METHODS OF OPTIMIZATION OF HAUSDORFF DISTANCE BETWEEN CONVEX ROTATING FIGURES

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Abstract: We studied the problem of optimizing the Hausdorff distance between two convex polygons. Its minimization is chosen as the criterion of optimality. It is believed that one of the polygons can make arbitrary movements on the plane, including parallel transfer and rotation with the center at any point. The other polygon is considered to be motionless. Iterative algorithms for the phased shift and rotation of the polygon are developed and implemented programmatically, providing a decrease in the Hausdorff distance between it and the fixed polygon. Theorems on the correctness of algorithms for a wide class of cases are proved. Moreover, the geometric properties of the Chebyshev center of a compact set and the differential properties of the Euclidean function of distance to a convex set are essentially used. When implementing the software package, it is possible to run multiple times in order to identify the best found polygon position. A number of examples are simulated.

1. INTRODUCTION

In problems of control theory [1] it is often required to find the location of sets that provide the greatest degree of proximity. The most frequently considered approximations of various sets on the plane are convex [2] polygons. Naturally, the question arises of finding such motions on the plane that translate the movable polygon to the position in which it is most close to another polygon. As a criterion for evaluating the position of polygons, it is natural to choose the Hausdorff distance [3] between them. Similar problems were considered, for example, in [4]. Approximations of planar figures by convex polygons are used in the theory of pattern recognition [5]. They play an important role in emphasizing the external borders of objects on the map [6]. Recently, the problems of approximating polygons have been solved in order to recognize signs of affected tissues and internal organs [7].

Let two arbitrary polygons $A, B \subset \mathbb{R}^2$ be given. It is required to find their mutual arrangement, ensuring the minimization of the Hausdorff distance between them:

$$d(A, B) = \max\{h(A, B), h(B, A)\}.$$

where

$$h(A, B) = \max_{a \in A} \min_{b \in B} \|a - b\| \tag{1}$$

is the Hausdorff deviation of the set A from B.

We assume that the polygon A is stationary, we can apply parallel translation and rotation centered at an arbitrary point to the second polygon B (although any movement can be realized as a sequential shift and rotation centered at some given point, for example, the origin). The problem is to calculate such $x^0 \in \mathbb{R}^2$ and $\varphi^0 \in [0, 2\pi]$ for which the relation holds

$$d\left(A, \{x^0\} + \Pi(\varphi^0)B\right) = \min_{x^0 \in \mathbb{R}^2, \, \varphi \in [0, 2\pi]} d\left(A, \{x\} + \Pi(\varphi)B\right), \tag{2}$$

where

$$\Pi(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, \quad \Pi(\varphi)B = \{\Pi(\varphi)b \colon b \in B\}, \quad \varphi \in [0, 2\pi].$$

The considered problem is an extension of the questions considered earlier by the authors in [4, 8] about finding such an oriented set $\{x^0\} + B$ that the Hausdorff distance between this set and the fixed set A was minimal. Naturally, the case with the rotation of the set is much more interesting.

2. METHODS FOR SOLVING THE PROBLEM

In the general case, the solution to the problem is possible only by numerical methods. To describe these methods, the authors needed to introduce a number of auxiliary concepts.

2.1. Parallel Shift Methods

We give some definitions necessary for describing an iterative algorithm for solving the problem formulated in §1.

Definition 1. *The projection $p(a, B)$ of a point a onto a convex compact set B is the point from B closest to a in the Euclidean metric.*

Note that if the set B is not convex, then there can be more than one [2] of points closest to a, but in the case of a convex set it is always unique.

Definition 2. *The Chebyshev center [9] of the compact set $M \subset \mathbb{R}^2$ is such a point $c(M)$ that*

$$h(M, \{c(M)\}) = \inf \left\{ h(M, \{x\}) \colon x \in \mathbb{R}^2 \right\}. \tag{3}$$

The value (3) is called the Chebyshev radius $r(M)$ of the compact set $M \subset \mathbb{R}^2$.

