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ESTIMATION OF PARAMETERS OF NADARAJAH-HAGHIGHI EXTENSION OF THE EXPONENTIAL DISTRIBUTION USING PERFECT AND IMPERFECT RANKED SET SAMPLE

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Abstract: The ranked set sampling (RSS) is a cost-effective method of sampling that can be used in a wide range of statistical problems. In this paper, the shape and the scale parameters of Nadarajah-Haghighi extension of the exponential distribution are estimated based on a simple random sample (SRS) and RSS. Three cases are considered: 1) the scale parameter is known; 2) the shape parameter is known; 3) both shape and scale parameters are unknown. Observations are done when the ranking mechanism in the ranked set sample is perfect and when it is not. Method of moments, the maximum likelihood method, and a modification of the maximum likelihood method are used. The obtained estimators are compared in terms of their biases and mean square errors (MSE). The results revealed that estimators based on RSS tend to show better properties (smaller bias and MSE) relative to their SRS counterparts, regardless of the quality of the ranking.

Keywords: Nadarajah-Haghighi Extension of the Exponential Eistribution, Ranked set Sampling, Simple Random Sampling, Imperfect Ranking.

MSC: 62F10, 62F07.

1. INTRODUCTION

In certain cases, obtaining measurements of a variable of interest is difficult, i.e., it is expensive or time-consuming. However, it is relatively straightforward to rank units. One can use the Ranked Set Sampling (RSS) as a statistical technique for data collection. The RSS was first proposed in [16].

The mechanism of Ranked Set Sampling can be summarized as the following: Simple random sample (SRS) of m units, which is called *set*, is selected from the population. The next step is to rank units of the sample without taking actual measurements (based on expert opinion, visual comparisons, etc; or using auxiliary variable). The smallest unit is selected and others are rejected. In the next step, a new SRS of size m is chosen and ranked. In this set, the second smallest unit is extracted, while the rest are discarded. The procedure is repeated m times so that in the i-th repetition, the i-ranked unit within the set is chosen. This procedure is called a cycle. In this way, a final sample of size m is obtained. If we need a larger sample, we can repeat the cycle k times and obtain a sample of size $n = mk$. Actual measurements are performed only on these units.

In most cases, estimators based on the RSS exhibit better qualities in comparison with estimators based on the SRS. The advantage of RSS estimators is shown for more theoretical distributions, see for example, [1], [4], [5], [22], [23]. The popularity of RSS increased during the last few decades, and many variations of the original model have been developed, see among others, [2], [3], [12], [21]. Initial papers assumed perfect ranking of units. The case when ranking is not perfect was first examined in [9], and this topic was, among others, later further examined in, [6], [10], [11]. The RSS procedure has been applied to wide range of problems: in market and consumer surveys [15], control charts [13], reliability analysis[20], studies about cost-effectiveness in the ecological and environmental field [17], etc. For a detailed explanation of the RSS, see [8] and [7].

Generalization of the exponential distribution is suggested by Nadarajah and Haghighi as an alternative to the gamma, Weibull, and exponentiated exponential distributions [18]. The cumulative distribution function of Nadarajah-Haghighi extension of the exponential distribution is

$$F(x) = 1 - \exp\{1 - (1 + \lambda x)^{\alpha}\},$$

and probability density function is

$$f(x) = \alpha\lambda(1 + \lambda x)^{\alpha-1} \exp\{1 - (1 + \lambda x)^{\alpha}\},$$

where $\lambda > 0$, $\alpha > 0$, and $x > 0$. The first and the second moment of variable X that has Nadarajah-Haghighi extension of the exponential distribution are:

$$E(X) = \frac{1}{\lambda}\left(-1 + e\Gamma\left(1 + \frac{1}{\alpha}, 1\right)\right)$$

and

$$E(X^2) = \frac{1}{\lambda^2}\left(1 - 2e\Gamma\left(1 + \frac{1}{\alpha}, 1\right) + e\Gamma\left(1 + \frac{2}{\alpha}, 1\right)\right),$$

where $\Gamma(\alpha, x) = \int\limits_x^{\infty} t^{\alpha-1}e^{-t}dt$ is complementary incomplete gamma function.

The remainder of this article is organized as following. In Section 2, we considered estimators of shape parameter α of Nadarajah-Haghighi extension of the exponential distribution in the case when scale parameter is known, $\lambda = 1$. In Subsection 2.1, we presented estimators of shape parameter based on the SRS, and in Subsection 2.2, those based on RSS. A comparison between the SRS and RSS estimators, when ranking is perfect and when it is not, is given in Subsection 2.3. We compared estimators by their bias and mean square error. In Section 3 we estimated scale parameter λ when the shape parameter is fixed, and in Section 4 the comparisons were done in the case when both parameters are unknown.

