

# THE ESTIMATION OF THE NUMBER OF OE-CHAINS AND REALIZABLE OE-ROUTES FOR CUTTING PLANS WITH COMBINED CONTOURS

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**Abstract:** Cuts combination of plane details leads to optimization of material use and shortening of cutting time. However, the problem of cutting plan realization arises. This plan has to satisfy the following restrictions: (1) part cut off a sheet does not require further cuts (constructing of OE-route); (2) there are some restrictions on placement of pierce points (constructing of PPOE-cover). In this paper we consider the problem of enumeration the OE-chains for cutting plan with combined cuts and necessary conditions of realizable cutting.

**Keywords:** Ordered Enclosing, Pierce point placement, Enumeration, Routing problem, Cutting, Technological restrictions, Cut realization.

**MSC:** .

## 1. INTRODUCTION

Laser cutting is one of the major cutting processes used to manufacture sheet metal products. Hence, the routing problem for a cutting tool is actual. The task of determining the trajectory is to define the exact sequence of cuts. The development of industrial automation has led to the emergence of numerical control technology

equipment used for cutting sheet materials. New technologies allow cutting along an arbitrary path with sufficient accuracy.

The advantage of using laser cutting is the minimum of such indicators as cutting width and thermal deformations.

The main constraints for laser cutting are:

(1) all elements of the inner contours must be cut before the outer contour is completely covered (OE-chains [1]);

(2) avoid crossings of the cutting trajectory, touches are allowed (NOE-chains [2, 3, 24]);

(3) thermal effects should be taken into account because heating of a metal sheet occurs during cutting [4];

(4) constraints on placing the pierce point (constructing of PPOE-cover [5]);

(5) the summary time need for cutting (time for cut, time for air moves, time for piercing etc.) should be shortened.

Papers [2] and [6] contain classification of routing problems for cutting tool, and their authors admit that ECP (Endpoint Cutting Problem) and ICP (Intermittent Cutting Problem) technologies allow to reduce the waste of material, shorten the lengths of cutting and air moves due to the possibility of combining the boundaries of the cut out parts. The problems of reducing material waste and maximizing combination of the detail fragment contours are solved at the stage of creating the cutting plan.

Despite the obvious advantages of the ECP and ICP technologies, currently the majority of papers (for example, [2, 7, 8, 9, 10, 11, 12, 13, 14]) are devoted to the development of the GTSP (General Travelling Salesman Problem) or CCP (Continuous Cutting Problem) technologies which do not involve the combination of the details contours. Thus, when using these technologies, the length of the trajectory is equal to the sum of the perimeters of all the contours, and the number of pierce points is equal to the number of contours.

A number of papers [15, 16, 17] discuss constructing the efficient algorithms for cutting plans in which combining the contours is allowed. To solve this problem the authors of [15] use graph theory, and the algorithm considered in this paper allows to construct additional edges between odd degree vertices. Nevertheless, algorithm from [15] is not enough formalized and proved to be the "one algorithm for solving routing problem with some restrictions". This algorithm works only for plane graphs admitting completion to a plane Euler graph. In [17] the author considers the same problem as in [15] but formalized it in details giving the definition of a route with restrictions for a plane graph (this type of routes is called OE-routes) and not only the algorithm for solving the problem for Eulerian graph but also for solving Chinese Postman Problem (CPP). Thus, algorithm for obtaining the solution of this problem is given for a plane connected graph.

The recent advances are given in our later papers [3, 5, 18]. In [3] we consider the algorithm that transforms the initial plane Eulerian graph to a plane 4-regular graph for which the routing problem can be solved in polynomial time by constructing an AOE-chain, i.e. the chain where adjacent edges are neighbours in the cyclic order relatively their common vertex, and the inner part of passed

cycle does not contain not passed edges. This transformation means splitting of all vertices of degree  $2k$ ,  $k \geq 3$  to  $k$  fictive ones and adding a cycle containing  $k$  fictive edges incident to these vertices. Then, we construct the AOE-chain for the obtained 4-regular graph. This trail is to be the NOE-chain (non-intersecting OE-chain) for the initial graph after absorbing all the fictive edges and vertices obtained while splitting vertices of the initial graph. Some relevant algorithms and proofs are discussed in [5, 18].

To solve the routing problem, the cutting plan may be represented as a plane graph as following [18].

Let plane  $S$  be a model of a metal sheet, and plane graph  $G$  be a model of a cutting plan. Let  $E(G)$  be a set of edges of graph  $G$ , which are plane Jordan curves with pairwise disjoint interiors homeomorphic to open segments. The set of vertices  $V(G)$  is represented by the set of bounding points of these curves.