For any convex set $A \subset \mathbb{R}^2$, the function of distance to it from the point $x \in \mathbb{R}^2$, $\rho(x, A) = \min\limits_{a \in A} \|x - a\|$, is convex. It is known from convex analysis (see, for example, [10]) that the maximum value of a convex function on a convex polygon is reached at vertices of this polygon. Thus, in the formula (1), as the maximum points of the Euclidean distance to another polygon, it is not possible to sort through all the points of the polygons, but only through vertices a_i and b_j, i.e.,

$$a_i \in A, \quad i = 1, \ldots, N_a; \quad b_j \in B, \quad j = 1, \ldots, N_b;$$

$$d(A, B) = \max \left\{ \max\limits_{i=1,\ldots,N_a} \rho(a_i, B), \max\limits_{j=1,\ldots,N_b} \rho(b_j, A) \right\}. \tag{4}$$

Here N_a and N_b are the number of vertices of the polygons A and B, respectively. In what follows a_i and b_j will be used to denote the vertices of the polygons A and B.

The basis of the iterative algorithm is a circuit consisting of alternately applying parallel translation and rotation of the polygon B. These transformations turn it into a figure congruent to it.

As in [8], the shift translating B into the new polygon $\overline{B}$ is carried out according to the formula

$$\overline{B} = B + \left\{ c\big(W(A, B)\big) \right\}, \tag{5}$$

where

$$W(A, B) = \left\{ (a_i - p(a_i, B)) \colon i = 1, \ldots, N_a \right\} \bigcup \left\{ -(b_j - p(b_j, A)) \colon j = 1, \ldots, N_b \right\}.$$

Theorem 3. *For arbitrary convex polygons A and B and the set $\overline{B}$ obtained by the formula (5), the following estimate holds*

$$d(A, \overline{B}) \leq d(A, B). \tag{6}$$

Proof. We introduce the function $F(x) = d(A, B + \{x\})$ — the Hausdorff distance between the polygons A and $B + \{x\}$. The equality (4) can be written as

$$F(\mathbf{0}) = \max \left\{ \max_{i=1,\dots,N_a} \rho(a_i, B), \max_{j=1,\dots,N_b} \rho(b_j, A) \right\}.$$

Consider the set $W(A, B)$. It consists of all vectors connecting the points of one polygon with their projections onto the second polygon. By its construction, the following equality holds

$$F(\mathbf{0}) = \max\{\|w_i\| : w_i \in W(A, B)\}.$$

We now proceed to estimate the value of $F(x)$. By construction

$$\forall i = 1, \dots, N_a \ \ (p(a_i, B) + x) \in (B + \{x\}), \quad \forall j = 1, \dots, N_b \ \ (b_j + x) \in (B + \{x\})$$

holds. Therefore, we can write the inequality for the Hausdorff distance

$$F(\mathbf{0}) = \max \left\{ \max_{i=1,\dots,N_a} \rho(a_i, B + \{x\}), \max_{j=1,\dots,N_b} \rho(b_j + x, A) \right\} \leq$$

$$\leq \max \left\{ \max_{i=1,\dots,N_a} \|a_i - (p(a_i, B) + x)\|, \max_{j=1,\dots,N_b} \|p(b_j, A) - (b_j + x)\| \right\}. \quad (7)$$

The right side of the inequality (7) can be written as

$$\zeta(x) = \max\{\|w_i - x\| : w_i \in W(A, B)\},$$

because by construction it is the maximum of the norms of vectors belonging to $W(A, B)$ and shifted by $-x$. In its structure, the function $\zeta(x)$ coincides with the Hausdorff deviation $h(W(A, B), \{x\})$. Therefore, the function reaches a strict minimum at the point $c(W(A, B))$. We obtain an inequality

$$F(x) \leq \zeta(x) \leq \zeta(\mathbf{0}) = d(A, B),$$

that coincides with (6). $\qquad\qquad\square$

Remark 4. *If $\overline{B} \neq B$, then the inequality (6) is strict. Indeed, in this case the point x is the Chebyshev center of the set $W(A, B)$ and does not coincide with* **0**. *For any point x^* that does not coincide with $x = c(W(A, B))$, the inequality $h(W(A, B), \{x^*\}) > h(W(A, B), \{x\})$ holds, since the Chebyshev center of the set in Euclidean space is unique. So, we can write inequality*

$$F(x) \leq \zeta(x) < \zeta(\mathbf{0}) = d(A, B).$$

2.2. Rotation methods

Consider first the rotation of the polygon B centered at a given point p^* by the angle φ counterclockwise. We denote the resulting polygon by:

$$\text{rotation}(B, p^*, \varphi) \triangleq \{p^*\} + \Pi(\varphi)(B - \{p^*\}),$$

and Hausdorff distance (for some given p^*) as follows:

$$\chi(\varphi) = d(A, \text{rotation}(B, p^*, \varphi)).$$

Let us single out the direction in which the polygon B should be rotated to minimize $\chi(\varphi)$.

