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GENERALIZED INVEXITY AND MATHEMATICAL PROGRAMS

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Abstract: In this paper, using generalized convexity assumptions, we show that M-stationary condition is sufficient for global or local optimality under some mathematical programming problem with equilibrium constraints(MPEC). Further, we formulate and study Wolfe type and Mond-Weir type dual models for the MPEC, and we establish weak and strong duality theorems.

Keywords: Constraint Qualification, Duality, Generalized Convex Function.

MSC: 90C30, 90C46.

1. INTRODUCTION

Mathematical program with equilibrium constraints (MPEC) includes the bilevel programming problem (see[6, 33]) as its special case and has wide range of applications such as engineering design, traffic control, and economic modeling(see[2, 12]). Some feasibility issues for MPEC are presented by Fukushima and Pang[8] later, Scheel and Scholtes [31] introduced several stationary concepts and presented a mathematical program with complementarity constraints(MPCC). In 2011, Henrion and Surowiec [13] compared two distinct calmness conditions on

MPEC and derived first order necessary optimality conditions by using tools of generalized differentiation introduced by Mordukhovich[25]. Moreover, Gfrerer [9] introduced the concept of strong M-stationarity, which makes a bridge between S-stationarity and M-stationarity for MPEC. For more literature on MPEC we refer to [32, 14, 1, 36, 19] and references therein.

In order to solve many practical problems, there have been many attempts to weaken the convexity assumptions. Therefore, a number of concepts on generalized convexity have been introduced and applied to mathematical programming problems in the literature[30]. Hanson[11] generalized the Karush-Kuhn-Tucker (KKT) type sufficient optimality condition with the help of a new class of generalized convex function for differentiable real valued functions which are defined on $\mathbb{R}^n$. Later, this class of functions was named by Craven[5] as the class of "invex" functions due to its property of invariance under convex transformations. The class of invex functions preserves many properties of the class of convex functions and has shown to be very useful in a variety of applications [21, 15, 16, 17].

For the last three decades, duality and optimality conditions in generalized convex optimization theory have been discussed by several authors[21, 4, 22, 23], especially due to the modern work in optimization such as economic science, theoretical physics, mathematical programming, critical point theory, game theory, nonconvex-nonsmooth analysis, variational analysis and in many other areas. In nonlinear programming problems, Wolfe[34] and Mond-Weir[24] type dual models are most popular. By using generalized convexity assumptions Pandey and Mishra[27], and Guu *et al.*[10] studied Wolfe and Mond-Weir type dual models for MPEC and presented weak and strong duality results. For recent developments in duality theory for MPEC, we refer to [28, 18] and references therein.

The organization of this paper is as follows: in Section 2, we give some preliminary, definitions and results, which are used in the sequel. In Section 3, we show that M-stationary condition is sufficient for global or local optimality under some MPEC generalized convexity assumptions. In Section 4, we formulate Wolfe and Mond-Weir type dual models for the MPEC and establish weak and strong duality theorems relating to the MPEC and the two dual models under generalized convexity assumptions. In Section 5, we conclude the results of this paper.

2. PRELIMIARIES

In this section, we give some preliminaries and definitions which will be used throughout the paper.

We consider the mathematical program with equilibrium constraints of the form:

$$\begin{aligned}
&\text{(MPEC)} \qquad \min F(v)\\
&\text{subject to}: \quad g(v) \leqslant 0, \quad h(v) = 0,\\
&\phi(v) \geqslant 0,\ \theta(v) \geqslant 0,\ \langle \phi(v), \theta(v) \rangle = 0,
\end{aligned}$$

where $F : \mathbb{R}^n \to \mathbb{R},\ g : \mathbb{R}^n \to \mathbb{R}^k,\ h : \mathbb{R}^n \to \mathbb{R}^p,\ \phi : \mathbb{R}^n \to \mathbb{R}^l$ and $\theta : \mathbb{R}^n \to \mathbb{R}^l$, assume that, $F,\ g,\ h,\ \phi$, and θ are continuously differentiable on $\mathbb{R}^n$. Note that, if we take $h(v) := 0, \phi(v) := 0, \theta(v) := 0$, then MPEC becomes the standard nonlinear programming problem, which is studied in the literature[20].

The feasible set of the problem (MPEC) is denoted by X, that is

$$X := \{v \in \mathbb{R}^n : g(v) \leqslant 0,\ h(v) = 0,\ \phi(v) \geqslant 0,\ \theta(v) \geqslant 0,\ \langle \phi(v), \theta(v) \rangle = 0\}.$$

Based on the definitions of generalized invex functions [3], we are introducing following definitions for higher order case.

Definition 2.1. *Let $X \subseteq \mathbb{R}^n$ be an open set. The differentiable function $F : X \to \mathbb{R}$ is said to be higher order strongly p-invex at $\tilde{v} \in X$ with respect to the kernel functions $\eta : X \times X \to \mathbb{R}^n$, and $\gamma : X \times X \to \mathbb{R}^n$, such that, there exist $\mu > 0$, and*

$$F(v) \geqslant F(\tilde{v}) + \frac{1}{p}\langle \nabla F(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1} \rangle + \mu \|\gamma(v,\tilde{v})\|^{\sigma}, \forall v \in X, \sigma > 0, p \neq 0.$$

Definition 2.2. *Let $X \subseteq \mathbb{R}^n$ be an open set. The differentiable function $F : X \to \mathbb{R}$ is said to be higher order strongly p-pseudoinvex at $\tilde{v} \in X$ with respect to the kernel functions $\eta : X \times X \to \mathbb{R}^n$, and $\gamma : X \times X \to \mathbb{R}^n$, such that, there exist $\mu > 0$, and*

$$\frac{1}{p}\langle \nabla F(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1} \rangle + \mu \|\gamma(v,\tilde{v})\|^{\sigma} \geqslant 0 \Rightarrow F(v) \geqslant F(\tilde{v}),\ \forall v \in X, \sigma > 0, p \neq 0.$$

Definition 2.3. *Let $X \subseteq \mathbb{R}^n$ be an open set. The differentiable function $F : X \to \mathbb{R}$ is said to be higher order strongly p-quasiinvex at $\tilde{v} \in X$ with respect to the kernel functions $\eta : X \times X \to \mathbb{R}^n$, and $\gamma : X \times X \to \mathbb{R}^n$, such that, there exist $\mu > 0$, and*

$$F(v) \leqslant F(\tilde{v}) \Rightarrow \frac{1}{p}\langle \nabla F(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1} \rangle + \mu \|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0, \forall v \in X, \sigma > 0, p \neq 0.$$

Now, we provide following examples in support of the above definitions.

