

(2) $\{j\in\mathbb{N}:\phi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\zeta,s)\le 1-\epsilon\}\in I$ *and* $\{j\in\mathbb{N}:\psi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\zeta,s)\ge\epsilon\}\in I$;

(3) $\{j\in\mathbb{N}:\phi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\zeta,s)>1-\epsilon$ *and* $\psi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\zeta,s)<\epsilon\}\in\mathcal{F}(I)$;

(4) $\{j\in\mathbb{N}:\phi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\zeta,s)>1-\epsilon\}\in\mathcal{F}(I)$ *and* $\{j\in\mathbb{N}:\psi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\zeta,s)<\epsilon\}\in\mathcal{F}(I)$;

(5) $I-\lim_{j\to\infty}\ \phi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\zeta,s)=1$ *and* $I-\lim_{j\to\infty}\ \psi(v_1,v_2,\dots,v_{n-1},\Delta^m u_j-\zeta,s)=0$.

Theorem 18. *Let $u=(u_j)$ be a sequence in IFnNS $(V,\phi,\psi,\bullet,\diamond)$. If (u_j) is generalized difference $I-$convergent with respect to intuitionistic fuzzy n-norm (ϕ,ψ), then $I_{(\phi,\psi)}-\lim\ \Delta^m(u_j)$ is unique.*

Proof. Let there are two different elements ζ_1 and ζ_2 such that $I_{(\phi,\psi)}-\lim\ \Delta^m(u_j)=\zeta_1$ and $I_{(\phi,\psi)}-\lim\ \Delta^m(u_j)=\zeta_2$. For a given $\epsilon>0$, choose $\gamma>0$ such that $(1-\gamma)\bullet(1-\gamma)>1-\epsilon$ and $\gamma\diamond\gamma<\epsilon$.
For $s>0$ and $v_1,v_2,\dots,v_{n-1}\in V$, we define

$$\begin{aligned}
A_1 &= \{j\in\mathbb{N}:\phi(v_1,v_2,\dots,v_{n-1},\Delta^m(u_j)-\zeta_1,s)\le 1-\gamma\},\\
A_2 &= \{j\in\mathbb{N}:\psi(v_1,v_2,\dots,v_{n-1},\Delta^m(u_j)-\zeta_1,s)\ge \gamma\},\\
A_3 &= \{j\in\mathbb{N}:\phi(v_1,v_2,\dots,v_{n-1},\Delta^m(u_j)-\zeta_2,s)\le 1-\gamma\},\\
A_4 &= \{j\in\mathbb{N}:\psi(v_1,v_2,\dots,v_{n-1},\Delta^m(u_j)-\zeta_2,s)\ge \gamma\} \text{ and}\\
A &= (A_1\cup A_3)\cap(A_2\cup A_4)
\end{aligned}$$

Sets A_1,A_2,A_3,A_4 and A must belong to I as (u_j) has two different generalized difference $I-$ limit with respect to intuitionistic fuzzy n-norm (ϕ,ψ) i.e. ζ_1 and ζ_2. Hence $A^c\in\mathcal{F}(I)$ then A^c is non empty. Let us say some $k\in A^c$ then either

$k \in {A_1}^c \cap {A_3}^c$ or $k \in {A_2}^c \cap {A_4}^c$.
If $k \in {A_1}^c \cap {A_3}^c$ which implies that

$$\phi\left(v_1, v_2, \ldots, v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{2}\right) > 1 - \gamma \text{ and } \phi\left(v_1, v_2, \ldots, v_{n-1}, \Delta^m(u_k) - \zeta_2, \frac{s}{2}\right) > 1 - \gamma$$

Hence

$$\begin{aligned}\phi(v_1, v_2, \ldots v_{n-1}, \zeta_1 - \zeta_2, s) &\\ \geq \;& \phi\left(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{2}\right) \bullet \phi\left(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_2, \frac{s}{2}\right)\\ > \;& (1-\gamma) \bullet (1-\gamma) > 1 - \epsilon\end{aligned}$$

As $\epsilon > 0$ was arbitrary hence $\phi(v_1, v_2, \ldots v_{n-1}, \zeta_1 - \zeta_2, t) = 1$ for all $s > 0$. So we have $\zeta_1 = \zeta_2$, which is a contradiction.
If $k \in {A_2}^c \cap {A_4}^c$ which implies that

$$\psi\left(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{2}\right) < \gamma \text{ and } \psi\left(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_2, \frac{s}{2}\right) < \gamma$$

