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A QUASI-POLYNOMIAL ALGORITHM FOR THE KNAPSACK PROBLEM

Valentin E. BRIMKOV

Institute of Mathematics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

Abstract: The main result of the paper is a quasi-polynomial algorithm for generating the extremal points (vertices) of the Knapsack Polytope. The complexity of this algorithm (for fixed dimension) is better than the complexity of all the known quasi-polynomial algorithms for this problem. The idea is similar to this one used by Haies and Larman [2]. Our improvement is based on obtaining of new upper bounds for the number of the vertices of the Knapsack Polytope.

The algorithm can be applied directly to solve the Knapsack Problem with arbitrary convex objective function.

Keywords: Knapsack, convex criterion, quasi-polynomial algorithm.

1. INTRODUCTION

The Knapsack Problem

(KP) $\max cx$ subject to $ax \le b,\ x \in Z_0^n$

where $a, c \in Z_+^n$, $b \in Z_+$ is one of the most intensively explored combinatorial optimization problems. It is a model of important real problems and, at the same time, because of its simplicity, is often used as a model one for searching and testing of new ideas and approaches for solving other NP-hard problems. Moreover, some algorithms for more general integer programming problems use algorithms for KP as a subroutine.

The KP is among the NP-hard problems for whose solving most notable successes are reached: pseudopolynomial algorithms [6], fully polynomial approximation schemes [3], high effective heuristic algorithms [7], etc. Amidst the basic theoretical

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achievements the so called quasi-polynomial algorithms can be indicated, whose complexity is polynomial if the problem' s dimension n is fixed.

An interesting and important object of research, connected to the KP is the knapsack polytope, defined as a convex hull of all admissible solutions of the problem. The extremal elements (vertices) of the knapsack polytope are of special importance. It is clear that a method for generating the vertices is also a method for solving the KP, even with arbitrary convex objective function.

The main result of this paper is sketched in the last section quasi-polynomial algorithm for finding the vertices of the knapsack polytope (and, consequently, for solving the KP), whose complexity (for fixed dimension) is better than the complexity of all the known quasi-polynomial algorithms.

The algorithm' s idea is similar to this one, used by Haies and Larman in [2]. The improvement is based on obtaining of new upper bounds for the number of vertices of the knapsack polytope (section 2.2). The bounds are improved bilaterally: by increasing the logarithm base in the evaluation function, and by decreasing the degree of this (polynomial) function.

Some of the results are presented in [1].

2. VERTICES OF THE KNAPSACK POLYTOPE

In this section we present some definitions and subsidiary results highlighting the matter, and obtain new upper bounds for the number of the vertices of the knapsack polytope.

2.1. Definitions and Subsidiary Results

Lemma 1. If P is an arbitrary knapsack polytope, it can be presented by an inequality $ax \le b$ whose minimal coefficient $a_{min} = \min_i a_i$ is arbitrary large.

Corollary 1. Every knapsack polytope can be given by arbitrary large input (independently on the dimension of the polytope).

Let us denote $P_Z(a,b) = conv\{\, x \in Z_0^n : a x \le b \,\}$ and let P be an arbitrary n-dimensional knapsack polytope. We define a class

$$K(P) = \{\, \langle a, b \rangle : a \in Z_+^n,\, b \in Z_+,\, GCD(a_1, \dots, a_n, b) = 1,\ P = P_Z(a,b) \,\}$$

(GCD – Greatest Common Divisor) of all knapsack inputs determining P (obviously, $K(P)$ is nonempty).

Usually, a concrete (individual) n-dimensional knapsack polytope P is given by a concrete couple $\langle a, b \rangle \in K(P)$, and it is naturally to evaluate the number of the vertices by a function of the parameters a, b and n.

Let $N(P)$ denotes the number of the vertices of a knapsack polytope P. We say that the function $f(n, a, b)$ is an upper bound for $N(P)$, if for every n-dimensional knapsack polytope P and every $\langle a, b\rangle \in K(P)$ $N(P) \le f(n, a, b)$.

LEMMA 2. Let $n \ge 2$ be a fixed natural number. Then, for every natural number $l \ge n+1$ there exist n-dimensional knapsack polytopes with l vertices.

The proof can be done in constructive way, by induction on n and l.

COROLLARY 2. There is no function $f: Z_+ \to Z_+$ such that for every fixed $n \ge 2$ and for every n-dimensional knapsack polytope P to be fulfilled $N(P) \le f(n)$.