We introduce several auxiliary functions (for some fixed center of rotation p^*):

$$f_i^a(\varphi) = \rho(a_i, \text{rotation}(B, p^*, \varphi)), \quad i = \overline{1, N_A},$$

$$f_j^b(\varphi) = \rho\big(\text{rotation}(b_j, p^*, \varphi), A\big), \quad j = \overline{1, N_B},$$

$$\sigma_i^a = \text{sign}\,\big((a_i - p^*) \wedge (p(a_i, B) - p^*)\big), \quad i = \overline{1, N_A}, \tag{8}$$

$$\sigma_j^b = \text{sign}\,\big((p(b_j, A) - p^*) \wedge (b_j - p^*)\big), \quad j = \overline{1, N_B}. \tag{9}$$

Here $(a_1, a_2) \wedge (b_1, b_2) = a_1 b_2 - a_2 b_1$ is the skew product of vectors,

$$\text{sign}(t) = \begin{cases} -1, & \text{if } t < 0, \\ 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Consider the differential properties of the functions $f_j^b(\varphi)$, $j = \overline{1, N_B}$, for which $f_j^b(0) > 0$, at the point $\varphi = 0$. In fact, these are functions of the Euclidean distance from the point located on the circle $\partial O(p^*, \|p^* - b_j\|)$ at an angle φ to the point b_j, to the set A (here $O(p^*, r)$ means a circle of radius r centered at p^*, and $\partial O(p^*, r)$ means the circle bounding it). Since the projection $p(b_j, A)$ is unique and does not coincide with b_j, the function $u_a(x) = \rho(x, A)$ in b_j has a gradient (for more details see [10])

$$\nabla u_a(b_j) = \frac{b_j - p(b_j, A)}{\|b_j - p(b_j, A)\|}.$$

Moreover, the vector v of the tangent to the circle $\partial O(p^*, r)$, where $r = \|p^* - b_j\|$, at the point b_j, codirectional with the direction of increasing the angle φ, is codirectional with the vector $b_j - p(b_j, A)$, rotated by $\pi/2$ counterclockwise (see fig. 1). Therefore, the derivative of the function in the direction v has the form (see [12])

$$\left.\frac{du_a(x)}{dv}\right|_{x=b_j} = \lim_{\lambda \to 0,\, \lambda > 0} \frac{u_a(b_j + \lambda v) - u_a(b_j)}{\lambda} =$$

$$= \langle \nabla u_a(b_j), v \rangle = \|\nabla u_a(b_j)\| \cdot \|v\| \cos \angle\big(v, \nabla u_a(b_j)\big) = \cos \angle(v, b_j - p^*). \tag{10}$$
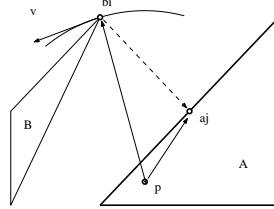
Figure 1: Polygons A, B and points b_j, $p(b_j, A)$, and p^*.

The sign $\langle \cdot, \cdot \rangle$ means the scalar product of vectors, $\angle(\cdot, \cdot)$ is the angle between them. The value (10) is positive if the rotation from the vector $b_j - p^*$ to $b_j - p(b_j, A)$ is counterclockwise, and negative if the rotation is clockwise. If $b_j - p^*$ and $b_j - p(b_j, A)$ are parallel, the expression (10) is equal to zero.

We now proceed from estimating the derivatives with respect to the directions of the function $u_a(x)$ to estimating the derivative of the function $f_j^b(\varphi)$. Consider the point $b(\varphi) = \text{rotation}(b_j, p^*, \varphi)$, into which b_j goes when it is rotated through the angle φ. For sufficiently small $\varphi > 0$, its projection $p(b(\varphi), L)$ onto the ray $L = \{l \in \mathbb{R}^2 \colon\ l = b_j + \lambda v, \lambda \geq 0\}$ lies on the perpendicular dropped from $b(\varphi)$ to a straight line containing the ray L. By construction, this means the fulfillment of equalities