Hence

$$\begin{aligned}\psi(v_1, v_2, \ldots v_{n-1}, \zeta_1 - \zeta_2, s) &\\ \leq \;& \psi\left(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{2}\right) \diamond \psi\left(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_2, \frac{s}{2}\right)\\ < \;& \gamma \diamond \gamma < \epsilon\end{aligned}$$

As $\epsilon > 0$ was arbitrary hence $\psi(v_1, v_2, \ldots v_{n-1}, \zeta_1 - \zeta_2, s) = 0$ for all $s > 0$. So we have $\zeta_1 = \zeta_2$, which is a contradiction. Hence (u_j) has unique generalized difference $I_{(\phi,\psi)}-$ limit. □

Theorem 19. *Let* $u = (u_j)$ *be any sequence in IFnNS* $(V, \phi, \psi, \bullet, \diamond)$ *such that* $(\phi, \psi) - \lim\ \Delta^m(u_j) = \zeta$, *then* $I_{(\phi,\psi)} - \lim\ \Delta^m(u_j) = \zeta$.

Proof. Given that $(\phi, \psi) - \lim\ \Delta^m(u_j) = \zeta$, hence for any $\epsilon > 0$, $v_1, v_2, \ldots v_{n-1} \in V$ and $s > 0$, we can find a natural number $k \in \mathbb{N}$ in such a way that

$$\phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_j) - \zeta, s) < 1 - \epsilon \text{ and } \psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_j) - \zeta, s) > \epsilon$$

for all $j \geq k$.
Now let

$$K = \{j \in \mathbb{N} : \phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_j) - \zeta, s) \leq 1 - \epsilon \text{ or } \psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_j) - \zeta, s) \geq \epsilon\}$$

As $K \subset \{1, 2, \ldots, k-1\}$ and I is an admissible ideal so $K \in I$. Hence $I_{(\phi,\psi)} - \lim\ \Delta^m(u_j) = \zeta$. □

The converse of theorem 3.3 is not true in general.

Example 20. *Let* $V=\mathbb{R}^n$ *with* $|u_1,u_2,\dots,u_n| = abs\left(\begin{vmatrix} u_{11} & u_{12} & . & . & u_{1n}\\ u_{21} & u_{22} & . & . & u_{2n}\\ . & . & . & . & .\\ . & . & . & . & .\\ u_{n1} & u_{n2} & . & . & u_{nn}\end{vmatrix}\right)$,
where $u_i=(u_{i1},u_{i2},\dots,u_{in})\in\mathbb{R}^n$ *for all* $1\le i\le n$*, let for all* $a,b\in[0,1]$*, t-norm is defined as* $a\bullet b\ =\ ab$ *and t-co-norm is defined as* $a\diamond b\ =\ min\{a+b,1\}$,

$$\phi(u_1,u_2,\dots,u_n,s)=\frac{s}{s+|u_1,u_2,\dots,u_n} \quad and \quad \psi(u_1,u_2,\dots,u_n,s)=\frac{|u_1,u_2,\dots,u_n|}{s+|u_1,u_2,\dots,u_n|}$$

Then $(\mathbb{R}^n,\phi,\psi,\bullet,\diamond)$ *forms an intuitionistic fuzzy n-normed space.*

Suppose $I=\{A\subset\mathbb{N}:\delta(A)=0\}$*, where* $\delta(A)$ *is the natural density of the set* A *in* $\mathbb{N}$ *which is defined as* $\delta(A)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\chi_A(i)$*, where* χ_A *is the characteristic function on* A*, hence* I *is a non-trivial admissible ideal.*

Now let us define a sequnece $u=(u_i)$ *where*

$$u_i \quad = \quad \begin{cases}(i,0,\dots,0)\in\mathbb{R}^n, & if\ i=k^2,\ (k\in\mathbb{N})\\ (0,0,...,0)\in\mathbb{R}^n, & otherwise.\end{cases}$$