Example 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(v) = v^2$ then at the point $\tilde{v} = 0$ the function is higher order strongly p-invex with respect to the kernel functions $\eta(v,\tilde{v}) = log \sin v$ and $\gamma(v,\tilde{v}) = \sin v\tilde{v}$.*

Example 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(v) = v^2$ then at the point $\tilde{v} = 0$ the function is higher order strongly p-pseudoinvex with respect to the kernel functions $\eta(v,\tilde{v}) = log \cos v$ and $\gamma(v,\tilde{v}) = v + \tilde{v}$.*

Example 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(v) = -v^2$ then at the point $\tilde{v} = 0$ the function is higher order strongly p-quasiinvex with respect to the kernel functions $\eta(v,\tilde{v}) = logv^2$ and $\gamma(v,\tilde{v}) = \tilde{v}e^v$.*

Given a feasible vector $\tilde{v} \in X$ for the problem (MPEC), we define the following index sets:

$$
\begin{aligned}
&I_g := I_g(\tilde{v}) := \{i = 1, 2, \ldots, k : g_i(\tilde{v}) = 0\},\\
&\delta := \delta(\tilde{v}) := \{i = 1, 2, \ldots, l : \phi_i(\tilde{v}) = 0, \theta_i(\tilde{v}) > 0\},\\
&\zeta := \zeta(\tilde{v}) := \{i = 1, 2, \ldots, l : \phi_i(\tilde{v}) = 0, \theta_i(\tilde{v}) = 0\},\\
&\alpha := \alpha(\tilde{v}) := \{i = 1, 2, \ldots, l : \phi_i(\tilde{v}) > 0, \theta_i(\tilde{v}) = 0\}.
\end{aligned}
$$

Here the set ζ is known as degenerate set and if ζ is empty, the vector $\tilde{v}$ is said to satisfy the strict complementarity condition.

In 1999, Outrata[26] introduced the following concept of M-stationary point.

Definition 2.7. *(M-stationary point) A feasible point $\tilde{v}$ of MPEC is said to be Mordukhovich stationary point if, $\exists\ \xi = (\xi^g, \xi^h, \xi^\phi, \xi^\theta) \in \mathbb{R}^{k+p+2l}$, such that following conditions hold:*

$$0 = \nabla F(\tilde{v}) + \sum_{i \in I_g} \xi_i^g \nabla g_i(\tilde{v}) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(\tilde{v}) - \sum_{i=1}^{l} [\xi_i^\phi \nabla \phi_i(\tilde{v}) + \xi_i^\theta \nabla \theta_i(\tilde{v})], \quad (1)$$

$$\xi_{I_g}^g \geqslant 0, \quad \xi_\alpha^\phi = 0, \quad \xi_\delta^\theta = 0, \quad (2)$$

$$\forall i \in \zeta, \ \textit{either}\ \xi_i^\phi > 0, \quad \xi_i^\theta > 0 \quad \textit{or} \quad \xi_i^\phi \xi_i^\theta = 0.$$

Definition 2.8. *(see [29]) (S-stationary point) A feasible point $\tilde{v}$ of MPEC is said to be strong stationary point if, $\exists\ \xi = (\xi^g, \xi^h, \xi^\phi, \xi^\theta) \in \mathbb{R}^{k+p+2l}$, such that, the following condition along with* (1) *and* (2) *hold:*

$$\forall i \in \zeta, \quad \xi_i^\phi \geqslant 0, \quad \xi_i^\theta \geqslant 0.$$

Definition 2.9. *(see [31]) (C-stationary point) A feasible point $\tilde{v}$ of MPEC is said to be Clarke stationary point if, $\exists\ \xi = (\xi^g, \xi^h, \xi^\phi, \xi^\theta) \in \mathbb{R}^{k+p+2l}$, such that the following condition along with* (1) *and* (2) *hold:*

$$\forall i \in \zeta, \quad \xi_i^\phi \xi_i^\theta \geqslant 0.$$

Definition 2.10. *(see [7]) (A-stationary point) A feasible point $\tilde{v}$ of MPEC is said to be alternatively stationary point if, $\exists\ \xi = (\xi^g, \xi^h, \xi^\phi, \xi^\theta) \in \mathbb{R}^{k+p+2l}$, such that the following condition along with* (1) *and* (2) *hold:*

$$\forall i \in \zeta, \quad \xi_i^\phi \geqslant 0 \ \textit{or}\ \xi_i^\theta \geqslant 0.$$

Remark 2.1 Strong stationarity implies M-, A- and C-stationarity and the intersection of A-stationarity and C-stationarity give M-stationarity.

Definition 2.11. *(Definition 2.10 [35]) Let $\tilde{v}$ be a feasible point of MPEC and all functions are continuously differentiable at $\tilde{v}$. We say that the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) is satisfied at $\tilde{v}$, if there is no nonzero vector $\xi = (\xi^g, \xi^h, \xi^\phi, \xi^\theta) \in \mathbb{R}^{k+p+2l}$, such that*

$$0 = \sum_{i \in I_g} \xi_i^g \nabla g_i(\tilde{v}) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(\tilde{v}) - \sum_{i=1}^{l} [\xi_i^\phi \nabla \phi_i(\tilde{v}) + \xi_i^\theta \nabla \theta_i(\tilde{v})],$$

$$\xi_{I_g}^g \geqslant 0, \quad \xi_\alpha^\phi = 0, \quad \xi_\delta^\theta = 0,$$

$$\forall i \in \zeta, \ \textit{either}\ \xi_i^\phi > 0, \ \ \xi_i^\theta > 0 \ \ \textit{or} \ \ \xi_i^\phi \xi_i^\theta = 0.$$

The next theorem gives the Fritz-John type M-stationary condition for a feasible solution to be a local solution of the MPEC.