Let for any  $J \subseteq G$  (part of a cutter trajectory) the theoretical-set union of its inner faces be designated as  $\text{Int}(G)$  (the union of all its components  $S \setminus J$  without outer face). Then  $\text{Int}(G)$  may be interpreted as a part cut of a sheet. Let the initial part of a route in graph  $G$  be considered as a part of a graph containing all vertices and edges belonging to this part of a route. This allows formalizing the claim to a cutter as a condition of absence of the initial route part inner faces of graph  $G$  intersection with graph  $G$  edges not yet included to any chain yet. Such type of routes is called OE-routes [19].

We define the following functions for each edge  $E \in E(G)$  to represent the image of cutting plan as a plane graph  $G = (V, E)$  [17]:

- $v_k(e)$ ,  $k = 1, 2$ , be vertices incident to edge  $e$ ;
- $f_k(e)$  be a face placed on the right-hand side when moving over edge  $e$  from vertex  $v_k(e)$  to vertex  $v_{3-k}(e)$ ,  $k = 1, 2$ ;
- $l_k(e)$  be the edge incident to face  $f_{3-k}(e)$  and  $v_k(e)$ ,  $k = 1, 2$ ;
- $r_k(e)$  be the edge incident to face  $f_k(e)$  and  $v_k(e)$ ,  $k = 1, 2$ .

As functions  $v_k(e)$ ,  $f_k(e)$ ,  $l_k(e)$ ,  $r_k(e)$ ,  $k = 1, 2$ , constructed on graph  $G = (V, E)$  edges define incident vertices for each edge, incident faces, and adjacent edges, the following statement holds. Functions  $v_k(e)$ ,  $f_k(e)$ ,  $l_k(e)$ ,  $r_k(e)$ ,  $k = 1, 2$  constructed on graph  $G = (V, E)$  edges define plane graph  $G = (V, E)$  up to homeomorphism [1].

The space complexity of such a representation is  $O(|E| \cdot \log_2 |V|)$ . There is no problem to get these functions. In fact, they are defined at stage of interpreting the cutting plan in terms of graph  $G$ . This is minimal information needed to represent of any plane graph up to homeomorphism. Using the known coordinates of images of graph  $G = (V, E)$  vertices, and nesting fragments of the cutting plan (the images of graph  $G = (V, E)$  edges) any route in graph  $G$  may be interpreted as a tool trajectory.

The data structures used for storage the cutting plan data and information of the plane graph is given in [27].

The aim of our paper is to consider the enumeration problem of the OE-chains for cutting plan with combined cuts and to give necessary conditions of realizable cutting.

Section 2 is devoted to main definitions, data representation, and brief review of basic algorithms for OE-chains constructing. Here we present the theorem on the cardinality of OE-cover for a plane graph, and discuss the parallel algorithm for getting the minimal possible cardinality OE-cover.

In section 3 we introduce the definitions of PPOE-chains and discuss the necessary conditions of cutting realization. Let us mention that by constructing PPOE-cover for graph  $G$ , we solve the routing problem for a cutter with restrictions on pierce point placement.

## 2. OE-ROUTES FOR PLANE GRAPHS

For the convenience of the reader, we give the basic definitions and key algorithms from our earlier papers.

### 2.1. OE-cycles for Eulerian graphs

**Definition 1.** [20, 19] Let chain  $C = v_1e_1v_2e_2\dots v_k$ ,  $0 < k < |E(G)|$  and  $\text{Int}(v_1e_1v_2e_2\dots e_l) \cap E(G) = \emptyset$ ,  $1 \leq l \leq k$  be called ordered enclosing (or OE) chain.

**Theorem 2.** [20, 19] Let  $G$  be a plane Eulerian graph. Eulerian OE-cycle  $C = v_1e_1v_2e_2\dots v_{|E(G)|-1}e_{|E(G)|}v$  exists for any vertex  $v \in V(G)$  incident to an outer face of  $G$ .

Let us consider the enumeration problem for Eulerian OE-chains in plane Eulerian graph. This problem is of practical interest when it is necessary to define all the possible starting pierce points when the sequence of cutting is fixed. So, after running algorithm of OE-chain construction we may choose any of the permissible pierce points to get the OE-chain has started from it.

**Definition 3.** Graph of allowed transitions  $T_G(v)$  for vertex  $v \in V(G)$  be a graph whose vertices are the edges incident to vertex  $v$ , i.e.  $V(T_G(v)) = E_G(v)$ , and the set of edges be represented by allowed transitions between the edges.

**Definition 4.** System of allowed transitions  $T_G$  (or shortly the system of transitions) be a set

$$\{T_G(v) \mid v \in V(G)\}$$

where  $T_G(v)$  be transitions graph of vertex  $v$ .

Let us consider some statements proved in [23] but now given in English.

Let  $G$  be a plane Eulerian graph and  $C(G)$  be OE-chain,  $T_{C(G)}$  be the system of transitions for chain  $C(G)$ . Proposition 5 gives the estimation of the number of OE-chains with a fixed transition system  $T_C(G)$ .