For $m=2$*, we have*

$$\Delta^2 u_i \quad = \quad \begin{cases}(i,0,...,0)\in\mathbb{R}^n, & if\ i=k^2,k^2-2,\ (k\in\mathbb{N})\\ (-2i,0,...,0)\in\mathbb{R}^n, & if\ i=k^2-1,\ (k\in\mathbb{N})\\ (0,0,...,0)\in\mathbb{R}^n, & otherwise.\end{cases}$$

Now for any $\epsilon>0$*,* $s>0$ *and* $v_1,v_2,\dots,v_{n-1}\in V$*, we define*

$$K(\epsilon,s)=\{i\in\mathbb{N}:\phi(v_1,v_2,\dots,v_{n-1},\Delta^2u_i,s)\le 1-\epsilon\ or$$
$$\psi(v_1,v_2,\dots,v_{n-1},\Delta^2u_i,s)\ge\epsilon\}$$

then

$$K(\epsilon,s)=\left\{i\in\mathbb{N}:\frac{s}{s+|v_1,v_2,\dots,v_{n-1},\Delta^2u_i|}\le 1-\epsilon\ or\right.$$
$$\left.\frac{|v_1,v_2,\dots,v_{n-1},\Delta^2u_i|}{s+|v_1,v_2,\dots,v_{n-1},\Delta^2u_i|}\ge\epsilon\right\}.$$

Therefore

$$\begin{aligned}K(\epsilon,s) &= \left\{i\in\mathbb{N}:|v_1,v_2,\dots,v_{n-1},\Delta^2u_i|\ge\frac{\epsilon s}{1-\epsilon}>0\right\}\\ &\subseteq \{i\in\mathbb{N}:\Delta^2u_i=(i,0,...,0)\in\mathbb{R}^n\}\cup\{i\in\mathbb{N}:\Delta^2u_i=(-2i,0,...,0)\in\mathbb{R}^n\}\\ &= \{i\in\mathbb{N}:i=k^2\}\cup\{i\in\mathbb{N}:i=k^2-2\}\cup\{i\in\mathbb{N}:i=k^2-1\},\end{aligned}$$

where $k \in \mathbb{N}$. Hence $\delta(K(\epsilon, s)) = 0$, which implies that $K(\epsilon, s) \in I$. Hence $I_{(\phi,\psi)} - \lim\ \Delta^2(u_i) = 0$. On the other hand, $\Delta^2(u_i)$ is not convergent with respect to the intuitionistic fuzzy n-norm (ϕ, ψ) as $\Delta^2(u_i)$ is not convergent in $(\mathbb{R}^n, |.|)$.

Theorem 21. *Let $u = (u_j)$ and $w = (w_j)$ be any two sequences in IFnNS $(V, \phi, \psi, \bullet, \diamond)$ such that $I_{(\phi,\psi)} - \lim\ \Delta^m(u_j) = \zeta_1$ and $I_{(\phi,\psi)} - \lim\ \Delta^m(w_j) = \zeta_2$ then*

(1) $I_{(\phi,\psi)} - \lim\ \Delta^m(u_j + w_j) = \zeta_1 + \zeta_2$,

(2) For any real number α, $I_{(\phi,\psi)} - \lim\ \Delta^m(\alpha u_j) = \alpha\zeta_1$.

Proof. (1) For any $\epsilon > 0$, we may find $\gamma > 0$ such that $(1-\gamma) \bullet (1-\gamma) > 1-\epsilon$ and $\gamma \diamond \gamma < \epsilon$. For $s > 0$ and $v_1, v_2, \ldots, v_{n-1} \in V$, we define

$$\begin{aligned}
P_1 &= \{j \in \mathbb{N} : \phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_j) - \zeta_1, s) \leq 1-\gamma\},\\
P_2 &= \{j \in \mathbb{N} : \psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_j) - \zeta_1, s) \geq \gamma\},\\
P_3 &= \{j \in \mathbb{N} : \phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(w_j) - \zeta_2, s) \leq 1-\gamma\},\\
P_4 &= \{j \in \mathbb{N} : \psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(w_j) - \zeta_2, s) \geq \gamma\} \text{ and}\\
P &= (P_1 \cup P_3) \cap (P_2 \cup P_4).
\end{aligned}$$