Theorem 2.12. *(Theorem 2.1 [35]) (Fritz-John type M-stationary condition) Let $\tilde{v}$ be a local solution of MPEC, where all functions are continuously differentiable at $\tilde{v}$. Then there exists $r \geqslant 0$, $\xi = (\xi^g, \xi^h, \xi^\phi, \xi^\theta) \in \mathbb{R}^{k+p+2l}$, not all zero, such that*

$$0 = r\nabla F(\tilde{v}) + \sum_{i \in I_g} \xi_i^g \nabla g_i(\tilde{v}) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(\tilde{v}) - \sum_{i=1}^{l} [\xi_i^\phi \nabla \phi_i(\tilde{v}) + \xi_i^\theta \nabla \theta_i(\tilde{v})],$$

$$\xi_{I_g}^g \geqslant 0, \quad \xi_\alpha^\phi = 0, \quad \xi_\delta^\theta = 0,$$

$$\forall i \in \zeta, \ \textit{either}\ \xi_i^\phi > 0, \ \ \xi_i^\theta > 0 \ \ \textit{or} \ \ \xi_i^\phi \xi_i^\theta = 0.$$

Suppose $r \neq 0$, and let us consider $r = 1$, then the following Kuhn-Tucker type M-stationary condition holds.

Corollary 2.13. *(Corollary 2.1 [35]) (Kuhn-Tucker type necessary M-stationary condition) Let $\tilde{v}$ be a local optimal solution for MPEC where, all the functions are continuously differentiable at $\tilde{v}$ and consider that NNAMCQ is satisfied at $\tilde{v}$, then $\tilde{v}$ is M-stationary.*

In the next section, it can be seen that M-stationary condition turns into a sufficient optimality condition under certain MPEC generalized invexity condition.

Note Throughout the paper $\{\}$ will denote an empty set.

3. SUFFICIENT M-STATIONARY CONDITION

Theorem 3.1. *Let $\tilde{v}$ be a feasible point of MPEC and M-stationary condition holds at $\tilde{v}$, i.e., $\exists\ \xi = (\xi^g, \xi^h, \xi^\phi, \xi^\theta) \in \mathbb{R}^{k+p+2l}$, such that*

$$0 = \nabla F(\tilde{v}) + \sum_{i \in I_g} \xi_i^g \nabla g_i(\tilde{v}) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(\tilde{v}) - \sum_{i=1}^{l} [\xi_i^\phi \nabla \phi_i(\tilde{v}) + \xi_i^\theta \nabla \theta_i(\tilde{v})], \quad (3)$$

$$\xi^g_{I_g} \geqslant 0, \quad \xi^\phi_\alpha = 0, \quad \xi^\theta_\delta = 0,$$

$$\forall i \in \zeta, \ \textit{either } \xi^\phi_i > 0, \quad \xi^\theta_i > 0 \quad \textit{or} \quad \xi^\phi_i \xi^\theta_i = 0.$$

Let

$$\begin{aligned}
&j^+ := \{i : \xi^h_i > 0\}, \quad j^- := \{i : \xi^h_i < 0\},\\
&\zeta^+ := \{i \in \zeta : \xi^\phi_i > 0, \ \xi^\theta_i > 0\},\\
&\zeta^+_\phi := \{i \in \zeta : \xi^\phi_i = 0, \xi^\theta_i > 0\}, \ \zeta^-_\phi := \{i \in \zeta : \xi^\phi_i = 0, \xi^\theta_i < 0\},\\
&\zeta^+_\theta := \{i \in \zeta : \xi^\theta_i = 0, \xi^\phi_i > 0\}, \ \zeta^-_\theta := \{i \in \zeta : \xi^\theta_i = 0, \xi^\phi_i < 0\},\\
&\delta^+ := \{i \in \delta : \xi^\phi_i > 0\}, \quad \delta^- := \{i \in \delta : \xi^\phi_i < 0\},\\
&\alpha^+ := \{i \in \alpha : \xi^\theta_i > 0\}, \quad \alpha^- := \{i \in \alpha : \xi^\theta_i < 0\},
\end{aligned}$$

and assume that for $p \neq 0$, F is higher order strongly p-pseudoinvex at $\tilde{v}$, with respect to the kernel η. Also assume that g_i $(i \in I_g)$, h_i $(i \in j^+)$, $-h_i$ $(i \in j^-)$, ϕ_i $(i \in \delta^- \cup \zeta^-_\theta)$, $-\phi_i$ $(i \in \delta^+ \cup \zeta^+_\theta \cup \zeta^+)$, θ_i $(i \in \alpha^- \cup \zeta^-_\phi)$ and $-\theta_i$ $(i \in \alpha^+ \cup \zeta^+_\phi \cup \zeta^+)$ are higher order strongly p-quasiinvex at $\tilde{v}$, with respect to the common kernel η. If, $\delta^- \cup \alpha^- \cup \zeta^-_\phi \cup \zeta^-_\theta = \{\}$, $\tilde{v}$ is a global optimal solution of MPEC; if, $\zeta^-_\phi \cup \zeta^-_\theta = \{\}$ or when $\tilde{v}$ is an interior point relative to the set,

$$X \cap \{v : \phi_i(v) = 0, \ \theta_i(v) = 0, \ i \in \zeta^-_\phi \cup \zeta^-_\theta\},$$

i.e., for any feasible point v which is close to $\tilde{v}$, one has

$$\phi_i(v) = 0, \qquad \theta_i(v) = 0, \quad \forall i \in \zeta^-_\phi \cup \zeta^-_\theta,$$

then $\tilde{v}$ is a local optimal solution of MPEC, where X is the set of feasible solutions of MPEC.

