Yugoslav Journal of Operations Research
32 (2022), Number 1, 3–27
DOI: https://doi.org/10.2298/YJOR200915018U

NONDIFFERENTIABLE GENERALIZED MINIMAX FRACTIONAL PROGRAMMING UNDER (Φ, ρ)-INVEXITY

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Received: September 2020 / Accepted: June 2021

Abstract: In this paper, a class of nonconvex nondifferentiable generalized minimax fractional programming problems is considered. Sufficient optimality conditions for the considered nondifferentiable generalized minimax fractional programming problem are established under the concept of (Φ, ρ)-invexity. Further, two types of dual models are formulated and various duality theorems relating to the primal minimax fractional programming problem and dual problems are established. The results established in the paper generalize and extend several known results in the literature to a wider class of nondifferentiable minimax fractional programming problems. To the best of our knowledge, these results have not been established till now.

Keywords: Nondifferentiable Minimax Fractional Problem, Duality, (Φ, ρ)-invexity, Sufficient Conditions.

MSC: 90C26, 90C32, 49K35.

1. INTRODUCTION

Minimax programming has been an interesting field of active research for a long time. These problems are of pivotal importance in many areas of modern research such as economics, engineering design, portfolio selection, game theory, rational Chebyshev approximations and financial planning, see [7, 8, 43] and the references cited therein. Necessary optimality conditions for finite-dimensional constrained minimax problems in terms of Lagrange multipliers have been originally investigated by Bram [11] and Danskin [15]. Schmitendorf [42] has established the necessary and sufficient optimality conditions for the following minimax programming problem:

$$(P) \quad \min \sup_{y \in Y} f(x, y)$$

$$\text{subject to } h(x) \leq 0,$$

where $f(.,.) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $h(.) : \mathbb{R}^n \to \mathbb{R}^p$ are differentiable convex functions and Y is a compact subset of $\mathbb{R}^m$. Bector and Bhatia [9] and Weir [48] relaxed the convexity assumption in proving the sufficient optimality conditions for problem (P) and formulated several dual models. They proved such results under pseudoconvexity and quasiconvexity assumptions imposed on the functions constituting problem (P) and its duals. Bector et al. [10] derived duality results for minimax programming problems involving V-invex functions.

The concept of invex functions introduced by Hanson [24] and named by Craven [14] is a significant generalization of convex functions. The theory of mathematical programming has grown remarkably, when further extensions of invexity have been introduced to establish the optimality conditions and duality results. Preda [41] introduced the concept of (F, ρ)-convexity as an extension of F-convexity defined by Hanson and Mond [25], whereas the concept of ρ-convexity was introduced by Vial [46]. Jeyakumar [28] generalized the Vial [46] notion of ρ-convexity to ρ-invexity. In a recent work, Yuan et al. [50] introduced a unified formulation of generalized convexity, called (C, α, ρ, d)-convexity. The (C, α, ρ, d)-convexity extends the (F, α, ρ, d)-convexity introduced by Liang et al. [31] by relaxing the sublinearity of the scale function to convexity. Caristi et al. [12] introduced the notion of (Φ, ρ)-invexity for differentiable scalar optimization problems, which generalizes invexity as well as (F, ρ)-convexity. Recently, Antczak and Stasiak [6] generalized the concept of (Φ, ρ)-invexity to a nondifferentiable case and they introduced the definition of a locally Lipschitz (Φ, ρ)-invex function. Very recently, nonsmooth semi-inifnite minimax programming problems with locally Lipschitz (Φ, ρ)-invex and generalized locally Lipschitz (Φ, ρ)-invex functions have been studied by Liu et al. [34] and Upadhyay and Mishra [45].

Many authors investigated the optimality conditions and duality results for minimax fractional programming problems using generalized convexity assumptions, see for example [1, 3, 4, 5, 13, 16, 17, 18, 19, 20, 21, 32, 33, 34, 38, 39, 44, 45] and the references cited therein. Lai et al. [30] established the necessary and sufficient optimality conditions and Lai and Lee [29] obtained duality results for a class of nondifferentiable minimax programming problems with generalized convex functions. Several authors have developed interesting results in nondifferentiable minimax fractional programming problems; see for example [26, 27, 33, 36, 37, 40] and the references therein. Recently, the concept of nondifferentiable (Φ, ρ)-invexity was used by Antczak [2] in proving the sufficient optimality conditions and various duality results for a class of nondifferentiable minimax programming problems.

Motivated by the works of Caristi et al. [12], Lai and Lee [29], Lai et al. [30], Upadhyay and Mishra [45] and Yuan et al. [50], we consider a class of nonconvex nondifferentiable generalized minimax fractional programming problems with (Φ, ρ)-invex functions. We establish the sufficient optimality conditions for such a nondifferentiable optimization problem. The necessary optimality criteria are then used to formulate two types of dual models for the considered nondifferentiable generalized minimax programming problem. Further, under (Φ, ρ)-invexity hypotheses, weak, strong and strict converse duality results are established between the considered nondifferentiable generalized minimax programming problem and its duals formulated in the paper.

2. DEFINITIONS and PRELIMINARES

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be the non-negative orthant of $\mathbb{R}^n$. Let $\emptyset \neq X_0 \subseteq \mathbb{R}^n$ and $\langle ., . \rangle$ denotes the Euclidean inner product.

We consider the following nondifferentiable generalized minimax fractional programming problem:

$$(P) \quad \inf_{x \in \mathbb{R}^n} \sup_{y \in Y} \left\{ \phi(x, y) := \frac{f(x,y) + \langle x, Ax \rangle^{1/2}}{g(x,y) - \langle x, Bx \rangle^{1/2}} \right\}$$

$$\text{subject to } h_j(x) \leq 0, j \in J := \{1, \ldots, p\},$$

where Y is a compact subset of $\mathbb{R}^m$, $f(.,.), g(.,.) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are C^1-mappings. Let A and B be $n \times n$ positive semidefinite matrices. The problem (P) is a nondifferentiable optimization problem if either A or B is nonzero. If A and B are null matrices, problem (P) is a differentiable generalized minimax fractional programming problem.

Let $X := \{x \in \mathbb{R}^n \mid h_j(x) \leq 0, j \in J\}$ be the set of all feasible solutions for (P). Assume that for each $(x, y) \in X \times Y$, $f(x, y) + \langle x, Ax \rangle^{1/2} \geq 0$ and $g(x, y) - \langle x, Bx \rangle^{1/2} > 0$. Let us define the following sets for every $x \in X$:

$$J(x) := \{j \in J | h_j(x) = 0\},$$

$$Y(x) := \left\{ y \in Y | \frac{f(x,y) + \langle x, Ax \rangle^{1/2}}{g(x,y) - \langle x, Bx \rangle^{1/2}} = \sup_{z \in Y} \frac{f(x,z) + \langle x, Ax \rangle^{1/2}}{g(x,z) - \langle x, Bx \rangle^{1/2}} \right\},$$

$$K(x) := \Bigg\{ (s,t,\bar{y}) \in N \times \mathbb{R}^s_+ \times \mathbb{R}^{ms} : 1 \leq s \leq n+1, t = (t_1, \ldots, t_s) \in \mathbb{R}^s_+$$

$$\text{with} \sum_{i=1}^{s} t_i = 1 \,\text{and}\, \bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \,\text{with}\, \bar{y}_i \in Y(x), i = 1, \ldots, s \Bigg\}.$$

Since f and g are continuously differentiable and Y is a compact subset of $\mathbb{R}^m$, it follows that, for each $x^* \in X, Y(x^*) \neq \emptyset$ and, for any $\bar{y}_i \in Y(x^*)$, we can find a positive constant k_0 such that $k_0 = \phi(x^*, \bar{y}_i)$. The following generalized Schwartz inequality will be needed in our considerations:

$$\langle x, A\nu \rangle \leq \langle x, Ax \rangle^{1/2} \langle \nu, A\nu \rangle^{1/2}, \forall\, x, \nu \in \mathbb{R}^n. \tag{1}$$

The equality holds if $Ax = \lambda A\nu$, for some $\lambda \geq 0$.
Hence, if $\langle \nu, A\nu \rangle^{1/2} \leq 1$, we have

$$\langle x, A\nu \rangle \leq \langle x, Ax \rangle^{1/2}. \tag{2}$$

Now on, we assume that an element of $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ is represented as the ordered pair (y, r) with $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Let $\Phi : X_0 \times X_0 \times \mathbb{R}^{n+1} \to \mathbb{R}$ and ρ be a real number such that $\Phi(x, u, .)$ is convex on $\mathbb{R}^{n+1}$ and $\Phi(x, u, (0, r)) \geq 0$, for all $(x, u) \in X_0 \times X_0$ and $r \geq 0$.

In order to relax the convexity assumptions, we use the following definitions from Ferrara and Stefanescu [24].

Definition 1. *A differentiable function $f : X_0 \to \mathbb{R}$ is said to be (Φ, ρ)-invex at $u \in X_0$ on X, if for all $x \in X$, we have*

$$f(x) - f(u) \geq \Phi(x, u, (\nabla f(u), \rho)).$$

The following example is taken from Upadhyay and Mishra [45]

Example 2. *Consider the function $f : D \to \mathbb{R}$, given by*

$$f(x) = (x_1 + 1)(x_2 + 2),$$

where $D = \,]-1, 1[\, \times \,]-1, 1[\,$. Define

$$\Phi(x, y, (z, \rho)) = 2(2^\rho - 1)|(x_1 - y_1)(x_2 - y_2)| + \langle z, x - y \rangle.$$

The function f is (Φ, ρ)-invex at $y = 0$ for $\rho = \frac{1}{2}$ on X, where $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 0\}$.