Sets P_1, P_2, P_3, P_4 and P must belong to I as $I_{(\phi,\psi)} - \lim\ \Delta^m(u_j) = \zeta_1$ and $I_{(\phi,\psi)} - \lim\ \Delta^m(w_j) = \zeta_2$. Hence $P^c \in \mathcal{F}(I)$ then P^c is non empty. Now we show that

$$\begin{aligned}
P^c \subset\ &\{j \in \mathbb{N} : \phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_j + w_j) - (\zeta_1 + \zeta_2), s) > 1-\epsilon\\
&\text{and } \psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_j + w_j) - (\zeta_1 + \zeta_2), s) < \epsilon\}
\end{aligned}$$

To show this we let $k \in P^c$. So we have,

$\phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{2}) > 1-\gamma$, $\phi(v_1, v_2, \ldots, v_{n-1}, \Delta^m(w_k) - \zeta_2, \frac{s}{2}) > 1-\gamma$, $\psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{2}) < \gamma$ and $\psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(w_k) - \zeta_2, \frac{s}{2}) < \gamma$.

Hence,

$$\begin{aligned}
&\phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k + w_k) - (\zeta_1 + \zeta_2), s)\\
&\geq \phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{2}) \bullet \phi(v_1, v_2, \ldots v_{n-1}, \Delta^m(w_k) - \zeta_2, \frac{s}{2})\\
&> (1-\gamma) \bullet (1-\gamma)\\
&> 1-\epsilon
\end{aligned}$$

and

$$\begin{aligned}
&\psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k + w_k) - (\zeta_1 + \zeta_2), s)\\
&\leq \psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{2}) \diamond \psi(v_1, v_2, \ldots v_{n-1}, \Delta^m(w_k) - \zeta_2, \frac{s}{2})\\
&< \gamma \diamond \gamma\\
&< \epsilon.
\end{aligned}$$

which implies that

$$P^c \subset \{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j + w_j) - (\zeta_1 + \zeta_2), s) > 1 - \epsilon \text{ and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j + w_j) - (\zeta_1 + \zeta_2), s) < \epsilon\}$$

As $P^c \in \mathcal{F}(I)$, then $P \in I$ which implies that $I_{(\phi,\psi)} - \lim\ \Delta^m(u_j + w_j) = \zeta_1 + \zeta_2$.

(2) If $\alpha = 0$ then for any $\epsilon > 0$, $v_1, v_2, \dots v_{n-1} \in V$ and $s > 0$, there exists $n_0 = 1$ in such a way that

$$\phi(v_1, v_2, \dots v_{n-1}, \Delta^m(0u_j) - (0\zeta_1), s) = \phi(v_1, v_2, \dots v_{n-1}, 0, s) = 1 > 1 - \beta$$

and

$$\psi(v_1, v_2, \dots v_{n-1}, \Delta^m(0u_j) - (0\zeta_1), s) = \psi(v_1, v_2, \dots v_{n-1}, 0, s) = 0 < \beta$$

for each $j \geq n_0$ which implies that $(\phi, \psi) - \lim\ \Delta^m(0u_j) = \theta$. Hence by Theorem 3.3, $I_{(\phi,\psi)} - \lim\ \Delta^m(0x_j) = \theta$.

If $\alpha(\neq 0) \in \mathbb{R}$. To prove the result, we will show that for any $\epsilon > 0$, $v_1, v_2, \dots v_{n-1} \in V$ and $s > 0$, the set

$$\{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_j) - (\alpha\zeta_1), s) > 1 - \epsilon \text{ and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_j) - (\alpha\zeta_1), s) < \epsilon\} \in \mathcal{F}(I),$$

for any $\alpha(\neq 0) \in \mathbb{R}$.