**Proposition 5.** *Let  $G(V, E)$  be a plane Eulerian graph without cut-vertices and  $C(G)$  be OE-chain with system of transitions  $T_C(G)$ . Then the number of OE-chains  $N$  for  $T_C(G)$  satisfies the inequality  $1 \leq N \leq 2 \cdot |V(f_0)|$ ,  $V(f_0) = \{v \mid v \in f_0\}$ , and both upper and lower bounds are reachable.*

*Proof.* The OE-chain  $C(G)$  exists due to theorem 2 [20]. Hence, the lower bound is reachable. Let us fix the transition system  $T_C(G)$  for OE-chain  $C(G)$ . All vertices of set  $V(f_0)$  for this transition system may be divided into two classes:

$$V_1 = \{v : E(T_G(v)) \in \{e_1, e_2\} : e_1, e_2 \in f_0\}$$

and

$$V_2 = \{v : E(T_G(v)) \in \{e_1, e_2\} : e_1, e_2 \notin f_0\}.$$

A transition system for  $V \in V_1$  allows not more than two OE-chains starting from the edges bounding outer face. If we assume that a chain starts from an edge not belonging to the outer face then it finishes at the edge not belonging to outer face which means that it is not an OE-chain. Vice versa, for vertices of set  $v \in V_2$  constructing OE-chain is possible only if we start by an edge not belonging to the outer face. Otherwise, after returning to a chosen vertex  $v$  at least one edge (that is not adjacent to the outer face) is to be enclosed. Thus, the fixed transition system allows not more than  $2 \cdot |V(f_0)|$  of OE-chains. Let us show that this estimation is reachable. Let us consider the graph in figure 1. OE-

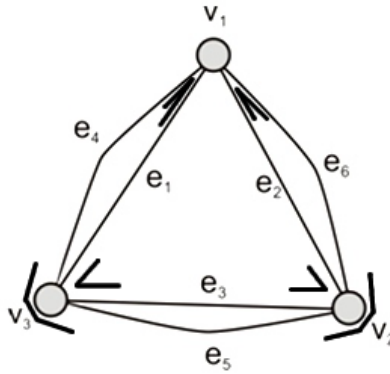


Figure 1: The example of graph with the system of non-intersected transitions

chain  $C_{1,1} = v_1 e_1 v_3 e_3 v_2 e_2 v_1 e_6 v_2 e_5 v_3 e_4 v_1$  for this graph induces transition system  $T_{C_{1,1}}(G) = \{T_G(v_1), T_G(v_2), T_G(v_3)\}$  where

$$V(T_G(v_1)) = \{e_1, e_2, e_4, e_6\}; E(T_G(v_1)) = \{\{e_1, e_4\}, \{e_2, e_6\}\};$$

$$V(T_G(v_2)) = \{e_2, e_3, e_5, e_6\}; E(T_G(v_2)) = \{\{e_2, e_3\}, \{e_5, e_6\}\};$$

$$V(T_G(v_3)) = \{e_1, e_3, e_4, e_5\}; E(T_G(v_3)) = \{\{e_1, e_3\}, \{e_4, e_5\}\}.$$

There exists one more *OE*-chain

$$C_{1,2} = v_1 e_2 v_2 e_3 v_3 e_1 v_1 e_4 v_3 e_5 v_2 e_6 v_1$$

for  $v_1$ . Wherein for  $v_2 \in f_0$  *OE*-chains

$$C_{2,1} = v_2 e_6 v_1 e_2 v_2 e_3 v_3 e_1 v_1 e_4 v_3 e_5 v_2$$

and

$$C_{2,2} = v_2 e_5 v_3 e_4 v_1 e_1 v_3 e_3 v_2 e_2 v_1 e_6 v_2$$

satisfy the transition system  $T_{C_{1,1}(G)}$ , and for vertex  $v_3$  the following *OE*-chains

$$C_{3,1} = v_3 e_4 v_1 e_1 v_3 e_3 v_2 e_2 v_1 e_6 v_2 e_5 v_3$$

and

$$C_{3,2} = v_3 e_5 v_2 e_6 v_1 e_2 v_2 e_3 v_3 e_1 v_1 e_4 v_3$$

satisfy it. Thus, a graph with three vertices has six  $T_C(G)$ -compatible *OE*-chains.

Let us consider the same graph with another transition system  $T_C(G)$  (fig. 2). The main difference of this transition system from the one in figure 1 is the inter-

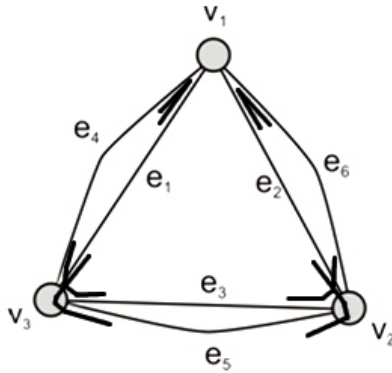


Figure 2: The example of graph transition system of which has intersections

sected transitions at vertices  $v_2$  and  $v_3$ .