Definition 3. *A differentiable function $f : X_0 \to \mathbb{R}$ is said to be strictly (Φ, ρ)-invex at $u \in X_0$ on X, if for all $x \in X$, we have*

$$f(x) - f(u) > \Phi\left(x, u, (\nabla f(u), \rho)\right).$$

Remark 4. *[12] If f_1 is (Φ, ρ_1)-invex and f_2 is (Φ, ρ_2) -invex, then $\lambda f_1 + (1-\lambda) f_2$ is $(\Phi, \lambda\rho_1 + (1-\lambda)\rho_2)$-invex, whenever $\lambda \in [0, 1]$.*

Remark 5. *The notion of (Φ, ρ)-invexity generalizes and extends a number of generalized convexity notions. Indeed, from Definition 1, there are the following special cases:*

(1) If $\Phi\left(x, u, (\nabla f(u), \rho)\right) = \langle \nabla f(u), \eta(x, u)\rangle$, where $\eta : X_0 \times X_0 \to \mathbb{R}^n$, then (Φ, ρ)-invexity reduces to the definition of invex function (with respect to η) introduced by Hanson [24].
(2) If $\Phi\left(x, u, (\nabla f(u), \rho)\right) = \frac{1}{b(x,u)} \langle \nabla f(u), \eta(x, u)\rangle$, where $b : X_0 \times X_0 \to \mathbb{R}_+/0$, and $\eta : X_0 \times X_0 \to \mathbb{R}^n$, then (Φ, ρ)-invexity reduces to the definition of b-invex function (with respect to η) (see, Mishra et al. [40]).
(3) If $\Phi\left(x, u, (\nabla f(u), \rho)\right) = F(x, u, \nabla f(u))$, where $F(x, u, .)$ is a sublinear function in the third argument, then the (Φ, ρ)-invexity reduces to the definition of F-convexity introduced by Hanson and Mond [25].
(4) If $\Phi\left(x, u, (\nabla f(u), \rho)\right) = \langle \nabla f(u), (x - u)\rangle + \rho \left\|x - u\right\|^2$, then (Φ, ρ)-invexity reduces to the definition of ρ-convex function introduced by Vial [46].
(5) If $\Phi\left(x, u, (\nabla f(u), \rho)\right) = \langle \nabla f(u), \eta(x, u)\rangle + \rho \left\|d(x, u)\right\|^2$, where $\eta, d : X_0 \times X_0 \to \mathbb{R}^n$ and $d(x, y) \neq 0$, whenever $x \neq y$, then (Φ, ρ)-invexity reduces to the definition of ρ-invex function (with respect to η and d) introduced by Jeyakumar [28].
(6) If $\Phi\left(x, u, (\nabla f(u), \rho)\right) = F(x, u, \nabla f(u)) + \rho \left\|d(x, u)\right\|^2$, where $F(x, u, .)$ is a sublinear functional in the third argument, then the (Φ, ρ)-invexity reduces to the definition of (Φ, ρ)-convexity introduced by Preda [41].
(7) Let $C : X_0 \times X_0 \times \mathbb{R}^n \to \mathbb{R},\ \alpha : X_0 \times X_0 \to \mathbb{R}_+\backslash\{0\}$ and $d : X_0 \times X_0 \to \mathbb{R}_+$ be the functions such that C is convex in the third argument satisfying $C(x, a, 0) = 0$, for all $(x, a) \in X_0 \times X_0$ and $d(x, a) = 0$ if and only if $x = a$. If $\Phi(x, a, (y, \rho)) = C(x, a, y)\alpha(x, a) + \rho d(x, a)$ and ρ is a constant, then the definition of a (Φ, ρ)-invex function reduces to the definition of (C, α, ρ, d)-convexity introduced by Yuan et al. [50].

3. OPTIMALITY CONDITIONS

In this section, we establish the sufficient optimality conditions for the problem (P) under (Φ, ρ)-invexity assumptions.

Before establishing the sufficient optimality conditions, we state the following necessary optimality conditions for the problem (P) established in Lai et al. [30]

Theorem 6. *(Necessary optimality conditions) Let x^* be an optimal solution for (P) satisfying $\langle x^*, Ax^*\rangle > 0, \langle x^*, Bx^*\rangle > 0$ and $\nabla h_j(x^*), j \in J(x^*)$ are linearly* *independent. Then, there exist* $(s,t^*,\bar{y})\in K(x^*), k_0\in\mathbb{R}_+, u,\nu\in\mathbb{R}^n$ *and* $\mu^*\in\mathbb{R}^p_+$, *such that*

$$\sum_{i=1}^{s} t_i^*\left\{\nabla f(x^*,\bar{y}_i)+Au-k_0\left(\nabla g(x^*,\bar{y}_i)-B\nu\right)\right\}+\nabla\left\langle \mu^*,h(x^*)\right\rangle=0, \tag{3}$$

$$f(x^*,\bar{y}_i)+\langle x^*,Ax^*\rangle^{1/2}-k_0\left(g(x^*,\bar{y}_i)-\langle x^*,Bx^*\rangle^{1/2}\right)=0, i=1,\ldots,s, \tag{4}$$

$$\langle \mu^*,h(x^*)\rangle=0, \tag{5}$$

$$t_i^*\geq 0, i=1,\ldots,s, \text{ with } \sum_{i=1}^{s} t_i^*=1, \tag{6}$$

$$\langle u,Au\rangle\leq 1, \langle \nu,B\nu\rangle\leq 1, \tag{7}$$

$$\langle x^*,Au\rangle=\langle x^*,Ax^*\rangle^{1/2}, \langle x^*,B\nu\rangle=\langle x^*,Bx^*\rangle^{1/2}. \tag{8}$$

In the above theorem, we observe that both the matrices A and B are positive definite at the solution x^*. If one of $\langle x^*,Ax^*\rangle$ and $\langle x^*,Bx^*\rangle$ is zero or both A and B are singular at x^*, then, for $s,t^*,\bar{y}\in K(x^*)$, we can take a set $Z_{\bar{y}}(x^*)$ as defined in Lai et al. [30] by

$$Z_{\bar{y}}(x^*):=\{z\in\mathbb{R}^n|\,\langle\nabla h_j(x^*),z\rangle\leq 0, j\in J(x^*)\}$$

satisfying any one of the following three assumptions:
(i)$\langle x^*,Ax^*\rangle>0, \langle x^*,Bx^*\rangle=0$

$$\Rightarrow\left\langle \sum_{i=1}^{s} t_i^*\nabla f(x^*,\bar{y}_i)+\frac{Ax^*}{\langle Ax^*,x^*\rangle^{1/2}}-k_0\nabla g(x^*,\bar{y}_i),z\right\rangle+\left\langle (k_0^2B)z,z\right\rangle^{1/2}<0,$$

(ii)$\langle x^*,Ax^*\rangle=0, \langle x^*,Bx^*\rangle>0$

$$\Rightarrow\left\langle \sum_{i=1}^{s} t_i^*\left(\nabla f(x^*,\bar{y}_i)-k_0\left(\nabla g(x^*,\bar{y}_i)-\frac{Bx^*}{\langle Bx^*,x^*\rangle^{1/2}}\right)\right),z\right\rangle+\langle Bz,z\rangle^{1/2}<0,$$

(iii)$\langle x^*,Ax^*\rangle=0, \langle x^*,Bx^*\rangle=0$

$$\Rightarrow\left\langle \sum_{i=1}^{s} t_i^*\nabla f(x^*,\bar{y}_i)-k_0\nabla g(x^*,\bar{y}_i),z\right\rangle+\langle (k_0B)z,z\rangle^{1/2}+\langle Bz,z\rangle^{1/2}<0.$$

If we assume that $Z_{\bar{y}}(x^*)=\emptyset$ in Theorem 6, then the result of Theorem 6 still holds.

In the next theorem, we denote

$$\phi_0(.):=\sum_{i=1}^{s} t_i^*\big\{\,(f(.,\bar{y}_i)+\langle .,Au\rangle)-k_0\left(g(.,\bar{y}_i)-\langle .,B\nu\rangle\right)\big\}.$$

Theorem 7. *(Sufficient optimality conditions) Let $x^* \in X$ be a feasible solution for (P) and there exist $k_0 \in \mathbb{R}_+, (s, t^*, \bar{y}) \in K(x^*), u, \nu \in \mathbb{R}^n$ and $\mu^* \in \mathbb{R}^p_+$ satisfying (3)-(8). Further, assume that $f(., \bar{y}_i) + \langle ., Au\rangle$ and $-g(., \bar{y}_i) + \langle ., B\nu\rangle, i = 1, \ldots, s$, are (Φ, ρ_i)-invex and $(\Phi, \bar{\rho}_i)$-invex at x^* on X, respectively, and $h_j(.), j = 1, \ldots, p$, is (Φ, ρ_j^*)-invex at x^* on X such that, the inequality*

$$\sum_{i=1}^{s} t_i^*(\rho_i + k_0\bar{\rho}_i) + \sum_{j=1}^{p} \mu_j^* \rho_j^* \geq 0 \tag{9}$$

holds. Then x^* is an optimal solution for (P).