As we have given that $I_{(\phi,\psi)} - \lim\ \Delta^m(u_j) = \zeta_1$ so we have for any $\epsilon > 0$, $v_1, v_2, \dots v_{n-1} \in V$ and $s > 0$, the set

$$K = \{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta_1, s) > 1 - \epsilon \text{ and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta_1, s) < \epsilon\} \in \mathcal{F}(I).$$

Choose any $k \in K$, hence we have $\phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, s) > 1 - \epsilon$ and $\psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, s) < \epsilon$. Now,

$$\begin{aligned} &\phi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_k) - (\alpha\zeta_1), s) \\ &\quad = \phi\left(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{|\alpha|}\right) \\ &\quad \geq \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, s) \bullet \phi(v_1, v_2, \dots v_{n-1}, 0, \frac{s}{|\alpha|} - s) \\ &\quad = \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, s) \bullet 1 \\ &\quad = \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, s) > 1 - \epsilon \end{aligned}$$

and,

$$\begin{aligned}
&\psi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_k) - (\alpha\zeta_1), s) \\
&\quad = \psi\left(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, \frac{s}{|\alpha|}\right) \\
&\quad \leq \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, s) \diamond \psi(v_1, v_2, \dots v_{n-1}, 0, \frac{s}{|\alpha|} - s) \\
&\quad = \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, s) \diamond 0 \\
&\quad = \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta_1, s) < \epsilon
\end{aligned}$$

which implies that

$$\begin{aligned}
k \in \{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_j) - (\alpha\zeta_1), s) > 1 - \epsilon \\
\text{and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_j) - (\alpha\zeta_1), s) < \epsilon\}.
\end{aligned}$$

Hence

$$\begin{aligned}
K \subset \{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_j) - (\alpha\zeta_1), s) > 1 - \epsilon \\
\text{and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_j) - (\alpha\zeta_1), s) < \epsilon\}
\end{aligned}$$

Since $K \in \mathcal{F}(I)$, hence the set

$$\begin{aligned}
\{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_j) - (\alpha\zeta_1), s) > 1 - \epsilon \\
\text{and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(\alpha u_j) - (\alpha\zeta_1), s) < \epsilon\} \in \mathcal{F}(I)
\end{aligned}$$

which implies that $I_{(\phi,\psi)} - \lim\ \Delta^m(\alpha x_j) = \alpha\zeta_1$. □

Theorem 22. *Let $u = (u_j)$ be any sequence in IFnNS $(V, \phi, \psi, *, \diamond)$ such that (u_j) is $\Delta^m - I_{(\phi,\psi)}$-convergent if and only if sequence (u_j) is $\Delta^m - I_{(\phi,\psi)}$-Cauchy.*

Proof. In V, let (u_j) is $\Delta^m - I_{(\phi,\psi)}$-convergent sequence to ζ, then for any given $\epsilon > 0$, we can choose $0 < \gamma < 1$ in such a way that $(1-\gamma) \bullet (1-\gamma) > 1 - \epsilon$ and $\gamma \diamond \gamma < \epsilon$. Then for any $s > 0$ and $v_1, v_2, \dots v_{n-1} \in V$, we define

$$\begin{aligned}
Q_1 &= \{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, \frac{s}{2}) \leq 1 - \gamma\}, \\
Q_2 &= \{j \in \mathbb{N} : \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, \frac{s}{2}) \geq \gamma\} \text{ and} \\
Q &= (Q_1 \cup Q_2)
\end{aligned}$$

Sets Q_1, Q_2 and Q must belong to I as $I_{(\phi,\psi)} - \lim\ \Delta^m(u_j) = \zeta$. Hence $Q^c \in \mathcal{F}(I)$ then Q^c is non empty. Let if $k \in Q^c$, choose a fixed $j \in Q^c$. So we have,

$$\begin{aligned}
&\phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \Delta^m(u_k), s) \\
&\quad \geq \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, \frac{s}{2}) \bullet \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta, \frac{s}{2}) \\
&\quad > (1-\gamma) \bullet (1-\gamma) \\
&\quad > 1 - \epsilon
\end{aligned}$$

and

$$\begin{aligned}\psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \Delta^m(u_k), s)&\\ \leq\ & \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, \frac{s}{2}) \diamond \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_k) - \zeta, \frac{s}{2})\\ <\ & \gamma \diamond \gamma\\ <\ & \epsilon.\end{aligned}$$

Hence,

$$\{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \Delta^m(u_k), s) > 1 - \epsilon \text{ and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \Delta^m(u_k), s) < \epsilon\} \in \mathcal{F}(I).$$