2. ESTIMATION OF α WHEN λ IS KNOWN

Let us assume, without loss of generality, that the scale parameter $\lambda = 1$. In that case, we get distribution with cumulative distribution function

$$F(x) = 1 - \exp\{1 - (1 + x)^{\alpha}\},$$

and probability density function

$$f(x) = \alpha(1 + x)^{\alpha-1} \exp\{1 - (1 + x)^{\alpha}\}.$$

2.1. SRS estimators

Let $X_1, ..., X_n$ be the SRS from Nadarajah-Haghighi extension of the exponential distribution with the scale parameter equal to 1 and the unknown shape parameter α.

- *Method of moments.* By equating the theoretical first moment $E(X) = -1 + e\Gamma\left(1 + \frac{1}{\alpha}, 1\right)$ with the sample first moment $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, we obtain

  $$\Gamma\left(1 + \frac{1}{\alpha}, 1\right) = \frac{\overline{X} + 1}{e}.$$

  This equation does not have closed-form solution, but the estimator of shape parameter by method of moments, $\hat{\alpha}_{mom,SRS}$, can be found by numerical methods.

- *Maximum likelihood method.* The maximum likelihood estimator is the solution of the equation:

  $$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(1 + X_i) - \sum_{i=1}^{n} (1 + X_i)^{\alpha} \ln(1 + X_i) = 0, \quad (1)$$

  where L is the maximum likelihood function. This equation does not have its analytical solution, so estimator $\hat{\alpha}_{mle,SRS}$ can also be obtained by numerical methods.

- *Modified Maximum likelihood method.* Solution of (1) can not be found because of the term $(1+X_i)^\alpha \ln(1+X_i)$, but we can replace this term with its expectation

$$E((1+X_i)^\alpha \ln(1+X_i)) = \frac{e}{\alpha}\int_1^\infty t\ln t e^{-t}dt.$$

The last integral can be evaluated through numerical integration and its estimated value is $I \approx 0.59$ with estimated absolute error 1.41×10^{-9}. The modified maximum likelihood estimator is

$$\hat{\alpha}_{mmle,SRS} = \frac{n(eI-1)}{\sum\limits_{i=1}^{n} \ln(1+X_i)}.$$

2.2. RSS estimators

Let $X_{11}, X_{21}, ..., X_{m1}, X_{12}, ..., X_{mk}$ be a RSS with set size m and the number of cycles k, i.e., the sample of total size $n = mk$, from Nadarajah-Haghighi extension of the exponential distribution where $\lambda = 1$.

- *Ad hoc estimators.* RSS estimators can be obtained by substituting SRS statistics with RSS statistics in estimators from the previous subsection. Estimator $\hat{\alpha}_{mom,ah}$ is the numerical solution of the equation

$$\Gamma\left(1+\frac{1}{\alpha},1\right) = \frac{\overline{X}_{RSS}+1}{e},$$

where $\overline{X}_{RSS} = \frac{1}{mk}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{k} X_{ij}$. In the same way, we got estimators $\hat{\alpha}_{mle,ah}$ and $\hat{\alpha}_{mmle,ah}$.

- *Modified Maximum likelihood method.* The probability density function for $i-$th order statistic from Nadarajah-Haghighi extension of the exponential distribution is [18]:

$$f_{(i)}(x) = m\binom{m-1}{i-1}(1-\exp\{1-(1+x)^\alpha\})^{i-1}(\exp\{1-(1+x)^\alpha\})^{m-i+1}\alpha(1+x)^{\alpha-1}.$$

The likelihood function for RSS, L, is given by

$$\prod_{i=1}^{m}\prod_{j=1}^{k} m\binom{m-1}{i-1}(1-\exp\{1-(1+X_{ij})^\alpha\})^{i-1}(\exp\{1-(1+X_{ij})^\alpha\})^{m-i+1}\alpha(1+X_{ij})^{\alpha-1},$$

so the Maximum likelihood estimator is the solution of the equation

$$\begin{aligned}\frac{\partial \ln L}{\partial \alpha} &= \frac{mk}{\alpha} + \sum_{i=1}^{m}\sum_{j=1}^{k} \ln(1+X_{ij}) \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{k} \frac{(1+X_{ij})^{\alpha}\ln(1+X_{ij})}{1-\exp\{1-(1+X_{ij})^{\alpha}\}}(m\exp\{1-(1+X_{ij})^{\alpha}\}-(m-i+1)) \\ &= 0.\end{aligned}$$

If we substitute term

$$\frac{(1+X_{ij})^{\alpha}\ln(1+X_{ij})}{1-\exp\{1-(1+X_{ij})^{\alpha}\}}(m\exp\{1-(1+X_{ij})^{\alpha}\}-(m-i+1))$$

with its expectation, the obtained estimator $\hat{\alpha}_{mmle,RSS}$ is equal to

$$\frac{mk + \left(1 + \sum_{i=1}^{m}\binom{m-1}{i-1}\sum_{s=0}^{i-2}(-1)^s\binom{i-2}{s}e\left(\frac{m(\Gamma(0,p)+e^{-p})}{p^2} - \frac{(m-i+1)(\Gamma(0,p-1)+e^{-p+1})}{(p-1)^2}\right)\right)}{-\sum_{i=1}^{m}\sum_{j=1}^{k}\ln(1+X_{ij})},$$

where $p = m - i + s + 2$.