Proof. We proceed by contradiction. Suppose that x^* is not an optimal solution for (P). Then, there exists $\tilde{x} \in X$, such that

$$\sup_{y\in Y} \frac{f(\tilde{x}, y) + \langle \tilde{x}, A\tilde{x}\rangle^{1/2}}{g(\tilde{x}, y) - \langle \tilde{x}, B\tilde{x}\rangle^{1/2}} < \sup_{y\in Y} \frac{f(x^*, y) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, y) - \langle x^*, Bx^*\rangle^{1/2}}.$$

We observe that

$$\sup_{y\in Y} \frac{f(x^*, y) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, y) - \langle x^*, Bx^*\rangle^{1/2}} = \frac{f(x^*, \bar{y}_i) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, \bar{y}_i) - \langle x^*, Bx^*\rangle^{1/2}} = k_0,$$

for $\bar{y}_i \in Y(x^*), i = 1, \ldots, s$ and

$$\frac{f(\tilde{x}, \bar{y}_i) + \langle \tilde{x}, A\tilde{x}\rangle^{1/2}}{g(\tilde{x}, \bar{y}_i) - \langle \tilde{x}, B\tilde{x}\rangle^{1/2}} \leq \sup_{y\in Y} \frac{f(\tilde{x}, y) + \langle \tilde{x}, A\tilde{x}\rangle^{1/2}}{g(\tilde{x}, y) - \langle \tilde{x}, B\tilde{x}\rangle^{1/2}}.$$

Thus, we have

$$\frac{f(\tilde{x}, \bar{y}_i) + \langle \tilde{x}, A\tilde{x}\rangle^{1/2}}{g(\tilde{x}, \bar{y}_i) - \langle \tilde{x}, B\tilde{x}\rangle^{1/2}} < k_0, i = 1, \ldots, s.$$

The above inequalities yield

$$f(\tilde{x}, \bar{y}_i) + \langle \tilde{x}, A\tilde{x}\rangle^{1/2} - k_0(g(\tilde{x}, \bar{y}_i) - \langle \tilde{x}, B\tilde{x}\rangle^{1/2}) < 0, i = 1, \ldots, s. \tag{10}$$

From $(1), (4), (6), (7) - (8)$ and (10), we get

$$\begin{aligned}
\phi_0(\tilde{x}) &= \sum_{i=1}^{s} t_i^* \left\{ \left(f(\tilde{x}, \bar{y}_i) + \langle \tilde{x}, Au\rangle^{1/2} \right) - k_0 \left(g(\tilde{x}, \bar{y}_i) - \langle \tilde{x}, B\nu\rangle^{1/2} \right) \right\} \\
&\leq \sum_{i=1}^{s} t_i^* \left\{ \left(f(\tilde{x}, \bar{y}_i) + \langle \tilde{x}, A\tilde{x}\rangle^{1/2} \right) - k_0 \left((g(\tilde{x}, \bar{y}_i) - \langle \tilde{x}, B\tilde{x}\rangle^{1/2} \right) \right\} \\
&< 0 = \sum_{i=1}^{s} t_i^* \left\{ \left(f(x^*, \bar{y}_i) + \langle x^*, Ax^*\rangle^{1/2} \right) - k_0 \left(g(x^*, \bar{y}_i) - \langle x^*, Bx^*\rangle^{1/2} \right) \right\} \\
&= \sum_{i=1}^{s} t_i^* \left\{ \, (f(x^*, \bar{y}_i) + \langle x^*, Au\rangle) - k_0 \left(g(x^*, \bar{y}_i) - \langle x^*, B\nu\rangle \right) \right\} = \phi_0(x^*).
\end{aligned}$$

Thus,

$$\sum_{i=1}^{s} t_i^* \{(f(x,\bar{y}_i) + \langle x, Au\rangle) - (f(x^*,\bar{y}_i) + \langle x^*, Au\rangle)\}$$
$$+ \sum_{i=1}^{s} t_i^* k_0 \{-(g(x,\bar{y}_i) - \langle x, B\nu\rangle) + (g(x^*,\bar{y}_i) - \langle x^*, B\nu\rangle)\} < 0.$$

By (5) and $x \in X$, it follows that

$$\sum_{i=1}^{s} t_i^* \{(f(x,\bar{y}_i) + \langle x, Au\rangle) - (f(x^*,\bar{y}_i) + \langle x^*, Au\rangle)\} + \sum_{i=1}^{s} t_i^* k_0 \{-(g(x,\bar{y}_i) -$$
$$\langle x, B\nu\rangle) + (g(x^*,\bar{y}_i) - \langle x^*, B\nu\rangle)\} + \sum_{j=1}^{p} \mu_j^* \left(h_j(x) - h_j(x^*)\right) < 0. \quad (11)$$

By the (Φ, ρ_i)-invexity of $(f(.,\bar{y}_i) + \langle ., Au\rangle)$ and $(\Phi, \bar{\rho}_i)$-invexity of $(g(.,\bar{y}_i) - \langle ., B\nu\rangle)$ at x^*, for $i = 1, \ldots, s$, and for all $\bar{x} \in X$, it follows that

$$f(\tilde{x},\bar{y}_i) + \langle \tilde{x}, Au\rangle - f(x^*,\bar{y}_i) - \langle x^*, Au\rangle \geq$$
$$\Phi\left(\tilde{x}, x^*, (\nabla f(x^*,\bar{y}_i) + Au, \rho_i)\right), i = 1, \ldots, s \quad (12)$$

and

$$-g(\tilde{x},\bar{y}_i) + \langle \tilde{x}, B\nu\rangle + g(x^*,\bar{y}_i) - \langle x^*, B\nu\rangle \geq$$
$$\Phi\left(\tilde{x}, x^*, (\nabla f(x^*,\bar{y}_i) + B\nu, \bar{\rho}_i)\right), i = 1, \ldots, s. \quad (13)$$

The (Φ, ρ_j^*)-invexity of $h_j(.)$ at x^* implies that

$$h_j(\tilde{x}) - h_j(x^*) \geq \Phi\left(\tilde{x}, x^*, (\nabla h_j(x^*)\rho_j^*)\right), j = 1, \ldots, p. \quad (14)$$

By $(12) - (14)$, the convexity of $\Phi(x, x^*, .)$ on $\mathbb{R}^{n+1}$, and the fact that $\tilde{x} \in X$, it follows that

$$\Phi\Bigg(\tilde{x}, x^*, \frac{1}{\gamma}\Bigg(\sum_{i=1}^{s} t_i^*\{\nabla f(x^*, \bar{y}_i) + Au - k_0(\nabla g(x^*, \bar{y}_i) - B\nu)\} + \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*), \sum_{i=1}^{s} t_i^*(\rho_i + k_0\bar{\rho}_i) + \sum_{j=1}^{p} \mu_j^* \rho_j^*\Bigg)\Bigg)$$

$$\leq \frac{1}{\gamma}\Bigg[\sum_{i=1}^{s} t_i^* \Phi(\tilde{x}, x^*, (\nabla f(x^*, \bar{y}_i) + Au, \rho_i)) + k_0 \sum_{i=1}^{s} t_i^* \Phi\big(\tilde{x}, x^*, \big(-\nabla g(x^*, \bar{y}_i) + B\nu, \bar{\rho}_i\big)\big) + \sum_{j=1}^{p} \mu_j^* \Phi\big(\tilde{x}, x^*, (\nabla h_j(x^*), \rho_j^*)\big)\Bigg]$$

$$\leq \frac{1}{\gamma}\Bigg[\sum_{i=1}^{s} t_i^*((f(\tilde{x}, \bar{y}_i) + \langle \tilde{x}, Au\rangle) - (f(x^*, \bar{y}_i) + \langle x^*, Au\rangle)) + \sum_{i=1}^{s} t_i^* k_0 \Bigg(-\big(g(\tilde{x}, \bar{y}_i) - \langle \tilde{x}, B\nu\rangle\big) + \big(g(x^*, \bar{y}_i) - \langle x^*, B\nu\rangle\big)\Bigg) + \sum_{j=1}^{p} \mu_j^*(h_j(\tilde{x}) - h_j(x^*))\Bigg] < 0, \quad (15)$$

where

$$\gamma = 1 + k_0 + \sum_{j=1}^{p} \mu_j^*.$$

By (3), (9) and the definition of Φ, we have

$$\Phi\Bigg(\tilde{x}, x^*, \frac{1}{\gamma}\Bigg(\sum_{i=1}^{s} t_i^*\left\{\nabla f(x^*, \bar{y}_i) + Au - k_0(\nabla g(x^*, \bar{y}_i) - B\nu)\right\} + \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*), \sum_{i=1}^{s} t_i^*(\rho_i + k_0\bar{\rho}_i) + \sum_{j=1}^{p} \mu_j^* \rho_j^*\Bigg)\Bigg) \geq 0,$$

which is a contradiction to (15).
This completes the proof. □

4. FIRST DUAL MODEL

In this section, for the considered nondifferentiable generalized minimax fractional programming problem (P), we formulate the first dual model (DI). Further, under (Φ, ρ)-invexity hypotheses, we establish various duality results between (P) and (DI).

Related to (P), we formulate the first dual model (DI) as follows:

$$\text{(DI)} \qquad \max_{(s,t,\bar{y})\in K(z)} \sup_{(z,\mu,k,u,\nu)\in\Omega_1(s,t,\bar{y})} k,$$

where $\Omega_1(s,t,\bar{y})$ denotes the set of all $(z,\mu,k,u,\nu) \in \mathbb{R}^n \times \mathbb{R}^p_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\sum_{i=1}^{s} t_i \left\{\nabla f(z,\bar{y}_i) + Au - k\left(\nabla g(z,\bar{y}_i) - B\nu\right)\right\} + \nabla \langle \mu, h(z)\rangle = 0, \tag{16}$$

$$\sum_{i=1}^{s} \nabla f(z,\bar{y}_i) + \langle z, Au\rangle - k\left(g(z,\bar{y}_i) - \langle z, B\nu\rangle\right) \geq 0, \tag{17}$$

$$\langle \mu, h(z)\rangle \geq 0, \tag{18}$$

$$(s,t,\bar{y}) \in K(z), \tag{19}$$

$$\langle u, Au\rangle \leq 1, \langle \nu, B\nu\rangle \leq 1. \tag{20}$$

For a triplet $(s,t,\bar{y}) \in K(z)$, if the set $\Omega_1(s,t,\bar{y}) = \emptyset$, then we define the supremum over it to be $-\infty$ Throughout this section, we denote

$$\phi_1(.) = \sum_{i=1}^{s} t_i \left\{f(.,\bar{y}_i) + \langle ., Au\rangle - k\left((.,\bar{y}_i) - \langle ., B\nu\rangle\right)\right\}.$$

Further, let $\text{pr}_{\mathbb{R}^n}\Omega_1(s,t,\bar{y})$ denote the projection of the set $\Omega_1(s,t,\bar{y})$ on $\mathbb{R}^n$.