Proof. Assume that v is any feasible point of MPEC, i.e., for any $i \in I_g$,

$$g_i(v) \leqslant 0 = g_i(\tilde{v}).$$

Using higher order strongly p-quasiinvexity of g_i at $\tilde{v}$ with respect to the common kernel η, it follows that

$$\frac{1}{p}\langle \nabla g_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1}\rangle + \mu^g_i \|\gamma(v,\tilde{v})\|^\sigma \leqslant 0, \quad \forall i \in I_g. \tag{4}$$

Similarly, we have

$$\frac{1}{p}\langle \nabla h_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1}\rangle + \mu^h_i \|\gamma(v,\tilde{v})\|^\sigma \leqslant 0, \quad \forall i \in j^+, \tag{5}$$

$$-\frac{1}{p}\langle \nabla h_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1}\rangle + \mu^h_i \|\gamma(v,\tilde{v})\|^\sigma \leqslant 0, \quad \forall i \in j^-. \tag{6}$$

Since, for any feasible point $v, -\phi(v) \leqslant 0, -\theta(v) \leqslant 0$, one also have

$$-\frac{1}{p}\langle \nabla \phi_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1}\rangle + \mu_i^{\phi} \|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0, \quad \forall i \in \delta^+ \cup \zeta_\theta^+ \cup \zeta^+, \tag{7}$$

$$-\frac{1}{p}\langle \nabla \theta_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1}\rangle + \mu_i^{\theta} \|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0, \quad \forall i \in \alpha^+ \cup \zeta_\phi^+ \cup \zeta^+. \tag{8}$$

Case 3.2. *First, we take $\delta^- \cup \alpha^- \cup \zeta_\phi^- \cup \zeta_\theta^- = \{\}$, multiplying (4)-(8) by $\xi_i^g \geq 0 \ (i \in I_g)$, $\xi_i^h > 0 \ (i \in j^+), -\xi_i^h > 0 \ (i \in j^-)$, $\xi_i^{\phi} > 0 \ (i \in \delta^+ \cup \zeta_\theta^+ \cup \zeta^+)$, $\xi_i^{\theta} > 0 \ (i \in \alpha^+ \cup \zeta_\phi^+ \cup \zeta^+)$ respectively and adding (4)-(8), we obtain*

$$\frac{1}{p}\left\langle \sum_{i\in I_g} \xi_i^g \nabla g_i(\tilde{v}) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(\tilde{v}) - \sum_{i=1}^{l} [\xi_i^{\phi} \nabla \phi_i(\tilde{v}) + \xi_i^{\theta} \nabla \theta_i(\tilde{v})], e^{p\eta(v,\tilde{v})} - \mathbf{1} \right\rangle$$
$$+\mu_i^g \|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^h \|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^{\phi} \|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^{\theta} \|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0.$$

Using equation (3), the above inequality, it follows that

$$\frac{1}{p}\langle \nabla F(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1}\rangle + \mu \|\gamma(v,\tilde{v})\|^{\sigma} \geqslant 0.$$

Applying, higher order strongly p-pseudoinvexity of F at $\tilde{v}$ with respect to the kernel η and for the same real number $p \neq 0$, we get $F(v) \geqslant F(\tilde{v})$ for all feasible point v. Hence $\tilde{v}$ is a global optimal solution of MPEC.

Case 3.3. *Now, we take $\delta^- \cup \alpha^- \neq \{\}$ and $\zeta_\phi^- \cup \zeta_\theta^- = \{\}$. For v sufficiently close to $\tilde{v}$, one has*

$$\phi_i(v) = \phi_i(\tilde{v}), \quad \forall i \in \delta.$$

Applying, higher order strongly p-quasiinvexity of $\phi_i (i \in \delta^-)$ at $\tilde{v}$ with respect to the common kernel η, i.e., for v sufficiently close to $\tilde{v}$, it holds that,

$$\frac{1}{p}\langle \nabla \phi_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1})\rangle + \mu \|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0, \quad \forall i \in \delta^-. \tag{9}$$

In the same manner, for v sufficiently close to $\tilde{v}$, one has

$$\frac{1}{p}\langle \nabla \theta_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1})\rangle + \mu \|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0, \quad \forall i \in \alpha^-. \tag{10}$$

Now, multiplying (4)-(10) by $\xi_i^g \geqslant 0 \ (i \in I_g)$, $\xi_i^h > 0 \ (i \in j^+)$, $-\xi_i^h > 0 \ (i \in j^-)$, $\xi_i^{\phi} > 0 \ (i \in \delta^+ \cup \zeta_\theta^+ \cup \zeta^+)$, $\xi_i^{\theta} > 0 \ (i \in \alpha^+ \cup \zeta_\phi^+ \cup \zeta^+)$, $-\xi_i^{\phi} > 0 \ (i \in \delta^-), -\xi_i^{\theta} > 0 \ (i \in \alpha^-)$ respectively and adding, we get

$$\frac{1}{p}\left\langle \sum_{i\in I_g} \xi_i^g \nabla g_i(\tilde{v}) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(\tilde{v}) - \sum_{i=1}^{l} [\xi_i^{\phi} \nabla \phi_i(\tilde{v}) + \xi_i^{\theta} \nabla \theta_i(\tilde{v})], e^{p\eta(v,\tilde{v})} - \mathbf{1} \right\rangle$$
$$+\mu_i^g \|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^h \|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^{\phi} \|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^{\theta} \|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0.$$

Using (3)*, for v sufficiently close to $\tilde{v}$, the above inequality follows that*

$$\frac{1}{p}\langle \nabla F(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1})\rangle + \mu\|\gamma(v,\tilde{v})\|^{\sigma} \geqslant 0.$$

Now applying higher order strongly p-pseudoinvexity of F at $\tilde{v}$ with respect to the kernel η and for the same real number $p \neq 0$, we get $F(v) \geqslant F(\tilde{v})$, i.e., $\tilde{v}$ is a local optimal solution of MPEC.