In this case, graph has only one *OE*-chain

$$C = v_2 e_3 v_3 e_4 v_1 e_1 v_3 e_5 v_2 e_2 v_1 e_6 v_2$$

for the defined  $X_C(G)$ . If we choose either vertex  $v_1$  or  $v_3$  we obtain that cycle  $v_1 e_1 v_3 e_5 v_2 e_2 v_1$  encloses edge  $e_3$ , which is not passed. Let us note that only  $e_3$  may be chosen as the starting edge in this case. Hence the lower bound is reachable.  $\square$

According to a practical point of view, graphs whose upper bound are reachable have particular interest. In the proof of proposition 5 (see [22]) it is clear that not every OE-cycle induces a transition system for which the upper bound is reachable. We also note that in order to find an appropriate transition system for which the upper bound is reached, it is not enough to know only the initial vertex and the initial edge.

Let us enumerate OE-chains for transition system of an A-chain [22].

**Definition 6.** [24] *Let for Eulerian chain*

$$T = v_0, k_1, v_1, \dots, k_n, v_n, v_n = v_0$$

*in graph  $G = (V, E)$  the cyclic order  $O^\pm(v)$  is given for each vertex  $v \in V$ . This order defines the transition system  $A_G(v) \subset O^\pm(v)$ . If  $\forall v \in V(G) A_G(v) = O^\pm(v)$ , let transition system  $A_G(v)$  be called **full transition system**.*

**Definition 7.** [24] *Eulerian  $A_G$ -compatible chain  $T$  be called **A-chain**. Hence the sequent edges of trail  $T$  (incident to vertex  $v$ ) be the neighbours in cyclic order  $O^\pm(v)$ .*

The following statement on OE-cycles [23] holds for transition system of A-chain.

**Theorem 8.** *Let plane graph  $G = (V, E)$  without cut-vertices has an A-chain  $C(G)$ , and  $T_C(G)$  be the transition system for this chain. If  $V(f_0)$  is a set of vertices adjacent to outer face, then there are  $2 \cdot |V(f_0)|$  of OE-cycles for  $T_C(G)$ .*

*Proof.* The proof that A-chain beginning and ending at vertex  $v_0 \in f_0$  is also OE-cycle is presented in [24].

Let us count the number of OE-cycles for the fixed transition system. Any OE-cycle starts from  $v \in f_0$  and finishes at  $e \in f_0$  [24]. According to the statement of this theorem no vertex  $v_j \in f_0, j = 1, \dots, |v(f_0)|$  is a cut-vertex, hence, there are exactly two incident edges adjacent to the outer face. As soon as the transition system corresponds to an A-chain and if vertex  $v_j$  can be reached by one of these edges, then the other edge is used for leaving this vertex. If both of these edges are used only to reach  $v_j$ , then the OE-condition is violated (in this case, one of these entering  $v_j$  edges is passed earlier than some inner edges). If both edges are used for leaving  $v_j$ , then the transition system  $X_T(G)$  will have intersections. Such a transition system does not satisfy any A-chain.

As soon as A-chain is a closed sequence of vertices and edges, it may start from any vertex  $v_j \in f_0$ . If  $v_j$  is the last vertex of an OE-chain, then it is necessary that  $e_{j-1} \in f_0$  is sequence  $e_{j-1}v_j e_j$ . Actually, otherwise, the last edge  $e_{j-1}$  of an OE-chain will be enclosed by a cycle of edges adjacent to the outer face. If we take some vertex  $v \in V(f_0)$  as a starting vertex of a chain, then according to the predefined cyclic order  $O^\pm(G)$  we may choose one of two incident edges to leave this vertex. Hence, there are two OE-cycles for any vertex  $v \in V(f_0)$ . Since

there are  $|V(f_0)|$  vertices adjacent to the outer face then the number of  $OE$ -cycles satisfying the transition system of an  $A$ -chain is equal to  $2 \cdot |V(f_0)|$ .  
 $\square$

If there are some cut-vertices for graph  $G(V, E)$ , then the following statement holds for transition system  $T_C(G)$  of  $A$ -chain for this graph [23].

**Theorem 9.** *Let plane graph  $G = (V, E)$  has  $K$  cut-vertices  $v_1, \dots, v_K \in f_0$  and let there exists  $A$ -chain  $T$ . Let  $T_C(G)$  be the transition system of chain  $C(G)$ , and  $V(f_0)$  be the set of vertices adjacent to outer face. There exist*

$$2 \cdot |V(f_0)| + \sum_{i=1}^K (\deg(v_i) - 2)$$

$OE$ -cycles for  $T_C(G)$ .