Now, we derive weak, strong and strict converse duality theorems.

Theorem 8. *(Weak duality) Let x and $(z,\mu,k,u,\nu,s,t,\bar{y})$ be any feasible solutions of* (P) *and* (DI)*, respectively. Further, assume that $f(.,\bar{y}) + \langle ., Au\rangle, i = 1,\dots,s$ are (Φ,ρ_i)-invex at z on $X \cup \text{pr}_{\mathbb{R}^n}\Omega_1(s,t,\bar{y})$, $-g(.,\bar{y}) + \langle ., B\nu\rangle, i = 1,\dots,s$ are $(\Phi,\bar{\rho}_i)$-invex at z on $X \cup \text{pr}_{\mathbb{R}^n}\Omega_1(s,t,\bar{y})$, $h_j(.), j = 1,...,p$ are (Φ,ρ_j^*)-invex at z on $X \cup \text{pr}_{\mathbb{R}^n}\Omega_1(s,t,\bar{y})$ and the inequality*

$$\sum_{i=1}^{s} t_i\left(\rho_i + k\bar{\rho}_i\right) + \sum_{j=1}^{p} \mu_j \rho_j^* \geq 0, \tag{21}$$

holds. Then,

$$\sup_{y\in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} \geq k.$$

Proof. We proceed by contradiction. Suppose that

$$\sup_{y\in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} < k.$$

Therefore, we get the following inequality

$$f(x,\bar{y}_i)+\langle x,Ax\rangle^{1/2}-k\left(g(x,\bar{y}_i)-\langle x,Bx\rangle^{1/2}\right)<0,\ \text{for all } \bar{y}_i\in Y.$$

From $t_i\geq 0, i=1,\ldots,s$ with $\sum_{i=1}^{s}t_i=1$, it follows that

$$t_i\left\{f(x,\bar{y}_i)+\langle x,Ax\rangle^{1/2}-k\left(g(x,\bar{y}_i)-\langle x,Bx\rangle^{1/2}\right)\right\}\leq 0, \tag{22}$$

with at least one strict inequality, because $t=(t_1,\ldots,t_s)\neq 0$.
From (1), (16), (19) and (22), we have

$$\begin{aligned}\phi_1(x)&=\sum_{i=1}^{s}t_i\left\{f(x,\bar{y}_i)+\langle x,Au\rangle-k\left(g(x,\bar{y}_i)-\langle x,B\nu\rangle\right)\right\}\\ &\leq\sum_{i=1}^{s}t_i\left\{f(x,\bar{y}_i)+\langle x,Ax\rangle^{1/2}-k\left(g(x,\bar{y}_i)-\langle x,Bx\rangle^{1/2}\right)\right\}\\ &<0\leq\sum_{i=1}^{s}t_i\left\{f(z,\bar{y}_i)+\langle z,Au\rangle-k\left(g(z,\bar{y}_i)-\langle z,B\nu\rangle\right)\right\}=\phi_1(z).\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{s}t_i\{(f(x,\bar{y}_i)+\langle x,Au\rangle)-(f(z,\bar{y}_i)+\langle z,Au\rangle)\}\\ &\qquad+\sum_{i=1}^{s}t_ik\{-(g(x,\bar{y}_i)-\langle x,B\nu\rangle)+(g(z,\bar{y}_i)-\langle z,B\nu\rangle)\}<0.\end{aligned}$$

By the feasibility of x for (P) and (18), we get

$$\begin{aligned}&\sum_{i=1}^{s}t_i\{(f(x,\bar{y}_i)+\langle x,Au\rangle)-(f(z,\bar{y}_i)+\langle z,Au\rangle)\}+\sum_{i=1}^{s}t_ik\{-(g(x,\bar{y}_i)-\langle x,B\nu\rangle)\\ &\qquad+(g(z,\bar{y}_i)-\langle z,B\nu\rangle)\}+\sum_{j=1}^{p}\mu_j(h_j(x)-h_j(z))<0.\end{aligned} \tag{23}$$

From (23) and convexity of $\Phi(x,z,.)$ on $\mathbb{R}^{n+1}$, it follows that

$$\Phi\Bigg(x,z,\frac{1}{\gamma}\Bigg(\sum_{i=1}^{s}t_i\{\nabla f(z,\bar{y}_i)+Au-k(\nabla f(z,\bar{y}_i)-B\nu)\}+\sum_{j=1}^{p}\mu_j\nabla h_j(z),$$
$$\sum_{i=1}^{s}t_i(\rho_i+k\bar{\rho}_i)+\sum_{j=1}^{p}\mu_j\rho_j^*\Bigg)\Bigg)$$
$$\leq\frac{1}{\gamma}\Bigg(\sum_{i=1}^{s}t_i\Phi(x,z,(\nabla f(z,\bar{y}_i)+Au,\rho_i))+k\sum_{i=1}^{s}t_i\Phi(x,z,(-\nabla g(z,\bar{y}_i)+B\nu,\bar{\rho}_i))$$
$$+\sum_{j=1}^{p}\mu_j\Phi(x,z,(\nabla h_j(z),\rho_j^*))\Bigg)$$
$$\leq\frac{1}{\gamma}\Bigg[\sum_{i=1}^{s}t_i((f(x,\bar{y}_i)+\langle x,Au\rangle)-(f(z,\bar{y}_i)+\langle z,Au\rangle))+\sum_{i=1}^{s}t_ik\Bigg(-\big(g(x,\bar{y}_i)$$
$$-\langle x,B\nu\rangle\big)+\big(g(z,\bar{y}_i)-\langle z,B\nu\rangle\big)\Bigg)+\sum_{j=1}^{p}\mu_j^*(h_j(x)-h_j(z))\Bigg]<0, \tag{24}$$

where

$$\gamma=1+k+\sum_{j=1}^{p}\mu_j.$$

By (3), (9) and the definition of Φ we get

$$\Phi\Bigg(x,z,\frac{1}{\gamma}\Bigg(\sum_{i=1}^{s}t_i\{\nabla f(z,\bar{y}_i)+Au-k(\nabla f(z,\bar{y}_i)-B\nu)\}+\sum_{j=1}^{p}\mu_j\nabla h_j(z),$$
$$\sum_{i=1}^{s}t_i(\rho_i+k\bar{\rho}_i)+\sum_{j=1}^{p}\mu_j\rho_j^*\Bigg)\Bigg)\geq 0,$$

which is a contradiction to (24). This completes the proof. □

Theorem 9. *(Strong duality) Assume that x^* is an optimal solution for (P) and that $\nabla h_j(x^*), j\in J(x^*)$, are linearly independent. Then, there exist $(\bar{s},\bar{t},\bar{y}^*)\in K(x^*)$ and $(x^*,\bar{\mu},\bar{k},\bar{u},\bar{\nu})\in\Omega_1(\bar{s},\bar{t},\bar{y}^*)$ such that $(x^*,\bar{\mu},\bar{k},\bar{u},\bar{\nu},\bar{s},\bar{t},\bar{y}^*)$ is a feasible solution for (DI). Further, if all hypotheses of weak duality (Theorem 8) are fulfilled, then $(x^*,\bar{\mu},\bar{k},\bar{u},\bar{\nu},\bar{s},\bar{t},\bar{y}^*)$ is optimal in dual problem (DI) and objective functions in problems (P) and (DI) have the same values.*

Proof. Since x^* is an optimal solution in the nondifferentiable minimax fractional programming problem (P), by Theorem 6, there exist $(\bar{s},\bar{t},\bar{y}^*)\in K(x^*)$ and $(x^*,\bar{\mu},\bar{k},\bar{u},\bar{\nu})\in\Omega_1(\bar{s},\bar{t},\bar{y}^*)$ such that $(x^*,\bar{\mu},\bar{k},\bar{u},\bar{\nu},\bar{s},\bar{t},\bar{y}^*)$ is feasible for (DI) and

$$\bar{k}=\frac{f(x^*,y_i^*)+\langle x^*,Ax^*\rangle^{1/2}}{g(x^*,y_i^*)-\langle x^*,Bx^*\rangle^{1/2}}.$$

Thus, if all the hypotheses of the weak duality (Theorem 8) are fulfilled, then the optimality of $(x^*, \bar{\mu}, \bar{k}, \bar{u}, \bar{\nu}, \bar{s}, \bar{t}, \bar{y}^*)$ follows directly from this theorem. □