$$p\big(b(\varphi), L\big) = b_j + r\sin(\varphi)v \tag{11}$$

and

$$\|b(\varphi) - p\big(b(\varphi), L\big)\| = r(1 - \cos(\varphi)). \tag{12}$$

Since the Euclidean function of the distance to the set is Lipschitz, with the Lipschitz constant 1 (see [12]), we can write the following equality holds

$$u_a\big(b(\varphi)\big) = u_a\big(p\big(b(\varphi), L\big)\big) + \omega(\varphi)\|b(\varphi) - p\big(b(\varphi), L\big)\|,$$

where $|\omega(\varphi)| \leq 1$ is true for $\forall \varphi \in \mathbb{R}$. Substituting the distance between the points from formula (12), we obtain

$$u_a\big(b(\varphi)\big) = u_a\big(p\big(b(\varphi), L\big)\big) + \omega(\varphi)(1 - \cos(\varphi))r. \tag{13}$$

If we consider the limit of the expression (13) when φ tends to zero, given the expression (11), we can write

$$\lim_{\varphi \to 0,\, \varphi > 0} \frac{u_a\big(b(\varphi)\big) - u_a(b_j)}{\varphi} =$$

$$= \lim_{\varphi \to 0,\, \varphi > 0} \frac{u_a\big(p\big(b(\varphi), L\big)\big) + \omega(\varphi)(1 - \cos(\varphi))r - u_a(b_j)}{\varphi} =$$

$$= \lim_{\varphi \to 0,\, \varphi > 0} \left(\frac{u_a\big(b_j + r\sin(\varphi)v\big) - u_a(b_j)}{\varphi} + \frac{\omega(\varphi)(1 - \cos(\varphi))r}{\varphi} \right) =$$

$$= \lim_{\varphi \to 0,\, \varphi > 0} \frac{u_a\big(b_j + r\sin(\varphi)v\big) - u_a(b_j)}{\varphi} + \lim_{\varphi \to 0,\, \varphi > 0} \frac{\omega(\varphi)(1 - \cos(\varphi))r}{\varphi}. \quad (14)$$

We individually convert each of the two limits on the right-hand side of the equality (14). For the first one, we introduce the auxiliary variable $\lambda = r\sin(\varphi)$, which has a meaning of the distance between the points b_j and $p\big(b(\varphi), L\big)$. Then, given the relation (10), we deduce

$$\lim_{\varphi \to 0,\, \varphi > 0} \frac{u_a\big(b_j + \lambda v\big) - u_a(b_j)}{\varphi} = \lim_{\varphi \to 0,\, \varphi > 0} \left(\frac{u_a\big(b_j + \lambda v\big) - u_a(b_j)}{\lambda} \cdot \frac{\lambda}{\varphi} \right) =$$

$$= \lim_{\varphi \to 0,\, \varphi > 0} \left(\frac{u_a\big(b_j + \lambda v\big) - u_a(b_j)}{\lambda} \cdot \frac{r\sin(\varphi)}{\varphi} \right) = \lim_{\varphi \to 0,\, \varphi > 0} \frac{u_a\big(b_j + \lambda v\big) - u_a(b_j)}{\lambda} \cdot$$

$$\cdot \lim_{\varphi \to 0,\, \varphi > 0} \frac{r\sin(\varphi)}{\varphi} = \lim_{\lambda \to 0,\, \lambda > 0} \frac{u_a\big(b_j + \lambda v\big) - u_a(b_j)}{\lambda} \cdot r = r\cos\angle(v, b_j - p^*).$$

For the second limit, on the right-hand side of the equality (14), having performed trigonometric transformations, we have

$$\lim_{\varphi \to 0,\, \varphi > 0} \frac{\omega(\varphi)(1 - \cos(\varphi))r}{\varphi} = \lim_{\varphi \to 0,\, \varphi > 0} \frac{\omega(\varphi)(2\sin^2(\varphi/2))r}{\varphi} =$$

$$= \lim_{\varphi \to 0,\, \varphi > 0} \omega(\varphi)\sin(\varphi/2) \cdot \lim_{\varphi \to 0,\, \varphi > 0} \frac{(\sin(\varphi/2))r}{\varphi/2} = 0 \cdot r = 0.$$

Accordingly, we can write the sum of the limits on the right-hand side of the equality (14) as a one-sided derivative (taking into account the fact that, by the definition of the functions $f_j^b(0) = u_a\big(b(0)\big) = u_a(b_j)$):

$$\lim_{\varphi \to 0,\, \varphi > 0} \frac{f_j^b(\varphi) - f_j^b(0)}{\varphi} = \lim_{\varphi \to 0,\, \varphi > 0} \frac{u_a\big(b(\varphi)\big) - u_a(b_j)}{\varphi} = r\cos\angle(v, b_j - p^*). \quad (15)$$