*Proof.* According to theorem 8, the plane graph  $G$  without cut-vertices has exactly  $2 \cdot |V(f_0)|$   $OE$ -cycles for transition system  $T_C(G)$  satisfying any  $A$ -chain. Let  $v_i \in V(F_0)$  be a cut-vertex with degree  $\deg(v_i) = 2 \cdot M_i$ . There are exactly  $M_i$  edges in a cyclic order by which chain reaches this vertex and the same number of edges by which it leaves this vertex. One pair of edges is already counted for  $|V(f_0)|$ , and  $M_i - 1$  ways of beginning the  $OE$ -chain are not taken into account. Summarizing by all cut-vertices, we get the expression from formulation of the theorem.  $\square$

Let us note that if  $T_C(G)$  does not satisfy any  $A$ -chain, then the upper bound cannot be reached even if chain  $C$  is a non-intersecting. This is confirmed by the example shown in the figure 3.

There is no  $A$ -chain in this graph, nevertheless, it is possible to define the system of non-intersecting transitions  $T_C(G)$ . For this graph, given the  $T_C(G)$  transition system shown in Figure 3, there are only five  $OE$ -chains starting at different vertices of the outer face:

$$C_1 = v_0 e_7 v_1 e_1 v_0 e_2 v_1 e_4 v_2 e_5 v_1 e_6 v_2 e_3 v_0 e_9 v_2 e_8 v_0;$$

$$C_2 = v_0 e_8 v_2 e_9 v_0 e_3 v_2 e_6 v_1 e_5 v_2 e_4 v_1 e_2 v_0 e_1 v_1 e_7 v_0;$$

$$C_3 = v_1 e_1 v_0 e_2 v_1 e_4 v_2 e_5 v_1 e_6 v_2 e_3 v_0 e_9 v_2 e_8 v_0 e_7 v_1;$$

$$C_4 = v_2 e_3 v_0 e_9 v_2 e_8 v_0 e_7 v_1 e_1 v_0 e_2 v_1 e_4 v_2 e_5 v_1 e_6 v_2;$$

$$C_5 = v_2 e_9 v_0 e_3 v_2 e_6 v_1 e_5 v_2 e_4 v_1 e_2 v_0 e_1 v_1 e_7 v_0 e_8 v_2.$$

When constructing a chain from the vertex  $v_1$ , it is possible to construct a chain starting either from the edge  $e_1$  (in this case the chain  $C_3$  will be constructed, the



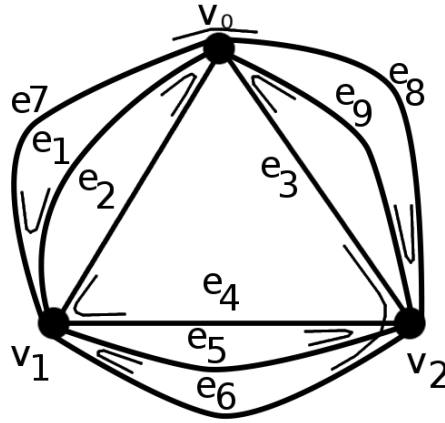


Figure 3: The example of graph with the system of non-intersecting transitions not permitting the  $A$ -chain

last edge of which will be  $e_7$ ), or from the edge  $e_5$  (in this case the last chain will have edge  $e_6$ , however the constructed chain

$$C_6 = v_1 e_5 v_2 e_4 v_1 e_2 v_0 e_1 v_1 e_7 v_0 e_8 v_2 e_9 v_0 e_3 v_2 e_6 v_1$$

will not be a  $OE$ -chain, because the edges  $e_9$  and  $e_3$ , by the time of their inclusion to the chain, will have already been enclosed by a cycle of passed edges).

A common transition system  $T_C(G)$  of any  $OE$ -chain may have intersections (the example of a chain satisfying the transition system with intersections is presented in figure 2). Thus, the number of  $OE$ -chains lays between 1 and  $2 \cdot |V(f_0)|$ . Hence, it is possible to choose any of these  $OE$ -chains as a route of a cutting tool.

2.2.  $OE$ -Routes for Any Plane Graphs

**Definition 10.** [19] Let the ordered sequence of edge-disjoint  $OE$ -chains be

$$C^0 = v^0 e_1^0 v_1^0 e_2^0 \dots e_{k_0}^0 v_{k_0}^0, \tag{1}$$

$$C^1 = v^1 e_1^1 v_1^1 e_2^1 \dots e_{k_1}^1 v_{k_1}^1, \dots, \tag{2}$$

$$C^{n-1} = v^{n-1} e_1^{n-1} v_1^{n-1} e_2^{n-1} \dots e_{k_{n-1}}^{n-1} v_{k_{n-1}}^{n-1}, \tag{3}$$

covering graph  $G$  and such that

$$(\forall m : m < n), \quad \left( \bigcup_{l=0}^{m-1} \text{Int}(C^l) \right) \cap \left( \bigcup_{l=m}^{n-1} C^l \right) = \emptyset,$$

be called a cover with ordered enclosing ( $OE$ -cover).