Suppose that $\tilde{v}$ is an interior point relative to the set $X \cap \{v : \phi_i(v) = 0, \theta_i(v) = 0,\ i \in \zeta_\phi^- \cup \zeta_\theta^-\}$. Then for any feasible point v sufficiently close to $\tilde{v}$, it holds that

$$\phi_i(v) = 0, \quad \theta_i(v) = 0, \quad \forall i \in \zeta_\phi^- \cup \zeta_\theta^-,$$

and hence by higher order strongly p-quasiinvexity of ϕ_i $(i \in \zeta_\theta^-)$ and θ_i $(i \in \zeta_\phi^-)$,

$$\frac{1}{p}\langle \nabla \phi_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1})\rangle + \mu\|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0, \quad \forall i \in \zeta_\theta^- \tag{11}$$

$$\frac{1}{p}\langle \nabla \theta_i(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1})\rangle + \mu\|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0, \quad \forall i \in \zeta_\phi^-. \tag{12}$$

Again multiplying (4)-(12) *by $\xi_i^g \geqslant 0$ $(i \in I_g)$, $\xi_i^h > 0$ $(i \in j^+)$, $-\xi_i^h > 0$ $(i \in j^-)$, $\xi_i^\phi > 0$ $(i \in \delta^+ \cup \zeta_\theta^+ \cup \zeta^+)$, $\xi_i^\theta > 0$ $(i \in \alpha^+ \cup \zeta_\phi^+ \cup \zeta^+)$, $-\xi_i^\phi > 0$ $(i \in \delta^-)$, $-\xi_i^\theta > 0$ $(i \in \alpha^-)$ $-\xi_i^\phi > 0$ $(i \in \delta^- \cup \zeta_\theta^-)$, $-\xi_i^\theta > 0$ $(i \in \alpha^- \cup \zeta_\phi^-)$, respectively and adding, we have*

$$\frac{1}{p}\left\langle \sum_{i\in I_g} \xi_i^g \nabla g_i(\tilde{v}) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(\tilde{v}) - \sum_{i=1}^{l}[\xi_i^\phi \nabla \phi_i(\tilde{v}) + \xi_i^\theta \nabla \theta_i(\tilde{v})], e^{p\eta(v,\tilde{v})} - \mathbf{1} \right\rangle$$
$$+\mu_i^g\|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^h\|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^\phi\|\gamma(v,\tilde{v})\|^{\sigma} + \mu_i^\theta\|\gamma(v,\tilde{v})\|^{\sigma} \leqslant 0.$$

By virtue of (3)*, for v sufficiently close to $\tilde{v}$, the above follows*

$$\frac{1}{p}\langle \nabla F(\tilde{v}), e^{p\eta(v,\tilde{v})} - \mathbf{1}\rangle + \mu\|\gamma(v,\tilde{v})\|^{\sigma} \geqslant 0.$$

By the higher order strongly p-pseudoinvexity of F at $\tilde{v}$, we have $F(v) \geqslant F(\tilde{v})$ for v sufficiently close to $\tilde{v}$. That is, $\tilde{v}$ is a local optimal solution of MPEC and this completes the proof. □

4. DUALITY

In this section, we formulate a Wolfe type dual problem and a Mond-Weir type dual problem for the MPEC under higher order strongly p-invexity assumptions.

$$\text{WDMPEC} \quad \max_{v,\xi} \left\{ F(v) + \sum_{i\in I_g} \xi_i^g g_i(v) + \sum_{i=1}^{p} \xi_i^h h_i(v) - \sum_{i=1}^{l}[\xi_i^\phi \phi_i(v) + \xi_i^\theta \theta_i(v)] \right\}$$

subject to:

$$0 = \nabla F(v) + \sum_{i\in I_g} \xi_i^g \nabla g_i(v) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(v) - \sum_{i=1}^{l} [\xi_i^\phi \nabla \phi_i(v) + \xi_i^\theta \nabla \theta_i(v)], \quad (13)$$

$$\xi_{I_g}^g \geqslant 0, \quad \xi_\alpha^\phi = 0, \quad \xi_\delta^\theta = 0,$$

$$\forall i \in \zeta, \text{ either } \xi_i^\phi > 0, \quad \xi_i^\theta > 0 \text{ or } \xi_i^\phi \xi_i^\theta = 0,$$

where, $\xi = (\xi^g, \xi^h, \xi^\phi, \xi^\theta) \in \mathbb{R}^{k+p+2l}$.

Remark 4.1. *If we take $h(v) := 0$, $\phi(v) := 0$, and $\theta(v) := 0$, then Wolfe type dual problem WDMPEC for MPEC coincides with the classical Wolfe type dual problem for nonlinear programming given by Wolfe [34].*

Theorem 4.2. *(Weak Duality) Let $\tilde{u}$ be feasible for MPEC, (v, ξ) be feasible for WDMPEC and index sets $I_g, \delta, \zeta, \alpha$ defined accordingly. Suppose that F, g_i $(i \in I_g)$, h_i $(i \in j^+)$, $-h_i$ $(i \in j^-)$, ϕ_i $(i \in \delta^- \cup \zeta_\theta^-)$, $-\phi_i$ $(i \in \delta^+ \cup \zeta_\theta^+ \cup \zeta^+)$, θ_i $(i \in \alpha^- \cup \zeta_\phi^-)$, and $-\theta_i$ $(i \in \alpha^+ \cup \zeta_\phi^+ \cup \zeta^+)$ are higher order strongly p-invex functions at v with respect to the common kernel η and for the same real number $p \neq 0$. If $\delta^- \cup \alpha^- \cup \zeta_\phi^- \cup \zeta_\theta^- = \{\}$ then, for any u feasible for the MPEC, we have*

$$F(u) \geqslant F(v) + \sum_{i\in I_g} \xi_i^g g_i(v) + \sum_{i=1}^{p} \xi_i^h h_i(v) - \sum_{i=1}^{l} [\xi_i^\phi \phi_i(v) + \xi_i^\theta \theta_i(v)].$$

Proof. Let us consider that u be any feasible point for MPEC. Then, we get

$$g_i(u) \leqslant 0, \quad \forall i \in I_g,$$

and

$$h_i(u) = 0, \quad i = 1, 2, \ldots, p.$$

Since, F is higher order strongly p-invex at v, with respect to the kernel η, then

$$F(u) - F(v) \geqslant \frac{1}{p}\langle \nabla F(v), e^{p\eta(u,v)} - \mathbf{1}\rangle + \mu \|\gamma(u,v)\|^\sigma. \quad (14)$$