Similar considerations regarding increments of the function $u_a\big(b(\varphi)\big)$ for small modulo values of $\varphi < 0$ give an equality

$$\lim_{\varphi \to 0,\, \varphi < 0} \frac{f_j^b(\varphi) - f_j^b(0)}{\varphi} = \lim_{\varphi \to 0,\, \varphi < 0} \frac{u_a\big(b(\varphi)\big) - u_a(b_j)}{\varphi} =$$

$$= -r\cos\angle(-v, b_j - p^*) = r\cos\angle(v, b_j - p^*). \quad (16)$$

As we see, from the equality of the one-sided limits (15) and (16) for the derivative on the left and right of the function of distance from the point $\text{rotation}(b_j, p^*, \varphi)$ to the polygon A it follows the formula

$$\left. \frac{df_j^b(\varphi)}{\varphi} \right|_{\varphi=0} = r\cos\angle(v, b_j - p^*) = \|b_j - p\|\cos\angle(v, b_j - p^*),$$

whose sign matches the sign of (10). At the same time, it is known from the properties of the skew product that its sign is positive if the rotation from the first vector is counterclockwise, and negative if the rotation is clockwise, i.e.

$$\operatorname{sign}\left(\left.\frac{f_j^b(\varphi)}{\varphi}\right|_{\varphi=0}\right) = \sigma_j^b. \tag{17}$$

We consider separately the situation when the center of rotation of p^* coincides with b_j. In this case, the point b_j transforms into itself during rotation, and therefore,

$$\frac{df_j^b(\varphi)}{\varphi} = 0.$$

At the same time, if $\|b_j - p^*\| = 0$, then the skew product of any vector by $b_j - p^*$ is equal to zero, which means that $\sigma_j^b = 0$. As you can see, formula (17) is true anyway.

Consider the differential properties of the functions $f_i^a(\varphi)$, $i = \overline{1, N_A}$, for which $f_i^a(0) > 0$, at the point $\varphi = 0$. Essentially, these are functions of the Euclidean distance from the point a_i to the set rotation(B, p^*, φ). Note that the figure consisting of the point a_i and the polygon rot(B, p^*, φ) can be turned by an angle $-\varphi$ around p^* into a figure consisting of the point rotation$(a_i, p^*, -\varphi)$ and the polygon B. Accordingly, we can write functions in the form

$$f_i^a(\varphi) = \rho(\operatorname{rotation}(a_i, p^*, -\varphi), B), \quad i = \overline{1, N_A}.$$

Applying to this Euclidean distance function construction the same reasoning as made above for $f_j^b(\varphi)$, we get the equalities

$$\operatorname{sign}\left(\left.\frac{f_i^a(\varphi)}{\varphi}\right|_{\varphi=0}\right) = \sigma_i^a \tag{18}$$

for all $i = \overline{1, N_A}$ for which $f_i^a(0) > 0$.

Denote by

$$I_A = \{i \colon \rho(a_i, B) = d(A, B)\}, \qquad I_B(x) = \{j \colon \rho(b_j, A) = d(A, B)\}$$

the sets of vertices of the polygons for which the Euclidean distances to the other polygon are equal to the Hausdorff distance between the polygons. Since the rotation operator is continuous and Lipschitzian (in the angle of rotation), this means that there is a constant $\varepsilon > 0$ such that for all $\varphi \in [-\varepsilon, \varepsilon]$ the Euclidean distance between A and rotation(B, p^*, φ) is reached at the vertices from the sets $\{a_i \colon i \in I_A\}$ and $\{\operatorname{rotation}(b_j, p^*, \varphi) \colon j \in I_B\}$.

We introduce the set

$$U(p^*, A, B) = \{\sigma_i^a \colon i \in I_A\} \cup \{\sigma_j^b \colon j \in I_B\},$$

consisting of the numbers $-1, 0, 1$, characterizing the differential properties of the functions $f_i^a(\varphi)$ and $f_j^b(\varphi)$. Since at least at one of the vertices of the polygons the distance to another is reached, then $U(p^*, A, B) \neq \emptyset$.