Theorem 10. *(Strict converse duality) Let x^* and $(\bar{z}, \bar{\mu}, \bar{k}, \bar{u}, \bar{\nu}, \bar{s}, \bar{t}, \bar{y}^*)$ be the optimal solutions for* (P) *and* (DI)*, respectively, and $\nabla h_j(x^*), j \in J(x^*)$ be linearly independent. Assume that $f(., y_i^*) + \langle ., A\bar{u}\rangle$ and $-g(., y_i^*) + \langle ., B\bar{\nu}\rangle$, $i = 1, \ldots, s$, are strictly (Φ, ρ_i)-invex and strictly $(\Phi, \bar{\rho}_i)$-invex functions at $\bar{z}$ on $X \cup \mathrm{pr}_{\mathbb{R}^n} \Omega_1(\bar{s}, \bar{t}, \bar{y}^*)$ respectively and $h_j(.), j = 1, \ldots, p$, is (Φ, ρ_j^*)-invex at $\bar{z}$ on $X \cup \mathrm{pr}_{\mathbb{R}^n} \Omega_1(\bar{s}, \bar{t}, \bar{y}^*)$ for all $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(\bar{z}, \bar{\mu}, \bar{k}, \bar{u}, \bar{\nu}) \in \Omega_1(\bar{s}, \bar{t}, \bar{y}^*)$, Further, assume that the inequality*

$$\sum_{i=1}^{s} \bar{t}_i(\rho_i + \bar{k}\bar{\rho}_i) + \sum_{j=1}^{p} \bar{\mu}_j \rho_j^* \geq 0 \tag{25}$$

holds. Then, $x^* = \bar{z}$, that is, $\bar{z}$ is optimal for (P) *and*

$$\sup_{y \in Y} \frac{f(\bar{z}, \bar{y}^*) + \langle \bar{z}, A\bar{z}\rangle^{1/2}}{g(\bar{z}, \bar{y}^*) - \langle \bar{z}, B\bar{z}\rangle^{1/2}} = \bar{k}.$$

Proof. Suppose, contrary to the result, that $x^* \neq \bar{z}$. By the hypotheses of the theorem and following along the lines of the proof of Theorem 8, we have

$$\begin{aligned}
0 \leq \Phi\Bigg(x^*, \bar{z}, \frac{1}{\gamma}\Bigg(&\sum_{i=1}^{s} \bar{t}_i\big\{\nabla f(\bar{z}, \bar{y}_i^*) + A\bar{u} - \bar{k}(\nabla g(\bar{z}, \bar{y}_i^*) - B\bar{\nu})\big\} + \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(\bar{z}), \\
&\sum_{i=1}^{s} \bar{t}_i(\rho_i + \bar{k}\bar{\rho}_i) + \sum_{j=1}^{p} \bar{\mu}_j \rho_j^*\Bigg)\Bigg) \\
\leq \frac{1}{\gamma}\Bigg(&\sum_{i=1}^{s} \bar{t}_i \Phi\big(x^*, \bar{z}, (\nabla f(\bar{z}, \bar{y}_i^*) + A\bar{u}, \rho_i)\big) + \bar{k}\sum_{i=1}^{s} \bar{t}_i \Phi\big(x^*, \bar{z}, \big(-\nabla g(\bar{z}, \bar{y}_i^*) \\
&+ B\bar{\nu}, \bar{\rho}_i\big)\big) + \sum_{j=1}^{p} \bar{\mu}_j \Phi(x^*, \bar{z}, (\nabla h_j(\bar{z}), \rho_j^*))\Bigg) \\
< \frac{1}{\gamma}\Bigg[&\sum_{i=1}^{s} \bar{t}_i\big\{(f(x^*, \bar{y}_i^*) + \langle x^*, A\bar{u}\rangle) - (f(\bar{z}, \bar{y}_i^*) + \langle \bar{z}, A\bar{u}\rangle)\big\} + \sum_{i=1}^{s} \bar{t}_i \bar{k} \\
&\big\{-(g(x^*, \bar{y}_i^*) - \langle x^*, B\bar{\nu}\rangle) + (g(\bar{z}, \bar{y}_i^*) - \langle \bar{z}, B\bar{\nu}\rangle)\big\} + \sum_{j=1}^{p} \bar{\mu}_j (h_j(x^*) - h_j(\bar{z}))\Bigg],
\end{aligned} \tag{26}$$

where

$$\gamma = 1 + \bar{k} + \sum_{j=1}^{p} \bar{\mu}_j.$$

By the feasibility of x^* and (18), we get

$$\sum_{j=1}^{p} \bar{\mu}_j \left[h_j(x^*) - h_j(\bar{z})\right] \leq 0. \tag{27}$$

From (26) and (27), we have

$$\sum_{i=1}^{s} \bar{t}_i \{(f(x^*, \bar{y}_i^*) + \langle x^*, A\bar{u}\rangle) - (f(\bar{z}, \bar{y}_i^*) + \langle \bar{z}, A\bar{u}\rangle)\} + \sum_{i=1}^{s} \bar{t}_i \bar{k} \{-(g(x^*, \bar{y}_i^*) - \langle x^*, B\bar{\nu}\rangle) + (g(\bar{z}, \bar{y}_i^*) - \langle \bar{z}, B\bar{\nu}\rangle)\} > 0.$$

By (16) and (2), the above inequality implies that

$$\sum_{i=1}^{s} \bar{t}_i \left\{ \left(f(x^*, \bar{y}_i^*) + \langle x^*, Ax^*\rangle^{1/2} \right) - \bar{k} \left(g(x^*, \bar{y}_i^*) + \langle x^*, Bx^*\rangle^{1/2} \right) \right\}$$
$$> \sum_{i=1}^{s} \bar{t}_i \left\{ \left(f(\bar{z}, \bar{y}_i^*) + \langle \bar{z}, A\bar{z}\rangle^{1/2} \right) - \bar{k} \left(g(\bar{z}, \bar{y}_i^*) + \langle \bar{z}, B\bar{z}\rangle^{1/2} \right) \right\} \geq 0.$$

From the above inequality, we conclude that there exists a certain $i_0 \in \{1, \ldots, s\}$, such that

$$\left\{ \left(f(x^*, \bar{y}_{i_0}^*) + \langle x^*, Ax^*\rangle^{1/2} \right) - \bar{k} \left(g(x^*, \bar{y}_{i_0}^*) + \langle x^*, Bx^*\rangle^{1/2} \right) \right\} > 0.$$

It follows that

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + \langle x^*, Ax^*\rangle^{1/2}}{g(x, \bar{y}^*) - \langle x^*, Bx^*\rangle^{1/2}} \geq \frac{f(x^*, \bar{y}_{i_0}^*) + \langle x^*, Ax^*\rangle^{1/2}}{g(x, \bar{y}^*) - \langle x^*, Bx^*\rangle^{1/2}} > \bar{k}. \tag{28}$$

By the strong duality theorem (Theorem 9), it follows that

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + \langle x^*, Ax^*\rangle^{1/2}}{g(x, \bar{y}^*) - \langle x^*, Bx^*\rangle^{1/2}} = \bar{k},$$

which contradicts (28). Therefore, $x^* = \bar{z}$. Hence, the proof is complete. □

5. SECOND DUAL MODEL

In this section, for the considered nondifferentiable generalized minimax fractional programming problem (P), we formulate the second dual model (DII). Further, under (Φ, ρ) -invexity hypotheses, we establish various duality results between (P) and (DII).

We state the modified version of Theorem 6, by replacing the parameter k_0 with $\frac{f(x^*,\bar{y}^*)+\langle x^*,Ax^*\rangle^{1/2}}{g(x,\bar{y}^*)-\langle x^*,Bx^*\rangle^{1/2}}$ and by rewriting the multiplier functions associated with the inequality constraints.

Theorem 11. *Let x^* be an optimal solution for (P) and let $\nabla h_j(x^*), j \in J(x^*)$ be linearly independent. Then, there exist $(\bar{s}, \bar{t}, \bar{y}) \in K(x^*)$ and $\bar{\mu} \in \mathbb{R}^p_+$, such that*

$$\sum_{i=1}^{\bar{s}} \bar{t}_i (g(x^*, \bar{y}_i) - \langle x^*, Bx^* \rangle^{1/2})(\nabla f(x^*, \bar{y}_i) + Au) - (f(x^*, \bar{y}_i) +$$

$$\langle x^*, Ax^* \rangle^{1/2})(\nabla g(x^*, \bar{y}_i) - B\nu) + \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(x^*) = 0,$$

$$\sum_{i=1}^{p} \bar{\mu}_j \nabla h_j(x^*) \geq 0,$$

$$\bar{\mu} \in \mathbb{R}^p_+, \bar{t}_i \geq 0, \sum_{i=1}^{\bar{s}} \bar{t}_i = 1, \bar{y}_i \in Y(\bar{x}), i = 1, \ldots, \bar{s}.$$

We formulate the second dual model as follows:

$$\text{(DII)} \qquad \max_{(s,t,\bar{y}) \in K(z)} \sup_{(z,\mu,u,\nu) \in \Omega_2(s,t,\bar{y})} F(z),$$

where $F(z) = \sup\limits_{y \in Y} \frac{f(z,y) + \langle z, Az \rangle^{1/2}}{g(z,y) - \langle z, Bz \rangle^{1/2}}$ and $\Omega_2(s, t, \bar{y})$ denote the set of all $(z, \mu, u, \nu) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\sum_{i=1}^{s} t_i \Big\{ (g(z, \bar{y}_i) - \langle z, Bz \rangle^{1/2})(\nabla f(z, \bar{y}_i) + Au) -$$

$$(f(z, \bar{y}_i) + \langle z, Az \rangle^{1/2})(\nabla g(z, \bar{y}_i) - B\nu) \Big\} + \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(z) = 0, \tag{29}$$

$$\sum_{j=1}^{p} \mu_j h_j(z) \geq 0, \tag{30}$$

$$(s, t, \bar{y}) \in K(z), \tag{31}$$

$$\langle z, Az \rangle^{1/2} = \langle z, Au \rangle, \langle z, Bz \rangle^{1/2} = \langle z, B\nu \rangle, \langle u, Au \rangle \leq 1, \langle \nu, B\nu \rangle \leq 1 \tag{32}$$

For a triplet $(s, t, \bar{y}) \in K(z)$, if the set $\Omega_2(s, t, \bar{y})$ is empty, then we define the supremum over it to be $-\infty$ Throughout this section, we denote

$$\phi_2(.) = \sum_{i=1}^{s} t_i \Big\{ (g(z, \bar{y}_i) - \langle z, B\nu \rangle)(f(., \bar{y}_i) + \langle ., Au \rangle) -$$

$$(f(z, \bar{y}_i) + \langle z, Au \rangle)(\nabla g(., \bar{y}_i) - \langle ., B\nu \rangle) \Big\}.$$

Further, let $\text{pr}_{\mathbb{R}^n} \Omega_2(s, t, \bar{y})$ denote the projection of the set on $\mathbb{R}^n$.