Similarly, we get

$$g_i(u) - g_i(v) \geqslant \frac{1}{p}\langle \nabla g_i(v), e^{p\eta(u,v)} - \mathbf{1}\rangle + \mu_i^g \|\gamma(u,v)\|^\sigma, \qquad \forall i \in I_g, \quad (15)$$

$$h_i(u) - h_i(v) \geqslant \frac{1}{p}\langle \nabla h_i(v), e^{p\eta(u,v)} - \mathbf{1}\rangle + \mu_i^h \|\gamma(u,v)\|^\sigma, \qquad \forall i \in j^+, \quad (16)$$

$$-h_i(u) + h_i(v) \geqslant -\frac{1}{p}\langle \nabla h_i(v), e^{p\eta(u,v)} - \mathbf{1}\rangle + \mu_i^h \|\gamma(u,v)\|^\sigma, \qquad \forall i \in j^-, \quad (17)$$

$$-\phi_i(u) + \phi_i(v) \geqslant -\frac{1}{p}\langle \nabla \phi_i(v), e^{p\eta(u,v)} - \mathbf{1}\rangle + \mu_i^\phi \|\gamma(u,v)\|^\sigma, \quad \forall i \in \delta^+ \cup \zeta_\theta^+ \cup \zeta^+, \quad (18)$$

$$-\theta_i(u)+\theta_i(v) \geqslant -\frac{1}{p}\langle \nabla\theta_i(v), e^{p\eta(u,v)}-\mathbf{1}\rangle+\mu_i^{\theta}\|\gamma(u,v)\|^{\sigma}, \quad \forall i \in \alpha^+\cup\zeta_\phi^+\cup\zeta^+. \quad (19)$$

If $\delta^-\cup\alpha^-\cup\zeta_\phi^-\cup\zeta_\theta^- = \{\}$, multiplying (15)-(19) by $\xi_i^g \geqslant 0\ (i \in I_g),\ \xi_i^h > 0\ (i \in j^+),\ -\xi_i^h > 0\ (i \in j^-),\ \xi_i^\phi > 0\ (i \in \delta^+\cup\zeta_\theta^+\cup\zeta^+),\ \xi_i^\theta > 0\ (i \in \alpha^+\cup\zeta_\phi^+\cup\zeta^+)$, respectively and adding (14)-(19), it follows that

$$\begin{aligned}
F(u) - F(v) &+ \sum_{i\in I_g}\xi_i^g g_i(u) - \sum_{i\in I_g}\xi_i^g g_i(v) + \sum_{i=1}^{p}\xi_i^h h_i(u) - \sum_{i=1}^{p}\xi_i^h h_i(v)\\
&- \sum_{i=1}^{l}\xi_i^\phi \phi_i(u) + \sum_{i=1}^{l}\xi_i^\phi \phi_i(v) - \sum_{i=1}^{l}\xi_i^\theta \theta_i(u) + \sum_{i=1}^{l}\xi_i^\theta \theta_i(v)\\
&\geqslant \frac{1}{p}\Bigg\langle \nabla F(v) + \sum_{i\in I_g}\xi_i^g \nabla g_i(v) + \sum_{i=1}^{p}\xi_i^h \nabla h_i(v) - \sum_{i=1}^{l}[\xi_i^\phi \nabla\phi_i(v)\\
&+ \xi_i^\theta \nabla\theta_i(v)], e^{p\eta(u,v)} - \mathbf{1}\Bigg\rangle + \mu\|\gamma(u,v)\|^\sigma + \mu_i^g\|\gamma(u,v)\|^\sigma\\
&+ \mu_i^h\|\gamma(u,v)\|^\sigma + +\mu_i^\psi\|\gamma(u,v)\|^\sigma + \mu_i^\theta\|\gamma(u,v)\|^\sigma.
\end{aligned}$$

Using (13), we get

$$\begin{aligned}
F(u) - F(v) &+ \sum_{i\in I_g}\xi_i^g g_i(u) - \sum_{i\in I_g}\xi_i^g g_i(v) + \sum_{i=1}^{p}\xi_i^h h_i(u) - \sum_{i=1}^{p}\xi_i^h h_i(v)\\
&- \sum_{i=1}^{l}\xi_i^\phi \phi_i(u) + \sum_{i=1}^{l}\xi_i^\phi \phi_i(v) - \sum_{i=1}^{l}\xi_i^\theta \theta_i(u) + \sum_{i=1}^{l}\xi_i^\theta \theta_i(v) \geqslant 0.
\end{aligned}$$

Using the feasibility of u for MPEC, that is $g_i(u) \leqslant 0,\ h_i(u) = 0,\ \phi_i(u) \geqslant 0$, and $\theta_i(u) \geqslant 0$, we obtain

$$F(u) - F(v) - \sum_{i\in I_g}\xi_i^g g_i(v) - \sum_{i=1}^{p}\xi_i^h h_i(v) + \sum_{i=1}^{l}[\xi_i^\phi \phi_i(v) + \xi_i^\theta \theta_i(v)] \geqslant 0.$$

Hence,

$$F(u) \geqslant F(v) + \sum_{i\in I_g}\xi_i^g g_i(v) + \sum_{i=1}^{p}\xi_i^h h_i(v) - \sum_{i=1}^{l}[\xi_i^\phi \phi_i(v) + \xi_i^\theta \theta_i(v)],$$

and the proof is complete. $\square$

Theorem 4.3. *(Strong Duality) If $\tilde{u}$ is a global optimal solution of MPEC, such that NNAMCQ is satisfied at $\tilde{u}$ and index sets $I_g, \delta, \zeta, \alpha$ defined accordingly. Let $F,\ g_i\ (i \in I_g),\ h_i\ (i \in j^+),\ -h_i\ (i \in j^-),\ \phi_i\ (i \in \delta^-\cup\zeta_\theta^-),\ -\phi_i\ (i \in \delta^+\cup\zeta_\theta^+\cup\zeta^+),\ \theta_i\ (i \in \alpha^-\cup\zeta_\phi^-)$ and $-\theta_i\ (i \in \alpha^+\cup\zeta_\phi^+\cup\zeta^+)$ fulfill the assumption of the Theorem 4.2. Then, there exists $\tilde{\xi}$, such that $(\tilde{u}, \tilde{\xi})$ is a global optimal solution of WDMPEC and corresponding objective values of MPEC and WDMPEC are equal.*