In [2], [8], [9] and [10] bounds are obtained for the number of vertices of the polytope of Integer Linear Programming Problem $P_Z(A, b) = conv\,\{ x \in Z^n : Ax = b \}$, where A is a $m \times n$ matrix with integer elements, and b is a n-dimensional integer vector. In particular, for the knapsack polytope, the best bounds are polynomials of power n of $\log_2 (b / a_{min})$, i.e. quasi-polynomials. For example, it is shown in [2] that

$$N(P_Z(a, b)) \le (\log_2 \frac{4b}{a_{\min}})^n, \quad \text{where} \quad a_{\min} = \min_i a_i. \tag{1}$$

Further, we will improve this result.

2.2. NEW UPPER BOUNDS FOR THE NUMBER OF THE KNAPSACK POLYTOPE VERTICES

First, we will show that the bound (1) can be improved by increasing the base of the logarithm in the evaluating function.

THEOREM 1. For every n-dimensional knapsack polytope $P_Z(a, b)$ there exists a positive number ε, such that

$$N(P_Z(a, b)) \le \prod_{j=1}^{n} [\, \log_{2+\varepsilon} Q_j \,], \quad \text{where} \quad Q_j = \frac{4b}{a_j} \quad \text{for } j = 1, \dots, n. \tag{2}$$

(here and further $[\, x\,]$ denotes the greatest integer in x).

PROOF. The idea used to prove (1) [2] is the following: construct in an appropriate way a set of $\prod_{j=1}^{n} [\, \log_2 Q_j \,]$ boxes covering $P_Z(a, b)$, and show that each of them contains not more than one vertex of $P_Z(a, b)$. We will show here that the sizes of these boxes can be enlarged, preserving the property "not more one vertex in a box".

Let β^ε be a set of boxes with length of the edgings – powers of $2 + \varepsilon$, where $\varepsilon > 0$ is a parameter:

$$\beta^\varepsilon = \{ x \in R_+^n : x \in \prod_{j=1}^{n} I_{k_j^\varepsilon},\ k_j^\varepsilon \text{ – integers}, 1 \le k_j^\varepsilon \le N_j^\varepsilon \quad \text{for} \quad j = 1, \dots, n \}$$

(here $\prod$ denotes Cartesian Product), where N_j^ε $(j = 1, \dots, n)$ are such that $x_{N_j^\varepsilon - 1} \le b / a_j < x_{N_j^\varepsilon}$, $I_{k_j^\varepsilon} = [\, x_{k_j^\varepsilon - 1}, x_{k_j^\varepsilon} \,)$, $x_{k_j^\varepsilon} = (2+\varepsilon)^{k_j^\varepsilon - 1}$, $x_0 = 0$ (if $\varepsilon = 0$ we obtain the boxes from [2]). We have

$$P_Z(a,b) \subset \bigcup_{B \in \beta^\varepsilon} B, \quad N_j^\varepsilon \le [\, \log_{2+\varepsilon} \frac{4b}{a_j} \,], \quad j = 1, \dots, n.$$

It is enough to show that there exists $\varepsilon > 0$ such that every box of β^ε contains not more than one vertex of $P_Z(a, b)$.

Let us suppose the obverse, i.e. for every $\varepsilon > 0$ there exists a box $B = \prod_{j=1}^n I_{k_j^\varepsilon} \in \beta^\varepsilon$ (for some set of numbers k_j^ε with $1 \le k_j^\varepsilon \le N_j^\varepsilon$), containing two vertices v and w of $P_Z(a,b)$. Because the boxes are "open from the right", v_j, $w_j \ne (2+\varepsilon)^{k_j^\varepsilon - 1}$ for $j = 1, \dots, n$. Now, following [2], we consider the hyperplane $H = \{\, x \in R^n : ax = b \,\}$. Let u be the normal vector to H, directed to the halfspace non-containing $P_Z(a, b)$, and let $\alpha \in \mathrm{R}$ be such that $ux = \alpha$ for all $x \in H$. Let $uv \le uw$. We will show that for some $\varepsilon > 0$ the vector $2w - v \in P_Z(a, b)$. Really, $2w - v$ is integer, and from $uv \le \alpha$, $uv \le uw$ follows $(2w - v)\, u \le \alpha$. It remains to show that for some $\varepsilon > 0$ $2w_j - v_j \ge 0$ is true for all $j = 1, \dots, n$. Having in mind that $v_j \le [\, (2+\varepsilon)^{k_j^\varepsilon - 1} \,]$, we consider two cases:

1. Let $w_j = (2+\varepsilon)^{k_j^\varepsilon - 2}$ be integer. Setting $m := k_j^\varepsilon - 2$, we obtain

$$2w_j = 2\,(2^m + \binom{m}{1} 2^{m-1} \varepsilon + \cdots + \varepsilon^m) = 2^{m+1} + \binom{m}{1} 2^m \varepsilon + \cdots + \binom{m}{m-1} 4\, \varepsilon^{m-1} + 2\, \varepsilon^m,$$

$$v_j \le [\, (2+\varepsilon)^{m+1} \,] = [\, 2^{m+1} + \binom{m+1}{1} 2^m \varepsilon + \cdots + \binom{m+1}{m} 2\, \varepsilon^m + \varepsilon^{m+1} \,].$$

Then $2\, w_j - v_j \ge 2\, w_j - [\, (2+\varepsilon)^{m+1} \,] \ge 0$ if, for example,

$$\binom{m+1}{1} 2^m \varepsilon + \cdots + \binom{m+1}{m} 2\, \varepsilon^m + \varepsilon^{m+1} < 1$$

which is true for ε enough small (let us mention also that m is restricted by a number not depending on ε, e. g. $m = k_j^\varepsilon - 2 \le N_j^\varepsilon - 2 \le N_j^0 - 2 \le \log_2(4b / a_j) - 2$ for $j = 1, \dots, n$).

2. If $w_j \ne (2+\varepsilon)^{k_j^\varepsilon - 2}$, then $w_j \ge [\, (2+\varepsilon)^{k_j^\varepsilon - 2} \,] + 1$. For $m := k_j^\varepsilon - 2$ we have

$$2\, w_j - v_j \ge 2\, ([\, (2+\varepsilon)^m \,] + 1) - [\, (2+\varepsilon)^{m+1} \,] =$$
$$2 + 2\, [\, 2^m + \varepsilon\, \mathrm{A}_1(\varepsilon, m) \,] - [\, 2^{m+1} + \varepsilon\, \mathrm{A}_2(\varepsilon, m) \,] = 2 > 0$$

for small enough ε (it is set $A_1(\varepsilon, m) := \binom{m}{1} 2^{m-1} + \binom{m}{2} 2^{m-2}\varepsilon + \cdots + \varepsilon^{m-1}$, $A_2(\varepsilon, m) := \binom{m+1}{1} 2^{m} + \binom{m+1}{2} 2^{m-1}\varepsilon + \cdots + \varepsilon^{m}$).

So we received that for small enough $\varepsilon > 0$ v and w are points of $P_Z(a,b)$. Then, for some such ε, we obtain $w = 1/2\,(v + (2w - v))$, i.e. w is not a vertex of $P_Z(a,b)$ – contradiction.

In addition, we will mention a sufficient condition for ε in order (2) to be the case.

THEOREM 2. (2) is true if

$$\varepsilon < \frac{1}{(N_j^0 - 1)[\frac{N_j^0 - 1}{3}]} \quad \text{for} \quad j = 1, \ldots, n.$$

The proof is based on the following:

LEMMA 3. The function $f : \{1, \ldots, m\} \to Z_+$, $f(k) = \binom{m}{k} 2^{m-k}$ (m – natural number) has a maximum for $k = [\, m/3 \,]$.

Let us also mention that $\varepsilon < 1$ (for $\varepsilon = 1$, $n = 2$, $a_1 = 4$, $a_2 = 5$, $b = 47$, the box $B = [\,3, 9\,) \times [\,3, 9\,)$ contains two vertices $v^{(1)} = (3,7)$ and $v^{(2)} = (8,3)$ of $P_Z(a,b)$).

The rest of this section gives a proof of the following:

THEOREM 3.

$$N(P_Z(a,b)) \le C(n)\, [\, \log_2 Q \,]^{n-1}, \tag{3}$$

where $Q = 4b / a_{min}$ and $C(n)$ is a constant for fixed n.

PROOF. First, we will modify the method for generating the set of boxes, covering $P_Z(a,b)$. For this purpose, let us define n sequences of rational numbers

$$\{ x_j^{(1)} \}_{j=0}^{\infty}, \ldots, \{ x_j^{(n)} \}_{j=0}^{\infty}$$

where $x_0^{(k)} = b / a_k$, $x_j^{(k)} = (b / a_k)\, 2^{-j}$ for $k = 1, \ldots, n$, $j = 1, 2, \ldots$.

For each of these sequences we determine an integer P_k with $0 < x_{P_k}^{(k)} < 1$, $x_{P_k - 1}^{(k)} \ge 0$. We have $P_k = N_k^0 - 1 \le [\, \log_2(2b / a_k) \,]$, where N_k^0 is a number defined as in the proof of Theorem 1.