Proposition 5. *Let convex polygons A and B be given such that $A \neq B$, and a point p^*. Then if*

$$\forall\, u_i \in U(p^*, A, B) \ \ u_i = 1, \tag{19}$$

then the function $\chi(\varphi)$ increases in a neighborhood of the point $\varphi = 0$.

Proof. We denote the set of functions

$$F_I = \{f_i^a(\varphi) \colon i \in I_A\} \cup \{f_j^b(\varphi) \colon j \in I_B\}.$$

As shown above, in some neighborhood $[-\varepsilon, \varepsilon]$ of the point $\varphi = 0$ the equality

$$\chi(\varphi) = \max\{\overline{f}_i'(\varphi) \colon \overline{f}_i' \in F_I\}.$$

By condition $d(A, B) > 0$, which means that functions from the set F_I have derivatives at the point $\varphi = 0$. Therefore, for the one-sided derivatives of the function $\chi(\varphi)$ at the point $\varphi = 0$, as for the derivatives of the maximum of a finite number of differentiable points whose values coincide $\varphi = 0$, the following formulas are true (see [13]):

$$\lim_{\varphi \to 0,\, \varphi > 0} \frac{d\chi(\varphi)}{d\varphi} = \max\{\overline{f}_i'(0) \colon \overline{f}_i \in F_I\}, \tag{20}$$

$$\lim_{\varphi \to 0,\, \varphi < 0} \frac{d\chi(\varphi)}{d\varphi} = \min\{\overline{f}_i'(0) \colon \overline{f}_i \in F_I\}. \tag{21}$$

As follows from formulas (17), (18) and the condition (19), the sign of the derivatives $\{\overline{f}_i'(0) \colon \overline{f}_i \in F_I\}$ is positive. Therefore, the one-sided derivatives $\chi'(-0)$ and $\chi'(+0)$ of the function $\chi(\varphi)$ at the point 0 are also positive. This is a sufficient condition for its increase in a certain neighborhood of the point $\varphi = 0$, embedded in $[-\varepsilon, \varepsilon]$. $\qquad\square$

Proposition 6. *Let convex polygons A and B be given such that $A \neq B$, and the point p^*. Then if*

$$\forall\, u_i \in U(p^*, A, B) \ \ u_i = -1, \tag{22}$$

then the function $\chi(\varphi)$ decreases in a neighborhood of the point $\varphi = 0$.

Proof. As follows from formulas (17), (18) and condition (22), the sign of all derivatives $\{\overline{f}_i'(0) \colon \overline{f}_i \in F_I\}$ is negative. From formulas (20), (21), it follows that the one-sided derivatives $\chi'(-0)$ and $\chi'(+0)$ of the function $\chi(\varphi)$ at the point 0 are also negative. This is a sufficient condition for it to decrease in some neighborhood of the point $\varphi = 0$, nested in $[-\varepsilon, \varepsilon]$. $\qquad\square$

Proposition 7. *Let convex polygons A and B be given such that $A \neq B$, and a point p^*. Then if*

$$\exists\, u_i \in U(p^*, A, B)\colon\ u_i = 1 \tag{23}$$

and

$$\exists\, u_j \in U(p^*, A, B)\colon\ u_j = -1, \tag{24}$$

then the function $\chi(\varphi)$ reaches at $\varphi = 0$ the local minimum.

Proof. As follows from formulas (17), (18) and conditions (23) and (24), among all the derivatives $\{\overline{f}'_i(0)\colon \overline{f}_i \in F_I\}$ there is at least one positive and one negative. From formulas (20), (21) it follows that in this case $\chi'(-0) < 0$ and $\chi'(+0) > 0$. This is a sufficient condition for the function $\chi(\varphi)$ to have a local minimum at the point $\varphi = 0$. $\square$

Remark 8. *Cases when the set $U(p^*, A, B)$ contains elements 0 and 1, 0 and -1, or only 0 need additional analysis. This is due to the fact that a function having a derivative equal to zero can either decrease, increase, or have an extremum point.*

In the framework of the algorithms implemented by the authors, the rotation that transfers B to the new polygon $\widetilde{B}$ is carried out according to the formula

$$\widetilde{B} = \mathrm{rotation}(B, p^*, \varphi^*), \tag{25}$$

where

$$\varphi^* = -k_\varphi (\max\{\Lambda(A, B)\} + \min\{\Lambda(A, B)\})/d(A, B), \tag{26}$$

$$\Lambda(A, B) = \{\|a_i - p(a_i, B)\|\sigma_i^a\colon i = 1, \ldots, N_a\} \bigcup$$

$$\bigcup \{\|b_j - p(b_j, A)\|\sigma_j^b\colon j = 1, \ldots, N_b\}.$$

The values of the functions σ_i^a, $i = 1, \ldots, N_a$, are determined by formula (8), σ_j^b, $j = 1, \ldots, N_b$ — by formula (9). The expression (25) implies that $A \neq B$, otherwise in (26) on the right side there will be a division by 0. But in the case of $A = B$ the problem of minimizing the Hausdorff distance is trivial.