Routes realizing *OE*-cover represent the ordered set of *OE*-chains and contain additional idle passes (edges) between the end of current chain and beginning of the next. Constructing of *OE*-cover for graph  $G$  solves the given routing problem. The most interesting covers are those with minimal number of chains, and their length, because of the transition from one chain to another, corresponds to an idle pass of a cutter.

**Definition 11.** [19] *Let minimal cardinality ordered sequence of edge-disjoint OE-chains for plane graph  $G$  be called **Eulerian cover with ordered enclosing** (Eulerian OE-cover).*

Constructing of an OE-route for graph  $G$  solves the above stated tool path problem when a part cut of a sheet does not require any further cuts (see constraint (1)). Routes with a minimum number of chains are of the major interest since the transition from one chain to another corresponds to the air move.

It is common that the estimation of Eulerian *OE*-cover cardinality is given by the following theorem.

**Theorem 12.** *Let  $G$  be a plane connected graph,  $V_{odd}(G)$  be the set of its odd degree vertices then the inequality*

$$k = \frac{|V_{odd}(G)|}{2} \leq N \leq |V_{odd}(G)| = 2k$$

*holds for cardinality  $N$  of Eulerian OE-cover of graph  $G$ . The upper and the lower bounds are reachable.*

*Proof.* According to Listing-Luke theorem, the lower bound cannot be less than  $k$ . This bound is reached for bridgeless graphs with at least one odd degree vertex adjacent to the outer face (see algorithm `OEcover`). There is an algorithm in [19] constructing the ordered sequence of *OE*-chains and covering bridgeless graphs by not more than  $k + 1$  chains. Routes realizing this cover contain additional edges, those between the end of a current chain and the beginning of the next one.

The reachability of the upper bound is illustrated by an example in figure 4. As a matter of fact, any of odd degree vertices  $v_1^*, v_2^*, \dots, v_{2k}^*$  may be only the beginning of a covering *OE*-chain because the route ending at any of these vertices cannot be an *OE*-route.

Thus, the cardinality of Eulerian *OE*-cover (i.e. minimal cardinality cover) for this example is less than  $2k$ .

Let us consider the process of constructing the *OE*-cover, where each of these vertices is a beginning of a chain, to prove that  $2k$  is the exact upper bound for Eulerian *OE*-cover cardinality.

Algorithm `Parallel OE-Cover` works parallel. Let us organize  $2k$  processes starting from vertices  $v_1^*, v_2^*, \dots, v_{2k}^*$ . We begin constructing *OE*-chains by procedure `ParallelFormChain()` from vertices  $v_1^*, v_2^*, \dots, v_{2k}^*$ . We use global variable `cur_rank` to synchronize these processes. Procedure waits for continuing of a chain constructing if rank of a current edge is less than `cur_rank`.

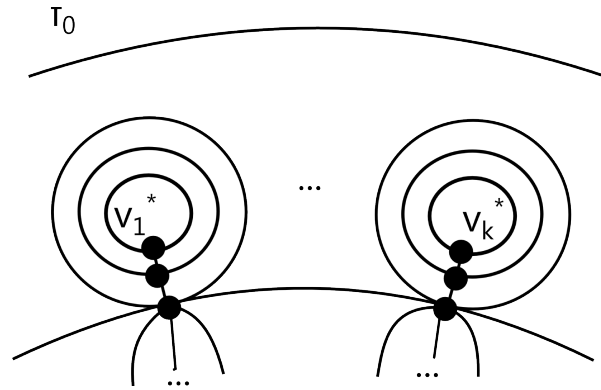


Figure 4: The example of graph all the vertices of odd degree of which are to be the beginnings of covering it OE-chains

**Procedure ParallelFormChain**

**Extern:** *cur\_rank* be a synchronizer by edges ranks; **In:** *w* be the first vertex of a chain; **Out:** *v* be the last vertex of a current chain.

*v* ← *w*; *e* ← *Stack*(*v*);

**do**

*e*<sub>1</sub> = arg max<sub>*e* ∈ *Q*(*v*)</sub> rank(*e*);

*e*<sub>2</sub> = arg max<sub>*e* ∈ *Q*(*v*): *f*<sub>1</sub>(*e*) = *f*<sub>2</sub>(*e*)</sub> rank(*e*);

// Find the maximal rank edge that is not bridge if possible

**If**(rank(*e*<sub>1</sub>) = rank(*e*<sub>2</sub>)) *e* ← *e*<sub>2</sub>;

**Else** *e* ← *e*<sub>1</sub>;

**EndIf**

**Wait**(rank(*e*) = *cur\_rank*);

**If**(*e* ∈ *E*(*G*))

*E*(*G*) ← *E*(*G*) \ {*e*}; // Delete edge *e* and unite faces separated by *e*

**If**(*v* = *v*<sub>1</sub>(*e*))

REPLACE(*e*); //Replacement of indices of functions for edge *e* from *k*

to 3*k*, *k* = 1, 2.