Let us consider the set of boxes

$$\beta = \{\, x \in \prod_{j=1}^{n} I_{k_j},\, k_j \text{ - integer}, 1 \le k_j \le P_j,\, j = 1, \ldots, n \,\}$$

Each of these boxes contains not more than one vertex of $P_Z(a,b)$. (Really, let $B \in \beta'$ and $v, w \in B$ be vertices of $P_Z(a,b)$. Let $H = \{x \in R^n : ax = b\} = \{x \in R^n : ux = \alpha\}$ where u is the normal vector to H directed to the halfspace not containing $P_Z(a,b)$. The vector $2w - v$ is integer, and if $uv \geq uw$ then $(2w - v)u \leq \alpha$. Besides, $|w_k - v_k| \leq x_{j-1}^{(k)} - x_j^{(k)} = (b/a_k)2^{-j+1} - (b/a_k)2^{-j} = x_j^{(k)}$, and $2w_k - v_k = w_k - (v_k - w_k) \geq w_k - |v_k - w_k| \geq x_j^{(k)} - x_j^{(k)} = 0$. Then, $w = 1/2\,(v + (2w - v))$, and hence w is not vertex of $P_Z(a,b)$ – contradiction.)

It is easy to check that the point $x^* = (b/(2^m a_1), \ldots, b/(2^m a_n))$ satisfies $ax^* \leq b$ if and only if $m \geq [\log_2 n] + 1$ for non-integer $\log_2 n$, and $m \geq [\log_2 n]$ for $\log_2 n$ – integer. Let us consider the first case (the another is analogical), and let m_0 be the minimal integer with this property. It is clear that all the vertices of $P_Z(a,b)$ belong to the area determined by the inequalities $ax \leq b$ and $\bar{a}x \geq \bar{b}$, where $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$, $\bar{a}_i = [b/a_1]\ldots[b/a_{i-1}][b/a_{i+1}]\ldots[b/a_n]$ $(i = 1, \ldots, n)$, $\bar{b} = \prod_{i=1}^n [b/a_i]$. We consider two cases:

1. $\bar{a}x^* \leq \bar{b}$.

For geometrical considerations, it is clear that in the set of boxes β'' whose points are majorated by x^* there is no one containing a vertex of $P_Z(a,b)$ different from $O(x = (x_1, \ldots, x_n)$ is said to be majorated by $y = (y_1, \ldots, y_n)$ if $x_i \leq y_i$ for $i = 1, \ldots, n)$.

The number of boxes in β'' is $\prod_{i=1}^n [\log_2(4b/(2^{m_0} a_i))]$. Then, the number of boxes in $\beta' \setminus \beta''$ which, eventually, could contain vertices, is

$$\prod_{i=1}^{n}[\log_2 \frac{4b}{a_i}] - \prod_{i=1}^{n}[\log_2 \frac{4b}{2^{m_0} a_i}] = \prod_{i=1}^{n}[\log_2 \frac{4b}{a_i}] - \prod_{i=1}^{n}[\log_2 \frac{4b}{a_i} - m_0] =$$
$$\prod_{i=1}^{n}[\log_2 \frac{4b}{a_i}] - \prod_{i=1}^{n}([\log_2 \frac{4b}{a_i}] - m_0) =$$
$$m_0 \Big(\prod_{i=1}^{n-1}[\log_2 \frac{4b}{a_i}] + \cdots + \prod_{i=2}^{n}[\log_2 \frac{4b}{a_i}]\Big) -$$
$$m_0^2 \Big(\prod_{i=1}^{n-2}[\log_2 \frac{4b}{a_i}] + \cdots + \prod_{i=3}^{n}[\log_2 \frac{4b}{a_i}]\Big) + \cdots \pm m_0^{n-1}\sum_{i=1}^{n}[\log_2 \frac{4b}{a_i}] \mp m_0^n \leq$$
$$m_0\binom{n}{n-1}[\log_2 \frac{4b}{a_{\min}}]^{n-1} - m_0^2\binom{n}{n-2}[\log_2 \frac{4b}{a_{\min}}]^{n-2} + \cdots \pm m_0^n \leq$$
$$C(n)[\log_2 \frac{4b}{a_{\min}}]^{n-1}$$

where $C(n)$ is a function of n.