Remark 9. *If the polygons A and B satisfy the conditions of the proposition 5, the rotation value φ^* is negative. Indeed, if all elements of the set $U(p^*, A, B)$ are equal to 1, then this means that all the largest modulo elements of the set $\Lambda(A, B)$ are positive. Therefore, the value of formula (26) is strictly negative. Similarly, it can be shown that if the polygons A and B satisfy the conditions of the proposition 6, then the rotation value is $\varphi^* > 0$. In both cases, by choosing a sufficiently small positive coefficient k_φ, it is possible to reduce $d(A, \widetilde{B})$ compared to $d(A, B)$.*

If the polygons A and B satisfy the conditions of the proposition 7, then the equality $\varphi^* = 0$ holds. In this case, among the set $\Lambda(A, B)$ there are at least two largest modulo numbers of the opposite sign. The value of (26) will be equal to their sum multiplied by some number, that is, zero.

3. IMPLEMENTATION OF THE SOFTWARE PACKAGE

As part of the work on creating a software implementation of the developed algorithms, the authors used the MATLAB application package. Previously, the authors have already used it to solve geometric problems related to minimizing the Hausdorff distance between sets, in particular, problems of constructing optimal coverings of plane and three-dimensional sets [14]. Due to this, many procedures have already been implemented (in particular, finding the projection of a point on a polygon and determining whether a point belongs to a polygon) [15].

The scheme of the iterative minimization algorithm for the Hausdorff distance for given convex polygons A and B and the coefficient k_φ is presented below.

Algorithm 1: Optimization of the location of polygons.

Step 1: The calculation of the value $d_0 = d(A, B)$ is performed.

Step 2: The polygon $B_1 = \overline{B}$ is constructed using the formula (5).

Step 3: The calculation of the value $d_1 = d(A, B_1)$ is performed.

Step 4: The Chebyshev center $c(B_1)$ of the polygon B_1 is calculated.

Step 5: The polygon $B_2 = \widetilde{B}_1$ is constructed by the formula (25), the center of rotation p^* is taken equal to $p^* = c(B_1)$.

Step 6: The calculation of the value $d_2 = d(A, B_2)$ is performed.

Step 7: If $d_2 < d_1$, then $B^* = B_2$ is taken as the output polygon, else $B^* = B_1$ is taken.

Remark 10. *In step 5 of the algorithm 1, the center of rotation of the polygon is selected not as some fixed point, but its Chebyshev center. This provides the smallest possible change in the coordinates of the vertices $\widetilde{B}_1$ compared to B_1 with the same change in the angle between vectors aligned with the sides of the polygons.*

When starting the software package, algorithm 1 is executed repeatedly in a cycle. The exit condition is that the Hausdorff distance $d(B, B^*)$ between the original polygon B and the resulting B^* value does not exceed the parameter δh set by the user. It is possible to reduce the coefficient k_φ if the condition $d_2 < d_1$ is not satisfied in step 7, that is, turning the polygon does not reduce the Hausdorff distance.

Within the framework of the software package for identifying non-optimal positions of the polygon B^*, which are locally stable with respect to algorithm 1, it is possible to construct the polygon $\widehat{B} = \{\widehat{x}\} + \Pi(\widehat{\varphi})B^*$, where $\widehat{x}$ and $\widehat{\varphi}$ are random variables. Then the cycle starts with the initial value of the moving polygon $\widehat{B}$, instead of B. The results of the cycles are compared to determine the optimal one.

In the general case, the of minimizing the Hausdorff distance between two convex polygons is easier, when their angles are smaller and they are closer in shape. For example, if polygons types A and B are similar, then it is easy to

find the optimal location B^* analytically. For this, it is necessary that Chebyshev center of the polygon B^* coincide with the Chebyshev center of the polygon A, A and B^* are homothetic.