**EndIf**

*Trail*<sub>*w*</sub> ← *Trail*<sub>*w*</sub> ∪ {*e*}; *v* ← *v*<sub>1</sub>(*e*);

**EndIf**

**While** ((*v* ∉ *V*<sub>odd</sub>) ∧ (*Q*(*v*) ≠ ∅));

**return** *v*;

**EndProcedure**

Let us add one edge to each chain at each stage.

Each of running processes returns either the vertex of odd degree or the vertex incident to outer face. One needs to order the chains obtained after finishing

Table 1: Algorithms for OE-routes constructing

Route type	Comp. complexity
Eulerian OE-cycle (alg. Recursive_OE) [20]	$O( V ^2)$
Eulerian OE-cycle (alg. OE-Cycle) [19], [21]	$O( E  \cdot \log_2  E )$
OE-Postman Route (alg. CPP_OE) [17]	$O( E  \cdot  V )$
OE-Router [1]	$O( E  \cdot \log_2  E )$
Optimal OE-Router [1]	$O( V ^2)$
MultiComponent (OE-Cover for Disconnected graph [18])	$O( E  \cdot \log_2  E )$
DoubleBridging (OE-Cover for Disconnected graph [18])	$O( E  \cdot \log_2  E )$

processes by decreasing the rank of starting vertex  $v_1^*, v_2^*, \dots, v_{2k}^*$ . This can be summarized as algorithm **Parallel OE-Cover**.

**Algorithm Parallel OE-Cover**

**In:**  $G = (V, E)$  be a plane graph;  $V_{odd} \subseteq V$  be a set of graph  $G$  odd degree vertices;

**Out:** *Trail* be *OE*-cover as an ordered array of edges;

Initiate();

Order();

SortOdd(); // *Sorting of odd degree vertices list by decreasing of their rank*

**For each**  $w \in V_{odd}$  **do parallel**

$cur\_rank \leftarrow \max_{v \in V_{odd}} \text{rank}(Q(v));$  // *Synchronising of processes*

ParallelFormChain( $w, v$ ); // *Constructing of OE-chain*

**EndFor**

$Trail \leftarrow Trail(v_1) \bullet Trail(v_2) \bullet \dots \bullet Trail(v_{2k});$

**End**

Thus, not more than  $2k$  chains are constructed.  $\square$

So, the cardinality of cover is affected by existing of bridges in graph. If we have a bridgeless graph, then the lower bound is reached if at least one odd degree vertex is adjacent to outer face. If there is no such vertices, then the cardinality of cover is one higher than the value of the lower bound.

Table 1 contains the list of algorithms used for constructing of *OE*-routes for different types of graphs.

It is easy to see that all these algorithms run in polynomial time. Nevertheless, they do not consider any constraints concerning starting vertices of any chain. So, the cover satisfying the considered restrictions and optimality criteria may not be unique. There are two ways of choosing the feasible one:

- to choose any of these covers;
- to introduce some additional criteria and choose the feasible cover satisfying them.

### 3. THE NECESSARY CONDITIONS OF CUTTING REALIZATION

The problem of feasibility of cutting with plasma cutting technology for solving the cutting-packing problem arises due to different restrictions. We consider the following restrictions: all elements of the inner contours must be cut before the outer contour is completely covered (OE-chains); and avoid crossings of the cutting trajectory, touches are allowed. The problem of minimization of pierce points number and the problem of their placement while constructing the cutter path also arises.

The minimal number of pierce points for a cutting map represented by a plane connected graph is greater than  $|V_{odd}|/2$  (see theorem 12). In this section we consider only graphs coverable by  $|V_{odd}|/2$  chains, i.e. bridgeless graphs with at least one vertex incident to outer face.

Let us formalize the problem of pierce points placement.

Let faces  $F_{in}(G) \subset F(G)$  allow placement of pierce point. Then let us designate odd vertices incident to  $F_{in}(G)$  as  $V_{in}(G) \subset V(G)$ . So, the partition  $V_{in}$  and  $V_{out}$  is defined on the set of vertices. We designate  $V_{in}$  as the set of pierce points, and  $V_{out}$  as the set of leaving points.

If a route in graph is *OE*-route and starting vertex  $v_1 \in V_{in}(G)$  then this route may be a base for constructing the cutting program for plasma cutting machine. The end of such chain is in  $V_{in} \cup V_{out}$ . This type of routes we call *PPOE*-routes [25, 26]. Constructing *PPOE*-cover for graph  $G$  solves the routing problem for a cutter with restrictions on pierce points placement.