Which implies that sequence (u_j) is $\Delta^m - I_{(\phi,\psi)}$-Cauchy.
Conversely, suppose sequence (u_j) is $\Delta^m - I_{(\phi,\psi)}$-Cauchy but it is not $\Delta^m - I_{(\phi,\psi)}$-convergent sequence. Then for any $\epsilon \in (0, 1)$, $s > 0$ and $v_1, v_2, \dots v_{n-1} \in V$, the set

$$A = \Big\{j \in \mathbb{N} : \phi(v_1, v_2, \dots, v_{n-1}, \Delta^m u_j - \zeta, s) > 1 - \epsilon \ \ or \ \ \psi(v_1, v_2, \dots, v_{n-1}, \Delta^m u_j - \zeta, s) < \epsilon\Big\}$$

belongs to I. Therefore, $A^c \in \mathcal{F}(I)$.
Simultaneously, there exists a $k = k(\epsilon) \in \mathbb{N}$ such that the set

$$B = \{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \Delta^m(u_k), s) \leq 1 - \epsilon \text{ or } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \Delta^m(u_k), s) \geq \epsilon\} \in I.$$

Consequently,

$$\phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \Delta^m(u_k), s) \geq 2\phi(v_1, v_2, \dots, v_{n-1}, \Delta^m u_j - \zeta, \frac{s}{2}) > 1 - \epsilon$$

and

$$\psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \Delta^m(u_k), s) \leq 2\psi(v_1, v_2, \dots, v_{n-1}, \Delta^m u_j - \zeta, \frac{s}{2}) < \epsilon,$$

if $\phi(v_1, v_2, \dots, v_{n-1}, \Delta^m u_j - \zeta, \frac{s}{2}) > \frac{1-\epsilon}{2}$ and $\psi(v_1, v_2, \dots, v_{n-1}, \Delta^m u_j - \zeta, \frac{s}{2}) < \frac{\epsilon}{2}$, respectively. Therefore $B^c \in I$ or $B \in \mathcal{F}(I)$. Which is a contradiction. □

4. $\Delta^m - I^*$-CONVERGENCE IN IFnNS

Definition 23. *A sequence (u_j) in IFnNS $(V, \phi, \psi, \bullet, \diamond)$ is said to be $\Delta^m - I^*$-convergent to $\zeta \in V$ with respect to the intuitionistic fuzzy n-norm (ϕ, ψ) if there exists a set $M = \{j_i \in \mathbb{N} : j_i < j_{i+1},\ for\ all\ i \in \mathbb{N}\}$ in such a way that $M \in \mathcal{F}(I)$ and $(\phi, \psi) - \lim\ \Delta^m u_{j_i} = \zeta$. In this case, we say $I^*_{(\phi,\psi)} - \lim\ \Delta^m u_j = \zeta$.*

Definition 24. *[15] An admissible ideal I is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{C_1, C_2, \dots\}$ belonging to I, there exists a countable family $\{D_1, D_2, \dots\}$ in I such that $C_k \ \Delta D_k$ is a finite set for each $k \in \mathbb{N}$ and $D = \cup_{k=1}^{\infty} D_k \in I$; where Δ is the symmetric difference.*

Theorem 25. *Let I be an admissible ideal and a sequence (u_j) in IFnNS $(V, \phi, \psi, *, \diamond)$ is such that $I^*_{(\phi,\psi)} - \lim \ \Delta^m u_j = \zeta$ then $I_{(\phi,\psi)} - \lim \ \Delta^m u_j = \zeta$.*

Proof. Since $I^*_{(\phi,\psi)} - \lim \ \Delta^m u_j = \zeta$ so there exists a subset $M = \{j_i \in \mathbb{N} : j_i < j_{i+1}, \text{ for all } i \in \mathbb{N}\}$ such that $M \in \mathcal{F}(I)$ and $(\phi, \psi) - \lim \ \Delta^m u_{j_i} = \zeta$. Hence for each $\epsilon > 0$, $s > 0$ and $v_1, v_2, \dots v_{n-1} \in V$, there exists $l \in \mathbb{N}$ in such a way that

$$\phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_{j_i}) - \zeta, s) > 1 - \epsilon \text{ and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_{j_i}) - \zeta, s) < \epsilon$$

for all $i \geq l$. As the set

$$\{j_i \in A : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_{j_i}) - \zeta, s) \leq 1 - \epsilon \quad \text{or } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_{j_i}) - \zeta, s) \geq \epsilon\}$$

is contained in $\{j_1, j_2, ..., j_{l-1}\}$.
Hence

$$\{j_i \in M : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_{j_i}) - \zeta, s) \leq 1 - \epsilon \quad \text{or } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_{j_i}) - \zeta, s) \geq \epsilon\} \in I$$