Now, we establish weak, strong and strict converse duality theorems.

Theorem 12. *(Weak duality) Let x and $(z, \mu, u, \nu, s, t, \bar{y})$ be the feasible solutions for* (P) *and* (DII)*, respectively. Further, assume that $f(., \bar{y}_i) + \langle ., Au \rangle$, $i = 1, \dots, s$ are (Φ, ρ_i)-invex at z on $X \cup \mathrm{pr}_{\mathbb{R}^n} \Omega_2(s, t, \bar{y})$, $-g(., \bar{y}_i) + \langle ., B\nu \rangle$, $i = 1, \dots, s$, are $(\Phi, \bar{\rho}_i)$-invex at z on $X \cup \mathrm{pr}_{\mathbb{R}^n} \Omega_2(s, t, \bar{y}), h_j(.), j = 1, \dots, p$ are (Φ, ρ_j^*)-invex at z on $X \cup \mathrm{pr}_{\mathbb{R}^n} \Omega_2(s, t, \bar{y})$ and the inequality*

$$\sum_{i=1}^{s} t_i \left\{ \left(g(z, \bar{y}_i) - \langle z, Bz \rangle^{1/2} \right) \rho_i + \left(\left(f(z, \bar{y}_i) + \langle z, Bz \rangle^{1/2} \right) \bar{\rho}_i \right) \right\} + \sum_{j=1}^{p} \mu_j \rho_j^* \geq 0, \tag{33}$$

holds. Then,

$$\sup_{y \in Y} \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}} \geq F(z).$$

Proof. Suppose, contrary to the result, that

$$\sup_{y \in Y} \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}} < F(z). \tag{34}$$

Since, $\bar{y}_i \in Y(z), i = 1, \dots, s$ we get

$$F(z) = \sup_{y \in Y} \frac{f(z, \bar{y}_i) + \langle z, Az \rangle^{1/2}}{g(z, \bar{y}_i) - \langle z, Bz \rangle^{1/2}}. \tag{35}$$

By (34) and (35), we get

$$\left(g(z, y_i) - \langle z, Bz \rangle^{1/2} \right) \left(f(x, \bar{y}_i) + \langle x, Ax \rangle^{1/2} \right) - \left(f(z, \bar{y}_i) + \langle z, Az \rangle^{1/2} \right) \left(g(z, \bar{y}_i) - \langle x, Bx \rangle^{1/2} \right) < 0,$$

for all $i = 1, \dots, s$ and $\bar{y}_i \in Y$.
From $\bar{y}_i \in Y(z) \subset Y$ and $t \in \mathbb{R}_+^s$ with $\sum_{i=1}^{s} t_i = 1$, it follows that

$$\sum_{i=1}^{s} t_i \left\{ \left(g(z, \bar{y}_i) - \langle z, Bz \rangle^{1/2}) \right) \left(f(x, \bar{y}_i) + \langle x, A \rangle^{1/2} \right) - \left(f(z, \bar{y}_i) + \langle z, Az \rangle^{1/2} \right) \left(g(z, \bar{y}_i) - \langle x, Bx \rangle^{1/2} \right) \right\} < 0. \tag{36}$$

From (1), (32) and (36), we have

$$\phi_2(x) = \sum_{i=1}^{s} t_i \Big\{ (g(z,\bar{y}_i) - \langle z, B\nu\rangle)(f(x,\bar{y}_i) + \langle x, Au\rangle) - (f(z,\bar{y}_i) + \langle z, Au\rangle)(g(z,\bar{y}_i) - \langle x, Bx\rangle) \Big\}$$
$$\leq \sum_{i=1}^{s} t_i \Big\{ (g(z,\bar{y}_i) - \langle z, Bz\rangle^{1/2})(f(x,\bar{y}_i) + \langle x, Ax\rangle^{1/2}) - (f(z,\bar{y}_i) + \langle z, Az\rangle^{1/2})(g(x,\bar{y}_i) - \langle x, Bx\rangle^{1/2}) \Big\} < 0 = \phi_2(z).$$

Hence,

$$\phi_2(x) < \phi_2(z).$$

By the feasibility of x for (P) and (30), we get

$$(\phi_2(x) - \phi_2(z)) + \sum_{j=1}^{p} \mu_j \left(h_j(x) - h_j(z)\right) < 0. \tag{37}$$

From the (Φ, ρ_i)-invexity of $(f(.,\bar{y}_i) + \langle ., Au\rangle)$ and $(\Phi, \bar{\rho}_i)$-invexity of $(-g(.,\bar{y}_i) + \langle ., B\nu\rangle)$ at z, for $i = 1, \ldots, s$, it follows that

$$f(x,\bar{y}_i) + \langle x, Au\rangle - f(z,\bar{y}_i) - \langle z, Au\rangle \geq \Phi(x, z, (\nabla f(z,\bar{y}_i) + Au, \rho_i)), i = 1, \ldots, s, \tag{38}$$

and

$$-g(x,\bar{y}_i) + \langle x, B\nu\rangle + g(z,\bar{y}_i) - \langle z, B\nu\rangle \geq \Phi(x, z, (-\nabla g(z,\bar{y}_i) + B\nu, \bar{\rho}_i)), i = 1, \ldots, s. \tag{39}$$

The (Φ, ρ_j^*)-invexity of $h_j(.)$ at z implies that

$$h_j(x) - h_j(z) \geq \Phi(x, z, (\nabla h_j(z), \rho_j^*)), j = 1, \ldots, p. \tag{40}$$

Multiplying (38) by $t_i \left(g(z,\bar{y}_i) - \langle z, Bz\rangle^{1/2}\right)$, (39) by $t_i \left(f(z,\bar{y}_i) - \langle z, Bz\rangle^{1/2}\right)$, and then summing up these inequalities to (40) along with the convexity of $\Phi(x, z, (.))$

on $\mathbb{R}^{n+1}$, we get

$$\frac{1}{\gamma}(\phi_2(x) - \phi_2(z) + \sum_{j=1}^{p} \mu_j h_j(x) - \sum_{j=1}^{p} \mu_j h_j(z))$$

$$\geq \frac{1}{\gamma} \sum_{i=1}^{s} t_i \Big\{ \Big(\Phi(x, z(\nabla f(z, \bar{y}_i) + Au, \rho_i)) \Big) (g(z, \bar{y}_i) \langle z, Bz \rangle^{1/2})$$

$$+ \Big(\Phi(x, z, (\nabla g(z) - B\nu, \bar{\rho}_i) \Big) \Big(f(z, \bar{y}_i) - \langle z, Bz \rangle^{1/2} \Big) \Big\}$$

$$+ \frac{1}{\gamma} \sum_{j=1}^{p} \mu_j \Phi(x, z, (\nabla h_j(z), \bar{\rho}_j^*))$$

$$\geq \Phi \Big(x, z, \frac{1}{\gamma} \Big(\sum_{i=1}^{s} t_i \Big(\Big(\nabla f(z, \bar{y}_i) + Au \Big) (g(z, \bar{y}_i) - \langle z, Bz \rangle^{1/2})$$

$$- \big(f(z, \bar{y}_i) + \langle z, Az \rangle^{1/2} \big) \big(\nabla g(z, \bar{y}_i) - B\nu \big) \Big) + \sum_{j=1}^{p} \mu_j \nabla h_j(z) \Big), \frac{1}{\gamma} \Big(\rho_i \big(f(z, \bar{y}_i)$$

$$- \langle z, Bz \rangle^{1/2} \big) + \bar{\rho}_i \big(f(z, \bar{y}_i) + \langle z, Az \rangle^{1/2} \big) \Big) + \sum_{j=1}^{p} \bar{\mu}_j \rho_j^* \Big) \Big) \geq 0, \tag{41}$$

where

$$\gamma = \sum_{i=1}^{s} \bar{t}_i \left\{ \Big(g(z, \bar{y}_i) - \langle z, Bz \rangle^{1/2} \Big) + \Big(f(z, \bar{y}_i) + \langle z, Bz \rangle^{1/2} \Big) \right\} + \sum_{j=1}^{p} \bar{\mu}_j.$$

which is a contradiction to (37). This completes the proof. □

Theorem 13. *(Strong duality) Let x^* be an optimal solution for (P) and $\nabla h_j(x^*)$, $j \in J(x^*)$ be linearly independent. Then, there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{u}, \bar{\nu}) \in \Omega_2(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{u}, \bar{\nu}, \bar{s}, \bar{t}, \bar{y}^*)$ is a feasible solution for (DII). Further, if all the hypotheses of the weak duality (Theorem 12) are fulfilled, then, $(x^*, \bar{\mu}, \bar{u}, \bar{\nu}, \bar{s}, \bar{t}, \bar{y}^*)$ is optimal in dual problem (DII) and objective functions in problems (P) and (DII) have the same values.*