Proof. Since, $\tilde{u}$ is a global optimal solution of MPEC and NNAMCQ is satisfied at $\tilde{u}$, therfore, there exist $\tilde{\xi} = (\tilde{\xi}^g, \tilde{\xi}^h, \tilde{\xi}^\phi, \tilde{\xi}^\theta) \in \mathbb{R}^{k+p+2l}$, such that M- stationarity conditions are satisfied for MPEC, i.e.,

$$0 = \nabla F(\tilde{u}) + \sum_{i \in I_g} \tilde{\xi}_i^g \nabla g_i(\tilde{u}) + \sum_{i=1}^{p} \tilde{\xi}_i^h \nabla h_i(\tilde{u}) - \sum_{i=1}^{l} [\tilde{\xi}_i^\phi \nabla \phi_i(\tilde{u}) + \tilde{\xi}_i^\theta \nabla \theta_i(\tilde{u})], \quad (20)$$

$$\tilde{\xi}_{I_g}^g \geqslant 0, \quad \tilde{\xi}_\alpha^\phi = 0, \quad \tilde{\xi}_\delta^\theta = 0,$$

$$\forall i \in \zeta, \quad \text{either} \quad \tilde{\xi}_i^\phi > 0, \quad \tilde{\xi}_i^\theta > 0, \quad \text{or } \tilde{\xi}_i^\phi \tilde{\xi}_i^\theta = 0.$$

Therefore, $(\tilde{u}, \tilde{\xi})$ is feasible for WDMPEC. By Theorem 4.2, we get

$$F(\tilde{u}) \geqslant F(v) + \sum_{i \in I_g} \xi_i^g g_i(v) + \sum_{i=1}^{p} \xi_i^h h_i(v) - \sum_{i=1}^{l} [\xi_i^\phi \phi_i(v) + \xi_i^\theta \theta_i(v)], \quad (21)$$

for any feasible solution (v, ξ) for WDMPEC. Now, using the feasibility condition of MPEC and WDMPEC, i.e, for $i \in I_g(\tilde{u})$, $g_i(\tilde{u}) = 0$, also $h_i(\tilde{u}) = 0, \phi_i(\tilde{u}) = 0, \forall i \in \delta \cup \zeta$ and $\theta_i(\tilde{u}) = 0, \forall i \in \zeta \cup \alpha$, then, it follows that

$$F(\tilde{u}) = F(\tilde{u}) + \sum_{i \in I_g} \tilde{\xi}_i^g g_i(\tilde{u}) + \sum_{i=1}^{p} \tilde{\xi}_i^h h_i(\tilde{u}) - \sum_{i=1}^{l} [\tilde{\xi}_i^\phi \phi_i(\tilde{u}) + \tilde{\xi}_i^\theta \theta_i(\tilde{u})]. \quad (22)$$

Using (21) and (22), we get

$$\begin{aligned} F(\tilde{u}) &+ \sum_{i \in I_g} \tilde{\xi}_i^g g_i(\tilde{u}) + \sum_{i=1}^{p} \tilde{\xi}_i^h h_i(\tilde{u}) - \sum_{i=1}^{l} [\tilde{\xi}_i^\phi \phi_i(\tilde{u}) + \tilde{\xi}_i^\theta \theta_i(\tilde{u})] \\ &\geqslant F(v) + \sum_{i \in I_g} \xi_i^g g_i(v) + \sum_{i=1}^{p} \xi_i^h h_i(v) - \sum_{i=1}^{l} [\xi_i^\phi \phi_i(v) + \xi_i^\theta \theta_i(v)]. \end{aligned}$$

Therefore, $(\tilde{u}, \tilde{\xi})$ is a global optimal solution for WDMPEC. Moreover, the corresponding objective values of MPEC and WDMPEC are equal. □

Now, we establish the duality relation between the MPEC and the following Mond-Weir type dual.

$$\text{MWDMPEC} \qquad \max_{v, \xi} F(v)$$

subject to:

$$0 = \nabla F(v) + \sum_{i \in I_g} \xi_i^g \nabla g_i(v) + \sum_{i=1}^{p} \xi_i^h \nabla h_i(v) - \sum_{i=1}^{l} [\xi_i^\phi \nabla \phi_i(v) + \xi_i^\theta \nabla \theta_i(v)], \quad (23)$$

$$\sum_{i\in I_g}\xi_i^g g_i(v)\geqslant 0,\quad \sum_{i=1}^{p}\xi_i^h h_i(v)\geqslant 0,$$

$$\sum_{i=1}^{l}\xi_i^\phi \phi_i(v)\leqslant 0,\quad \sum_{i=1}^{l}\xi_i^\theta \theta_i(v)\leqslant 0,$$

$$\xi_{I_g}^g\geqslant 0,\quad \xi_\alpha^\phi=0,\quad \xi_\delta^\theta=0,$$

$$\forall i\in\zeta,\ \text{either}\ \xi_i^\phi>0,\quad \xi_i^\theta>0\ \text{ or } \xi_i^\phi\xi_i^\theta=0,$$

where, $\xi=(\xi^g,\xi^h,\xi^\phi,\xi^\theta)\in\mathbb{R}^{k+p+2l}$.

Theorem 4.4. *(Weak Duality) Let $\tilde{u}$ be feasible for MPEC, (v,ξ) be feasible for MWDMPEC and the index sets I_g,δ,ζ,α defined accordingly. Suppose that $F,\ g_i\ (i\in I_g),\ h_i\ (i\in j^+),-h_i\ (i\in j^-),\ \phi_i\ (i\in\delta^-\cup\zeta_\theta^-),\ -\phi_i\ (i\in\delta^+\cup\zeta_\theta^+\cup\zeta^+),\ \theta_i\ (i\in\alpha^-\cup\zeta_\phi^-),\ -\theta_i\ (i\in\alpha^+\cup\zeta_\phi^+\cup\zeta^+)$ are higher order strongly p-invex functions at v with respect to the common kernel η and for the same real number $p\neq 0$. If $\delta^-\cup\alpha^-\cup\zeta_\phi^-\cup\zeta_\theta^-=\{\}$, then, for any u feasible for the MPEC, we have*

$$F(u)\geqslant F(v).$$

Proof. Let us consider that u be any feasible point for MPEC. Then, we have

$$g_i(u)\leqslant 0,\quad \forall i\in I_g,$$

and

$$h_i(u)=0,\quad i=1,2,\ldots,p.$$

Since, F is higher order strongly p-invex at v with respect to the kernel η, we have

$$F(u)-F(v)\geqslant \frac{1}{p}\langle\nabla F(v),e^{p\eta(u,v)}-\mathbf{1}\rangle+\mu\|\gamma(u,v)\|^\sigma. \tag{24}$$