As I is an admissible ideal. Also $M \in \mathcal{F}(I)$, then by the definition of $\mathcal{F}(I)$ there exists a set $B \in I$ such that $M = \mathbb{N} \setminus B$. So

$$\begin{aligned} \{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) &\leq 1 - \epsilon \\ \text{or } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) &\geq \epsilon\} \\ \subset \quad & B \cup \{j_1, j_2, ..., j_{l-1}\}. \end{aligned}$$

Therefore,

$$\{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \leq 1 - \epsilon \quad \text{or } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \geq \epsilon\} \in I$$

which implies that $I_{(\phi,\psi)} - \lim \ \Delta^m u_j = \zeta$. □

Remark 3.2. The converse of the above theorem does not hold in general.

Example 26. *Let $V = \mathbb{R}^n$ and $|u_1, u_2, \dots, u_n| = abs \left(\begin{vmatrix} u_{11} & u_{12} & . & . & u_{1n} \\ u_{21} & u_{22} & . & . & u_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ u_{n1} & u_{n2} & . & . & u_{nn} \end{vmatrix} \right)$, where $u_i = (u_{i1}, u_{i2}, \dots, u_{in}) \in \mathbb{R}^n$ for all $1 \leq i \leq n$, let for all $a, b \in [0, 1]$, t-norm is defined as $a \bullet b \ = \ ab$ and t-co-norm is defined as $a \diamond b \ = \ min\{a + b, 1\}$,*

$$\phi(u_1, u_2, \dots, u_n, s) = \frac{s}{s + |u_1, u_2, \dots, u_n|} \quad \text{and} \quad \psi(u_1, u_2, \dots, u_n, s) = \frac{|u_1, u_2, \dots, u_n|}{s + |u_1, u_2, \dots, u_n|}$$

Then $(\mathbb{R}^n, \phi, \psi, \bullet, \diamond)$ *is an IFnNS.*
Now we take a decomposition of $\mathbb{N}$ *as* $\mathbb{N} = \cup A_i$, *where every* A_i *is an infinite set and* $A_i \cap A_j = \Phi$, *for* $i \neq j$. *Suppose* $I = \{N \subset \mathbb{N} : N \subset \cup_{i=1}^{s} A_i$, *for some finite natural number* $s\}$ *then* I *is a non-trivial admissible ideal.*
Now we define a sequence (u_j) *in such a way that*
if $j \in A_i$, $\Delta^m(u_j) = (\frac{1}{i}, 0, \dots, 0) \in \mathbb{R}^n$, $(j = 1, 2, \dots)$. *Then for* $s > 0$ *and* $v_1, v_2, \dots v_{n-1} \in \mathbb{R}^n$,

$$\phi(v_1, v_2, \dots v_{n-1}, \Delta u_j, s) = \frac{s}{s + |v_1, v_2, \dots v_{n-1}, \Delta u_j|} \to 1$$
$$\text{and } \psi(v_1, v_2, \dots v_{n-1}, \Delta u_j, s) = \frac{|v_1, v_2, \dots v_{n-1}, \Delta u_j|}{s + |v_1, v_2, \dots v_{n-1}, \Delta u_j|} \to 0, \text{ as } j \to \infty.$$

Hence $I_{(\phi,\psi)} - \lim\ \Delta u_j = 0$.

Let on contrary that $I^*_{(\phi,\psi)} - \lim\ \Delta u_j = 0$, *then there exists* $B = \{k_i \in \mathbb{N} : k_i < k_{i+1}$, *for all* $i \in \mathbb{N}\}$ *such that* $B \in \mathcal{F}(I)$ *and* $(\phi, \psi) - \lim\ \Delta(u_{k_i}) = 0$. *As* $B \in \mathcal{F}(I)$, *there exists* $C = \mathbb{N} \setminus B$ *and* $C \in I$. *Then there exists a natural numer* r *such that* $C \subset \cup_{i=1}^{r} A_i$. *Then* $A_{r+1} \subset B$, *so we have*

$$\Delta^m(u_{k_i}) = (\frac{1}{r+1}, 0, \dots, 0) \in \mathbb{R}^n, \text{ for infinitely many values of } k_i \text{ in } B.$$

Which is a contradiction. Therefore, $I^*_{(\phi,\psi)} - \lim\ \Delta u_j \neq 0$.