Proof. Since x^* is an optimal solution in the considered nondifferentiable minimax fractional programming problem (P), then by Theorem 6 there exist $\bar{u}, \bar{\nu} \in \mathbb{R}^n$ and $\bar{\mu} \in \mathbb{R}^p_+$ to satisfy the expression (29) obtained by substituting

$$k_0 = \frac{f(x^*, \bar{y}_i^*) + \langle x^*, Ax^* \rangle^{1/2}}{g(x, \bar{y}_i^*) - \langle x^*, Bx^* \rangle^{1/2}},$$

in (3). It follows that there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{u}, \bar{\nu}) \in \Omega_2(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{u}, \bar{\nu}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (DII). Thus, if all the hypotheses of the weak duality (Theorem 12) are fulfilled, then, the optimality of $(x^*, \bar{\mu}, \bar{u}, \bar{\nu}, \bar{s}, \bar{t}, \bar{y}^*)$ follows directly from this theorem. □

Theorem 14. *(Strict converse duality) Let x^* and $(\bar{z}, \bar{\mu}, \bar{k}\bar{u}, \bar{\nu}, \bar{s}, \bar{t}, \bar{y}^*)$ be the optimal solutions for (P) and (DII), respectively and let $\nabla h_j(x^*), j \in J(x^*)$ be linearly independent. Assume that $f(., \bar{y}_i^*) + \langle ., A\bar{u}\rangle$ and $-g(., \bar{y}_i^*) + \langle ., B\bar{\nu}\rangle$, for $i = 1, \ldots, s$ are strictly (Φ, ρ_i)-invex and strictly $(\Phi, \bar{\rho}_i)$-invex at $\bar{z}$ on $X \cup \mathrm{pr}_{\mathbb{R}^n}\Omega_2(\bar{s}, \bar{t}, \bar{y}^*)$, respectively and let $h_j(.)$ for $j = 1, \ldots, p$ be (Φ, ρ_j^*)-invex at $\bar{z}$ on $X \cup \mathrm{pr}_{\mathbb{R}^n}\Omega_2(\bar{s}, \bar{t}, \bar{y}^*)$, for all $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(\bar{z}, \bar{\mu}, \bar{k}\bar{u}, \bar{\nu}) \in \Omega_2(\bar{s}, \bar{t}, \bar{y}^*)$. Further, assume that the inequality*

$$\sum_{i=1}^{s} \bar{t}_i \left(\left(g(\bar{z}, \bar{y}_i) - \langle \bar{z}, B\bar{z}\rangle^{1/2} \right) \rho_i + \left(f(\bar{z}, \bar{y}_i) + \langle \bar{z}, B\bar{z}\rangle^{1/2} \right) \bar{\rho}_i \right) + \sum_{j=1}^{p} \bar{\mu}_j \rho_j^* \geq 0, \quad (42)$$

holds. Then, $x^* = \bar{z}$, that is, $\bar{z}$ is optimal for (P) and

$$\sup_{y \in Y} \frac{f(\bar{z}, \bar{y}^*) + \langle \bar{z}, A\bar{z}\rangle^{1/2}}{g(\bar{z}, \bar{y}^*) - \langle \bar{z}, B\bar{z}\rangle^{1/2}} = F(\bar{z}).$$

Proof. Suppose, contrary to the result, that $x^* \neq z$. Using the hypotheses of the theorem and proceeding as in Theorem 12, we get

$$\frac{1}{\gamma} \left(\phi_2(x^*) - \phi_2(\bar{z}) + \sum_{j=1}^{p} \bar{\mu}_j h_j(x^*) - \sum_{j=1}^{p} \bar{\mu}_j h_j(\bar{z}) \right)$$

$$> \frac{1}{\gamma} \sum_{i=1}^{s} \bar{t}_i \Big\{ \left(\Phi\left(x^*, \bar{z}, (\nabla f(\bar{z}, \bar{y}_i^*) + A\bar{u}, \rho_i)\right) \right) \left(g(\bar{z}, \bar{y}_i^*) - \langle \bar{z}, B\bar{z}\rangle^{1/2} \right)$$

$$+ \left(\Phi\left(x^*, \bar{z}, (\nabla g(\bar{z}) + B\bar{\nu}, \bar{\rho}_i)\right) \right) \left(f(\bar{z}, \bar{y}_i^*) + \langle \bar{z}, A\bar{z}\rangle^{1/2} \right) \Big\} + \frac{1}{\gamma} \sum_{j=1}^{p} \bar{\mu}_j$$

$$\Phi\left(x^*, \bar{z}, \left(\nabla h_j(\bar{z}), \rho_j^*\right)\right)$$

$$\geq \Phi\Bigg(x^*, \bar{z}, \frac{1}{\gamma} \Bigg(\sum_{i=1}^{s} \bar{t}_i \Bigg(\left(\nabla f(\bar{z}, \bar{y}_i^*) + A\bar{u}\right) \left(g(\bar{z}, \bar{y}_i^*) - \langle \bar{z}, B\bar{z}\rangle^{1/2} \right)$$

$$- \left(f(\bar{z}, \bar{y}_i^*) + \langle \bar{z}, A\bar{z}\rangle^{1/2} \right) \left(\nabla g(\bar{z}, \bar{y}_i^*) - B\bar{\nu} \right) \Bigg) + \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(\bar{z}) \Bigg),$$

$$\rho_i \left(g(\bar{z}, \bar{y}_i^*) - \langle \bar{z}, B\bar{z}\rangle^{1/2} \right) + \bar{\rho}_i \left(f(\bar{z}, \bar{y}_i^*) + \langle \bar{z}, A\bar{z}\rangle^{1/2} \right) + \sum_{j=1}^{p} \bar{\mu}_j \rho_j^* \Bigg) \Bigg) \geq 0,$$

where

$$\gamma = \sum_{i=1}^{s} \bar{t}_i \left\{ \left(g(\bar{z}, \bar{y}_i) - \langle \bar{z}, B\bar{z}\rangle^{1/2} \right) + \left(f(\bar{z}, \bar{y}_i) + \langle \bar{z}, B\bar{z}\rangle^{1/2} \right) \right\} + \sum_{j=1}^{p} \bar{\mu}_j.$$

That is,

$$\frac{1}{\gamma}\left(\phi_2(x^*) - \phi_2(\bar{z}) + \sum_{j=1}^{p}\bar{\mu}_j h_j(x^*) - \sum_{j=1}^{p}\bar{\mu}_j h_j(\bar{z})\right) > 0. \tag{43}$$

By the feasibility of x^* and (30), we get

$$\sum_{j=1}^{p}\bar{\mu}_j\left[h_j(x^*) - h_j(\bar{z})\right] \leq 0. \tag{44}$$

From (43) and (44), we have

$$\phi_2(x^*) - \phi_2(\bar{z}) > 0.$$

That is,

$$\sum_{i=1}^{\bar{s}}\bar{t}_i\Big\{(g(\bar{z},\bar{y}_i^*) - \langle\bar{z},B\bar{\nu}\rangle(f(x^*,\bar{y}_i^*)+\langle x^*,A\bar{u}\rangle) - (f(\bar{z},\bar{y}_i^*)+ \langle\bar{z},A\bar{u}\rangle)(g(x^*,\bar{y}_i^*) - \langle x^*,B\bar{\nu}\rangle)\Big\}$$
$$> \sum_{i=1}^{\bar{s}}\bar{t}_i\Big\{(g(\bar{z},\bar{y}_i^*) - \langle\bar{z},B\bar{\nu}\rangle)(f(\bar{z},\bar{y}_i^*)+\langle\bar{z},A\bar{u}\rangle) - (f(\bar{z},\bar{y}_i^*)+ \langle\bar{z},A\bar{u}\rangle)(g(\bar{z},\bar{y}_i^*) - \langle\bar{z},B\bar{\nu}\rangle)\Big\} \geq 0.$$

Therefore, there exists a certain i_0, such that

$$\sum_{i=1}^{\bar{s}}\bar{t}_i\Big\{(g(\bar{z},\bar{y}_{i_0}^*) - \langle\bar{z},B\bar{\nu}\rangle)(f(x^*,\bar{y}_{i_0}^*) + \langle x^*,A\bar{u}\rangle) - (f(\bar{z},\bar{y}_{i_0}^*)+\langle\bar{z},A\bar{u}\rangle)(g(x^*,\bar{y}_{i_0}^*) - \langle x^*,B\bar{\nu}\rangle)\Big\} > 0.$$

From the above inequality and (32), it follows that

$$\sup_{y\in Y}\frac{f(x^*,\bar{y}^*) + \langle x^*,Ax^*\rangle^{1/2}}{g(x^*,\bar{y}^*) - \langle x^*,Bx^*\rangle^{1/2}} \geq \frac{f(x^*,\bar{y}_{i_0}^*) + \langle x^*,Ax^*\rangle^{1/2}}{g(x^*,\bar{y}_{i_0}^*) - \langle x^*,Bx^*\rangle^{1/2}} > F(\bar{z}). \tag{45}$$

By the strong duality theorem (Theorem 13), it follows that

$$\sup_{y\in Y}\frac{f(x^*,\bar{y}^*) + \langle x^*,Ax^*\rangle^{1/2}}{g(x^*,\bar{y}^*) - \langle x^*,Bx^*\rangle^{1/2}} = F(\bar{z}). \tag{46}$$

Thus, inequality (46), contradicts (45). Therefore, $x^* = z$.
Hence, the proof is complete. □

The follwing example illustrates the significance of Theorems 6-14.