2. Let now $\bar{a}x^* \geq \bar{b}$.

Let the straight lines through x^* and parallel to the co-ordinate axes intersect the hyperplane H in the points $M^{(1)}, \ldots, M^{(n)}$. It is not hard to calculate that if

$$\frac{1}{2}x_i^* > x_i^* - M_i^{(i)} \qquad \text{for} \quad i = 1, \ldots, n, \tag{4}$$

then all boxes, whose points are majorated by $1/2x^*$, do not contain vertices of $P_Z(a,b)$ different from O (here x_i^* and $M_i^{(i)}$ denote the i-th co-ordinates of x^* and $M^{(i)}$, respectively). We have

$$M_i^{(i)} = [\frac{b}{a_i}] - \frac{1}{m_0}[\frac{b}{a_i}](\frac{\frac{b}{a_1}}{[\frac{b}{a_1}]} + \cdots + \frac{\frac{b}{a_{i-1}}}{[\frac{b}{a_{i-1}}]} + \frac{\frac{b}{a_{i+1}}}{[\frac{b}{a_{i+1}}]} + \cdots + \frac{\frac{b}{a_n}}{[\frac{b}{a_n}]}),$$

and $\quad x_i^* = \dfrac{b}{2^{m_0}a_i}$.

After substitution in (4) and some trivial simplifications, we get that the condition $b/a_i \geq 2(n-1)$ for $i = 1, \ldots, n$ is sufficient one for (4). Under this condition, we can obtain (3) in analogical manner as in the case 1. Otherwise, (i.e. if $b/a_i \leq 2(n-1)$ for some indexes i), we trivially have the result of the theorem.

Combining the results of the Theorems 1 and 3, we receive

COROLLARY 3. For every n-dimensional knapsack polytope $P_Z(a,b)$ there exists a positive number ε, such that $N(P_Z(a,b)) \leq C(n)[\log_{2+\varepsilon} Q]^{n-1}$, where $C(n)$ is a constant for fixed n.

It is shown in [8] that if the condition

$$b \geq a_{\max}(a_{\max} - 1), \quad a_{\max} = \max_i a_i \tag{5}$$

takes place, then the following bound, depending only on the coefficients, is the case:

$$N(P_Z(a,b)) \leq 1 + \sum_{i=1}^{n}\binom{\delta_i + n - 2}{n-1},$$

where $\delta_i = 1 + \lfloor \log_2 a_i \rfloor$ for $i = 1, \ldots, n$.

Obviously, for fixed n this bound is a polynomial of $n-1$ degree of the input length. From Theorem 3 we can conclude that such bound is true, too, if the condition (5) is not satisfied.

COROLLARY 4. If $b \leq a_{max}(a_{max} - 1)$, then $N(P_Z(a,b)) \leq C(n)[\log_2 a_{max}]^{n-1}$, $C(n)$ – constant for fixed n.

3. THE ALGORITHM

In this section we propose an improvement of the Haies-Larman algorithm for generating the vertices of the knapsack polytope, based on the Theorems 1 and 3.

Let us consider the following procedure:

1. Determine $\varepsilon > 0$ so that (2) is fulfilled (see Theorem 1);

2. For every box $B \in \beta^{\varepsilon}$ such that $\bar{a}x^{*} \geq \bar{b}$. (see the proof of Theorem 3) solve the Integer Linear Programming Problem

$$\max ax \qquad \text{subject to} \qquad ax \leq b, \;\; x \in B, \;\; x\text{ - integer} \qquad (6)$$

(it could be done in quasi-polynomial time; e.g. see [5]);

After $O(C(n)\,[\log_{2+\varepsilon}(4b/a_{min})]^{n-1}$ steps (see Corollary 3) we obtain a list L of vectors $v^{(1)}, \ldots, v^{(t)}$. The vertices of $P_Z(a,b)$ are among these vectors (see the proof of (1) in [2]).

3. Check which of the vectors in L are vertices of $P_Z(a,b)$. For this purpose, the solvability of not more than $t-1$ systems of linear equations (determined in appropriate way) is to be checked (see again [2]). Each of these examinations could be done by the polynomial Khachiyan' s algorithm (see [4]).

So we see that for fixed dimension n, the vertices of $P_Z(a,b)$ can be generated in $O([\log_{2+\varepsilon}(4b/a_{min})]^{n-1} g(n,a,b)\, h(n,a,b)$ time, where $g(n,a,b)$ denotes the complexity of the quasi-polynomial algorithm for (6), and $h(n,a,b)$ is the complexity of the Chachian' s algorithm performed on step 3.

The described algorithm can be applied for solving the knapsack problem with arbitrary convex objective function. More precisely, it is the case:

Corollary 5. Let f be a convex function such that the value of f for every $x \in Z_0^n$ can be calculated in polynomial time $T_f(n)$ for fixed n. Then, there exists a quasi-polynomial algorithm with complexity

$$O([\log_{2+\varepsilon}(4b/a_{min})]^{n-1} T_f(n) g(n,a,b) h(n,a,b)$$

for solving the problem

$$\max f(n) \qquad \text{subject to} \qquad ax \leq b, \;\; x \in Z_o^n.$$

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