Theorem 27. *Let* $u = (u_j)$ *be a sequence in IFnNS* $(V, \phi, \psi, \bullet, \diamond)$ *such that* $I_{(\phi,\psi)} - \lim_{j\to\infty}\ \Delta^m u_j = \zeta$ *and ideal* I *satisfies condition* (AP). *Then* $I^*_{(\phi,\psi)} - \lim_{j\to\infty}\ \Delta^m u_j = \zeta$.

Proof. As $I_{(\phi,\psi)} - \lim_{j\to\infty}\ \Delta^m u_j = \zeta$. Then for each $\epsilon > 0$, $s > 0$ and $v_1, v_2, \dots v_{n-1} \in V$, we have

$$\{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \leq 1 - \epsilon \text{ or } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \geq \epsilon\} \in I$$

For $k \in \mathbb{N}$ and $s > 0$, we define

$$A_k = \{j \in \mathbb{N} : 1 - \frac{1}{k} \leq \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) < 1 - \frac{1}{k+1} \text{ or } \frac{1}{k+1} < \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \leq \frac{1}{k}\}$$

Now, it can be easily seen that a countable family of mutually disjoint sets $\{C_1, C_2, \dots\}$ belongs to I and therefore by property (AP) there is a countable family of sets $\{D_1, D_2, \dots\}$ in I in such a way that $C_i \Delta D_i$ is a finite set for each $i \in \mathbb{N}$ and $D = \cup_{i=1}^{\infty} D_i$. Since $D \in I$ so by definition of associate filter $\mathcal{F}(I)$ there is set $K \in \mathcal{F}(I)$ such that $K = \mathbb{N} \setminus D$. Now, to show the theorem, it is sufficient to show that the subsequence $(u_j)_{j\in\mathbb{N}} \in K$ is ordinary convergent to with respect to the intuitionistic fuzzy n-norm (ϕ,ψ). For this, let $\eta > 0$, $s > 0$ and $v_1, v_2, \dots v_{n-1} \in V$. Choose a positive integer q such that $\frac{1}{q} < \eta$. Then we have

$$\{j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \leq 1 - \eta \text{ or } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \geq \eta\}$$

$$\subset \left\{ j \in \mathbb{N} : \phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \leq 1 - \frac{1}{q} \text{ or } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) \geq \frac{1}{q} \right\}$$

$$\subset \cup_{i=1}^{q+1} C_i$$

Since $C_i \Delta D_i$ is a finite set for each $i = 1, 2, \dots q+1$, there exists a natural number j_0 such that

$$(\cup_{i=1}^{q+1} D_i) \cap \{j \in \mathbb{N} : j \geq j_0\} = (\cup_{i=1}^{q+1} A_i) \cap \{j \in \mathbb{N} : j \geq j_0\}.$$

If $j \geq j_0$ and $j \in K$, then $j \notin D$. This implies that $j \in \cup_{i=1}^{q+1} D_i$ and therefore $j \notin \cup_{i=1}^{q+1} C_i$. Hence for every $j \geq j_0$ and $j \in K$, we have

$$\phi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) > 1 - \eta \text{ and } \psi(v_1, v_2, \dots v_{n-1}, \Delta^m(u_j) - \zeta, s) < \eta$$

As this holds for every $\eta > 0$, $s > 0$ and $v_1, v_2, \dots v_{n-1} \in V$, so $I^*_{(\phi,\psi)} - \lim_{j\to\infty} \Delta^m u_j = \zeta$. □

5. CONCLUSIONS AND SUGGESTIONS

The main aim of this paper is to investigate the behaviour of generalized difference ideal convergent sequence and generalized difference ideal Cauchy sequence in intuitionistic fuzzy n-normed space. Furthermore, article also presents $\Delta^m - I^*$ convergence. A parallel has been drawn between the concepts of I and I^* generalized difference convergent sequences. This novel study and its results bring forward various new methods to tackle convergence problems of sequences arising in various areas of science and technology.

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