4. COMPUTATIONAL EXPERIMENTS

The authors developed a software package in MATLAB based on Theorem 3 and Propositions 5–7.

Example 11. *Let a convex heptagon A with a set of vertices*

$$\{a_i\}_{i=1}^7 = \{(0,0),(1,1),(2,1),(3,-0),(3,-1),(2,-2),(0,-2)\}$$

and a hexagon B with a set of vertices

$$\{b_i\}_{i=1}^6 = \{(0,2),(-3,3),(-4,2),(-5,0),(-3,-2),(-1,-1)\}.$$

be given. It is required to find such $x^0 \in \mathbb{R}^2$ and $\varphi \in [0,2\pi]$ for which the value of (2) is minimal.

The Hausdorff distance between the original polygons is $d(A,B) \approx 4.4995$. The result obtained during 13 iterations of the algorithm 1 : $x^0 \approx (-0.5044, 3.9354)$, $\varphi^0 = 3.3965$. The Hausdorff distance between the polygons A and $B^* = \{x^0\} + \Pi(\varphi^0)B$ is $d(A,B^*) \approx 0.9716$. The polygons A, B and B^* are shown in Fig. 2.
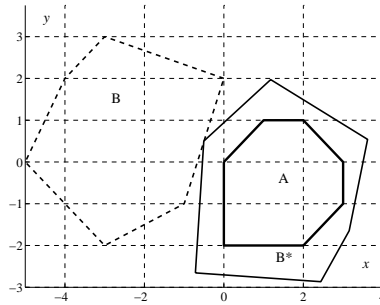

Figure 2: Polygons A, B and B^* in example 11

Example 12. *Let a convex pentagon A with a set of vertices*

$$\{a_i\}_{i=1}^5 = \{(-1,-3),(0,-3),(3,-0),(2,2),(1,3),(0,3)\}$$

and a hexagon B with a set of vertices

$$\{b_i\}_{i=1}^6 = \{(0,0),(-1,2),(-3,2),(-4,0),(-3,-3),(-1,-2)\}.$$

be given. It is required to find such $x^0 \in \mathbb{R}^2$ and $\varphi \in [0,2\pi]$ for which the value of (2) is minimal.

The Hausdorff distance between the original polygons is $d(A,B) \approx 3.0242$. The result obtained during 19 iterations of the algorithm $1 : x^0 \approx (2.1279, 0.361)$, $\varphi^0 = 5.7961$. The Hausdorff distance between the polygons A and $B^* = \{x^0\} + \Pi(\varphi^0)B$ is $d(A, B^*) \approx 1.0981$. The polygons A, B and B^* are shown in Fig. 3.
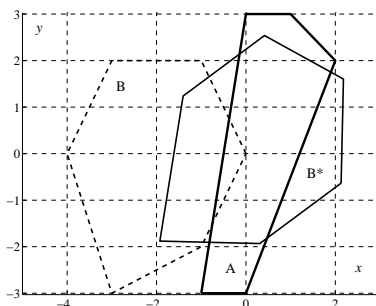

Figure 3: Polygons A, B and B^* in example 12

Problem solving was performed by repeatedly launching the software package. Accuracy estimates, upon reaching it, the algorithm is stopped, $\delta h = 0.001$.

Of the two examples considered in the present article, less computer time was required in Example 11. This is due to the fact that in Example 11 the polygon A is significantly smaller in size than the polygon B. Therefore, after the first iteration, the algorithm constructed a polygon $B*$ into which the polygon A was strictly nested. But in example 12, figures A and B differ significantly in shape. Therefore, the result of the algorithm produced a polygon B^* such that some of its vertices do not lie on A, and some of the vertices of A lie outside B^*.

5. CONCLUSIONS and SUGGESTIONS

The optimization algorithms developed by the authors for the Hausdorff distance between convex polygons have shown their effectiveness. The software package implemented on their basis for an acceptable time finds the shift and rotation values of one of the polygons, providing an approximation of their best overlap with high accuracy. The obtained results can be used in the future to solve more complex problems, in particular, to find the optimal location of convex bodies in the three-dimensional space [11, 14] and to minimize some generalization of the Hausdorff distance in the non-Euclidean metric, which can arise in problems of economics and logistics [16, 17]. The described algorithms can be developed to solve problems of finding the optimal arrangement of sets of large number of polygons [18].

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