Example 15. *Consider the following nondifferentiable generalized minimax fractional programming problem:*

$$(P1) \qquad \inf_{x\in\mathbb{R}} \sup_{y\in Y} \left\{ \phi(x,y) := \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} \right\}$$
$$\text{subject to} \quad h(x) \leq 0,$$

where $Y = [0,1]$, $A = B = 1$. *The functions* $f, g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *and* $h : \mathbb{R} \to \mathbb{R}$ *are defined as*

$$f(x,y) = x^2 + x + y^2, \quad g(x,y) = 2x + 1,$$

and

$$h(x) = 1 - x.$$

Note that the set of all feasible solutions is $X = \{x \in \mathbb{R} : 1 - x \leq 0\} = [1, \infty)$. *The sets* $Y(x)$ *and* $K(x)$ *are given as*

$$Y(x) = \{1\} \text{ and } K(x) = \{(1,1,1)\}.$$

Now, for all $(x,y) \in X \times Y = \{(x,y) : x \geq 1,\ 0 \leq y \leq 1\}$, *we have*

$$f(x,y) + \langle x, Ax\rangle^{1/2} = x^2 + x + y^2 + |x| > 0$$

and

$$g(x,y) - \langle x, Bx\rangle = 2x + 1 - |x| > 0.$$

Now, we will find a minimax solution of (P) for $x^* \in [1,\infty)$, *we have considered the following cases.*

Case 1 *Let* $x^* = 1$. *From* (3)-(8), *we get* $x^* = 1$, $u = 1$, $v = 1$, $k_0 = 2$ *and* $\mu^* = 2$. *Hence, necessary optimality conditions (Theorem 6) are satisfied at* x^*.

Case 2 *Let* $x^* > 1$. *From* (4)-(8), *we get* $u = 1$, $v = 1$, $k_0 = x^*+1$ *and* $\mu^* = 0$. *Putting these values in* (3), *we get* $x^*+1 = 0$, *which is not possible for any* $x^* > 1$. *Hence, necessary optimality conditions (Theorem 6) are not satisfied for* $x^* > 1$.

Define

$$\Phi(x, y, (z,\rho)) = \langle z, x - y\rangle + \rho\|x - y\|^2.$$

We can check that the functions $f(x,\bar{y}) + \langle x, Au\rangle = x^2 + x + \bar{y}^2 + xu$ *and* $-g(x,\bar{y}) + \langle x, Bv\rangle = -2x - 1 + xv$ *are* (Φ,ρ)*-invex for* $\rho = 1/2$ *and* $(\Phi,\bar{\rho})$*-invex at* x^* *for* $\bar{\rho} = -1/8$ *on* X, *respectively, and* $h(x)$ *is* (Φ,ρ^*)*-invex at* x^* *for* $\rho^* = -1/8$ *on* X. *Furhtermore,*

$$t^*(\rho + k_0\bar{\rho}) + \mu^*\rho* = 0.$$

By setting $s = 1$, $t^* = 1$, $\bar{y} = 1$, $u = 1$, $v = 1$, $k_0 = 2$ *and* $\mu^* = 2$, *the sufficient optimality conditions (Theorem 7) are satisfied. Hence,* $x^* = 1$ *is the optimal solution for (P1).*

Now, we will formulate the first dual model for (P1).

$$(DI') \qquad \max_{(s,t,\bar{y})\in K(z)} \ \sup_{(z,\mu,k,u,v)\in\Omega_1(s,t,\bar{y})} k,$$

where Ω_1 *denotes the set of all* $(z,\mu,k,u,v)\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ *satisfying*

$$\begin{aligned} 2z+1+u-k(2-v)-\mu=0,\\ 2z+1+zu-k(2z+1-zv)\geq 0\\ \mu(1-z)\geq 0,\\ u^2\leq 1,\ v^2\leq 1. \end{aligned}$$

We can verify that, $\Omega_1=\{(z,\mu,k,u,v):0\leq z\leq 1,\,0\leq\mu\leq 4,\,0\leq k\leq 2,\,-1\leq u,v\leq 1\}$.

We can check that the functions $f(x,\bar{y})+\langle x,Au\rangle = x^2+x+\bar{y}^2+xu$ *and* $-g(x,\bar{y})+\langle x,Bv\rangle=-2x-1+xv$ *are strictly* (Φ,ρ)*-invex for* $\rho=1/2$ *and strictly* $(\Phi,\bar{\rho})$*-invex at* x^* *for* $\bar{\rho}=-1/8$ *on* $X\cup_{pr_{\mathbb{R}}}\Omega_1(\bar{s},\bar{t},\bar{y}^*)$*, respectively, and* $h(x)$ *is strictly* (Φ,ρ^*)*-invex at* x^* *for* $\rho^*=-1/8$ *on* $X\cup_{pr_{\mathbb{R}}}\Omega_1(\bar{s},\bar{t},\bar{y}^*)$*. Furhtermore,*

$$t^*(\rho+k_0\bar{\rho})+\mu^*\rho*=0,$$

and

$$\sup_{y\in Y}\frac{f(x,y)+\langle x,Ax\rangle^{1/2}}{g(x,y)-\langle x,Bx\rangle^{1/2}}\geq k,\ \forall x\in X\,\text{and}\,k\in\Omega_1(s,t,\bar{y}).$$

Hence, the weak duality conditions (Theorem 8) are satisfied.

By setting $\bar{s}=1$, $\bar{t}=1$, $\bar{y}^*=1$, $\bar{u}=1$, $\bar{v}=1$, $\bar{k_0}=2$ *and* $\bar{\mu}=2$*, the strong converse duality conditions (Theorem 9) and strict converse duality conditions (Theorem 10) are satisfied. Hence,* $\bar{z}=x^*=1$ *is the optimal solution for (P1) and*

$$\sup_{y\in Y}\frac{f(\bar{z},\bar{y}^*)+\langle\bar{z},A\bar{z}\rangle^{1/2}}{g(\bar{z},\bar{y}^*)-\langle\bar{z},B\bar{z}\rangle^{1/2}}=\bar{k}.$$

Now, we will formulate the second dual model for (P1).

$$(DII')\qquad \max_{(s,t,\bar{y})\in K(z)}\ \sup_{(z,\mu,u,v)\in\Omega_2(s,t,\bar{y})} F(z),$$

where $F(z)=\sup\limits_{y\in Y}\frac{f(z,y)+\langle z,Az\rangle^{1/2}}{g(z,y)-\langle z,Bz\rangle^{1/2}}$ *and* Ω_2 *denotes the set of all* $(z,\mu,u,v)\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ *satisfying*

$$\begin{aligned} (2z+1-|z|)(2z+1+u)-(z^2+z+1+|z|)(2-v)-\mu=0,\\ \mu(1-z)\geq 0,\\ u^2\leq 1,\ v^2\leq 1,\ |z|=zu,\ |z|=zv. \end{aligned}$$

We can verify that, $\Omega_2=\{(z,\mu,u,v):z\in(-\infty,\frac{-1-\sqrt{10}}{3}]\cup[0,1],\ \mu\in[1,4],\ u=v\in\{-1,0,1\}\}$.

We can check that the functions $f(x,\bar{y})+\langle x,Au\rangle = x^2+x+\bar{y}^2+xu$ *and* $-g(x,\bar{y})+\langle x,Bv\rangle=-2x-1+xv$ *are strictly* (Φ,ρ)*-invex for* $\rho=1/2$ *and strictly*

$(\Phi, \bar{\rho})$*-invex at* x^* *for* $\bar{\rho} = -1/8$ *on* $X \cup_{pr_{\mathbb{R}}} \Omega_2(\bar{s}, \bar{t}, \bar{y}^*)$*, respectively, and* $h(x)$ *is strictly* (Φ, ρ^*)*-invex at* x^* *for* $\rho^* = -1/8$ *on* $X \cup_{pr_{\mathbb{R}}} \Omega_2(\bar{s}, \bar{t}, \bar{y}^*)$*. Furhtermore,*

$$t^*\{(g(z, \bar{y}) - \langle z, Bz\rangle^{1/2})\rho + (f(z, \bar{y} + \langle z, Bz\rangle^{1/2})\bar{\rho})\} + \mu\rho* \geq 0, \ \forall z \in [0, 1].$$

and

$$\sup_{y \in Y} \frac{f(x, y) + \langle x, Ax\rangle^{1/2}}{g(x, y) - \langle x, Bx\rangle^{1/2}} \geq F(z), \ \forall x \in X \, \text{and} \, z \in \Omega_2(s, t, \bar{y}).$$

Hence, weak duality conditions (Theorem 12) are satisfied.

By setting $\bar{z} = 1$, $\bar{s} = 1$, $\bar{t} = 1$, $\bar{y}^* = 1$, $\bar{u} = 1$, $\bar{v} = 1$ *and* $\bar{\mu} = 2$, *the strong converse duality conditions (Theorem 13) and strict converse duality conditions (Theorem 14) are satisfied. Hence,* $\bar{z} = x^* = 1$ *is the optimal solution for (P1) and*

$$\sup_{y \in Y} \frac{f(\bar{z}, \bar{y}^*) + \langle \bar{z}, A\bar{z}\rangle^{1/2}}{g(\bar{z}, \bar{y}^*) - \langle \bar{z}, B\bar{z}\rangle^{1/2}} = F(\bar{z}).$$

6. CONCLUSIONS and FUTURE DIRECTIONS

In the paper, a new class of nonconvex nondifferentiable generalized minmax programming problems (P) has been considered. The sufficient optimality results have been established for such nonsmooth optimization problems under hypotheses that the functions involved are (Φ, ρ)-invex. Further, two dual models (DI) and (DII) have been formulated for the considered nondifferentiable generalized minimax programming problem (P) and several duality results have been established between the primal optimization problem and its duals also under (Φ, ρ)-invexity. Note that, in the light of Remark 2.5, the results of the paper extend and generalize several results of Lai and Lee [29], Lai et al. [30], Liang and Shi [32], Liu and Wu [33], Liu et al. [35] and Mishra et al. [37, 40]. Results of the paper may be extended for real valued nondifferentiable locally Lipschitz functions on real Banach spaces using the tools of Michel Penot subdifferentials or convexificators, which will orient the future research of the authors.

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