Similarly, we have

$$g_i(u)-g_i(v)\geqslant \frac{1}{p}\langle\nabla g_i(v),e^{p\eta(u,v)}-\mathbf{1}\rangle+\mu_i^g\|\gamma(u,v)\|^\sigma,\qquad \forall i\in I_g, \tag{25}$$

$$h_i(u)-h_i(v)\geqslant \frac{1}{p}\langle\nabla h_i(v),e^{p\eta(u,v)}-\mathbf{1}\rangle+\mu_i^h\|\gamma(u,v)\|^\sigma,\qquad \forall i\in j^+, \tag{26}$$

$$-h_i(u)+h_i(v)\geqslant -\frac{1}{p}\langle\nabla h_i(v),e^{p\eta(u,v)}-\mathbf{1}\rangle+\mu_i^h\|\gamma(u,v)\|^\sigma,\qquad \forall i\in j^-, \tag{27}$$

$$-\phi_i(u)+\phi_i(v)\geqslant -\frac{1}{p}\langle\nabla \phi_i(v),e^{p\eta(u,v)}-\mathbf{1}\rangle+\mu_i^\psi\|\gamma(u,v)\|^\sigma,\quad \forall i\in\delta^+\cup\zeta_\theta^+\cup\zeta^+, \tag{28}$$

$$-\theta_i(u)+\theta_i(v)\geqslant -\frac{1}{p}\langle\nabla \theta_i(v),e^{p\eta(u,v)}-\mathbf{1}\rangle+\mu_i^\theta\|\gamma(u,v)\|^\sigma,\quad \forall i\in\alpha^+\cup\zeta_\phi^+\cup\zeta^+. \tag{29}$$

If $\delta^- \cup \alpha^- \cup \zeta_\phi^- \cup \zeta_\theta^- = \{\}$, multiplying(25)-(29) by $\xi_i^g \geqslant 0$ $(i \in I_g)$, $\xi_i^h > 0$ $(i \in j^+)$, $-\xi_i^h > 0$ $(i \in j^-)$, $\xi_i^\phi > 0$ $(i \in \delta^+ \cup \zeta_\theta^+ \cup \zeta^+)$, $\xi_i^\theta > 0$ $(i \in \alpha^+ \cup \zeta_\phi^+ \cup \zeta^+)$, respectively and adding (24)-(29), we obtain

$$\begin{aligned}
F(u) - F(v) + &\sum_{i \in I_g} \xi_i^g g_i(u) - \sum_{i \in I_g} \xi_i^g g_i(v) + \sum_{i=1}^p \xi_i^h h_i(u) - \sum_{i=1}^p \xi_i^h h_i(v) \\
&- \sum_{i=1}^l \xi_i^\phi \phi_i(u) + \sum_{i=1}^l \xi_i^\phi \phi_i(v) - \sum_{i=1}^l \xi_i^\theta \theta_i(u) + \sum_{i=1}^l \xi_i^\theta \theta_i(v) \\
&\geqslant \frac{1}{p} \Bigg\langle \nabla F(v) + \sum_{i \in I_g} \xi_i^g \nabla g_i(v) + \sum_{i=1}^p \xi_i^h \nabla h_i(v) \\
&- \sum_{i=1}^l [\xi_i^\phi \nabla \phi_i(v) + \xi_i^\theta \nabla \theta_i(v)], e^{p\eta(u,v)} - \mathbf{1} \Bigg\rangle \\
&+ \mu \|\gamma(u,v)\|^\sigma + \mu_i^g \|\gamma(u,v)\|^\sigma + \mu_i^h \|\gamma(u,v)\|^\sigma \\
&+ \mu_i^\psi \|\gamma(u,v)\|^\sigma + \mu_i^\theta \|\gamma(u,v)\|^\sigma.
\end{aligned}$$

Using (23), it follows that

$$\begin{aligned}
F(u) - F(v) + &\sum_{i \in I_g} \xi_i^g g_i(u) - \sum_{i \in I_g} \xi_i^g g_i(v) + \sum_{i=1}^p \xi_i^h h_i(u) - \sum_{i=1}^p \xi_i^h h_i(v) \\
&- \sum_{i=1}^l \xi_i^\phi \phi_i(u) + \sum_{i=1}^l \xi_i^\phi \phi_i(v) - \sum_{i=1}^l \xi_i^\theta \theta_i(u) + \sum_{i=1}^l \xi_i^\theta \theta_i(v) \geqslant 0.
\end{aligned}$$

Using the feasibility of u and v for MPEC and MWDMPEC, respectively, we obtain

$$F(u) \geqslant F(v),$$

and the proof is complete. □

Theorem 4.5. *(Strong Duality) If $\tilde{u}$ is a global optimal solution of MPEC such that the NNAMCQ is satisfied at $\tilde{u}$ and index sets $I_g, \delta, \zeta, \alpha$ defined accordingly. Let F, g_i $(i \in I_g)$, h_i $(i \in j^+)$, $-h_i$ $(i \in j^-)$, ϕ_i $(i \in \delta^- \cup \zeta_\theta^-)$, $-\phi_i$ $(i \in \delta^+ \cup \zeta_\theta^+ \cup \zeta^+)$, θ_i $(i \in \alpha^- \cup \zeta_\phi^-)$ and $-\theta_i$ $(i \in \alpha^+ \cup \zeta_\phi^+ \cup \zeta^+)$ fulfill the assumption of Theorem 4.4. Then, there exists $\tilde{\xi}$, such that $(\tilde{u}, \tilde{\xi})$ is a global optimal solution of MWDMPEC and corresponding objective values of MPEC and MWDMPEC are equal.*

Proof. The proof is similar to the proof of Theorem 4.3. □

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