**Definition 13.** Let chain  $C = v_1e_1v_2e_2 \dots v_k$  be called *PPOE-chain* if it is an *OE*-chain and starting vertex  $v_1 \in V_{in}(G)$ .

**Definition 14.** Let *PPOE-cover* of graph  $G$  be the *OE*-cover of graph  $G$  consisting of *PPOE*-chains.

**Definition 15.** Minimal cardinality ordered sequence of edge-disjoint *PPOE*-chains in plane graph  $G$  is called **Eulerian PPOE-cover**.

The problem of defining the possibility of cutting by plasma instrument may be stated as a problem of checking the existence of Eulerian *PPOE*-cover for a given graph. According to the mentioned earlier restrictions we may formulate the following necessary condition of *PPOE*-cover existence [26].

**Proposition 16.** Let  $G$  be a plane graph with  $2k$  odd degree vertices. If there exists Eulerian *PPOE*-cover, then  $|V_{in}(G)| \geq k$ .

Paths realizing *PPOE*-cover can be represented by a special way ordered set of *PPOE*-chains with additional idle paths (edges) between the end of the current chain and beginning of the next one. Such transitions form a matching on a bipartite oriented graph  $D = (V_{in} \cup v_{out} \rightarrow V_{in}, E)$ , where  $V_{in}$  is a set of odd degree vertices allowed to be the beginnings of trails (pierce points);  $V_{out}$  is a set of odd degree vertices allowed to be only the ends of constructed chains (leaving points).

**Proposition 17.** *It is necessary for the existing PPOE-cover of a mixed graph  $G \cup D$  to have a cycle wherein all additional arcs belong*

$$\{(v, u) : v \in V_{out} \cup V_{in}, u \in V_{in}\}.$$

**Proposition 18.** *It is necessary for the existence of a PPOE-cover of a plane connected graph  $G$  that cardinality of minimal  $\{V_{in}, V_{out}\}$ -cut be not greater than  $|V_{out}|$ .*

The proofs of these propositions are considered in [26], hence, we shall omit them.

### 3.1. Minimizing the Number of Pierce Points

The technology of plasma cutting claims presence of some space to place a pierce point. The image of cutting plan is a plane graph  $G$  with specified list  $L$  of faces allowing piercing. The outer face  $f_0 \in L$ .

Let us consider the following sequence of steps to minimize the number of pierce points for the given cutting plan.

**Proposition 19.** *If the number of odd degree vertices  $|V_{odd}(L)|$  incident to each  $f \in L$  is even, then it is possible to add edges without losing graph  $G$  planarity.*

*Proof.* When each  $f \in L$  is a connected set, then there exist Jordan curves connecting any pair of points of  $f$ , i.e. it is possible to put in correspondence any matching on the set of adjacent to  $f$  odd vertices without losing the planarity of the initial graph  $G$ .  $\square$

Proposition 19 means that it is possible to minimize the number of pierce points by connecting the odd degree vertices in  $L$  without vanishing the planarity of  $G$ . After that we get modified planar Eulerian graph  $\tilde{G}$  for which  $OE$ -cycles exist. Hence the  $OE$ -cover for  $G$  may be obtained from  $OE$ -cycle of  $\tilde{G}$  by replacing the additional edges between the odd degree vertices.

If there exist faces  $F^* \in L$  so that for any  $f \in F^*$   $|V_{odd}(f)| = 2k + 1$ ,  $k = 0, 1, 2, \dots$ , then after applying proposition 19 to each  $f \in F^*$  we obtain a planar graph  $G^*$  whose each face does not have more than one vertex of odd degree.

Hence, the following proposition holds.

**Proposition 20.** *Let  $G^*$  be a plane graph whose each face  $f \in L$  does not have more than one vertex  $v \in V_{odd}$  of odd degree. Then minimal number of  $OE$ -chains of cover constructed for  $G^*$  is equal to  $|V_{odd}(G^*)|/2$ .*

The proof of this proposition is obvious, and the minimal number of chains for  $G^*$  is equal to the lower bound mentioned in theorem 12.  $OE$ -chains may be constructed by any algorithm from table 1.

#### 4. CONCLUSIONS and SUGGESTIONS

Nowadays lots of researches on sheet metal cutting are devoted to constructing the cutting routes. There are some known technologies of cutting leading to usage of different routing algorithms. In our paper we discussed the technology allowing combination of cuts, which is poorly discussed in other works. We put the technological constraints that part cut off a sheet does not require further cuts and require putting pierce points only in predefined places of the sheet. Such a problem is not discussed earlier. We showed that the problem of constructing the technologically realizable routes with minimal number of pierce points may be solved in polynomial time. Moreover, the number of pierce points may be minimized by adding the edges connecting odd vertices incident to one face. So, the obtained results are new for sheet metal cutting.

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