2.3. Comparison of estimators

Bias and mean square error using Monte Carlo simulation with 10,000 iterations were used to compare SRS and RSS estimators.

We considered different values of the shape parameter $\alpha \in \{0.8, 2, 4\}$ and of the sample size, $n \in \{12, 24\}$. For the RSS, we considered different combinations of the set sizes and the number of cycles: for $n = 12$ combinations $(m, k) \in \{(2, 6), (3, 4), (4, 3)\}$ are investigated, and for $n = 24$ combinations $(m, k) \in \{(2, 12), (3, 8), (4, 6)\}$. Results for $n = 24$ are given in Appendix.

The model of imperfect ranking proposed by [6] was implemented in the simulations. In this model

$$F_{[i]}(x) = \sum_{i=1}^{n} p_{ij} F_{(i)}(x),$$

where $F_{(i)}(x)$ is CDF of i-th order statistic, $F_{[i]}$ is CDF of i-th judgment order statistic (i.e. statistic that was selected to be i-th) and $P_m(p) = \{p_{ij}\}$ is a doubly stochastic matrix. We use the form of matrix $P_m(p)$ as follows:

$$P_m(p) = \frac{1}{m-1}\begin{bmatrix}(m-1)p & 1-p & \cdots & 1-p \\ \vdots & \vdots & \ddots & \vdots \\ 1-p & 1-p & \cdots & (m-1)p\end{bmatrix}.$$

The previous simple parameterization of $P_m(p)$ was used in [14] and [19]. This model may be reliable for a small m. The considered values of the parameter p are $p \in \{1, 0.9, 0.8\}$. The lower the parameter p, the worse the ranking.

We chose a relative bias for the comparison, defined as:

$$\frac{|E(\hat{\theta})-\theta|}{\theta}\cdot 100\%.$$

The biases of obtained estimators of the parameter α for $n=12$ and $n=24$ are presented in Table 1 and Table 10 in Appendix, respectively.

α	m	k	p	$\hat{\alpha}_{mom,SRS}$	$\hat{\alpha}_{mle,SRS}$	$\hat{\alpha}_{mmle,SRS}$	$\hat{\alpha}_{mom,ah}$	$\hat{\alpha}_{mle,ah}$	$\hat{\alpha}_{mmle,ah}$	$\hat{\alpha}_{mmle,RSS}$
			1				4.36	5.52	3.03	3.53
0.8	2	6	0.9	5.68	6.7	4.32	4.8	5.88	3.49	3.09
			0.8				5.17	6.22	3.83	2.78
			1				3.55	4.69	2.34	0.4
0.8	3	4	0.9	5.68	6.7	4.32	4.05	5.15	2.8	0.86
			0.8				4.59	5.64	3.34	1.39
			1				2.97	4.1	1.8	1.21
0.8	4	3	0.9	5.68	6.7	4.32	4.99	6.16	3.72	3.12
			0.8				6.53	7.73	5.13	4.53
			1				3.31	5.41	2.9	3.65
2	2	6	0.9	4.87	6.75	4.45	3.95	5.93	3.54	3.05
			0.8				4.38	6.33	3.97	2.65
			1				2.64	4.68	2.27	0.34
2	3	4	0.9	4.87	6.75	4.45	3.19	5.17	2.82	0.87
			0.8				3.77	5.74	3.38	1.43
			1				2.17	4.1	1.85	1.26
2	4	3	0.9	4.87	6.75	4.45	3.95	6	3.55	2.95
			0.8				5.44	7.6	4.97	4.37
			1				3.23	5.48	3.05	3.5
4	2	6	0.9	4.78	6.91	4.58	3.89	6.11	3.7	2.9
			0.8				4.08	6.26	3.89	2.72
			1				2.45	4.72	2.28	0.35
4	3	4	0.9	4.78	6.91	4.58	3.03	5.22	2.86	0.92
			0.8				3.58	5.71	3.41	1.46
			1				2.01	4.17	1.85	1.27
4	4	3	0.9	4.78	6.91	4.58	3.76	6.01	3.58	2.99
			0.8				5.38	7.73	5.17	4.56

Table 1: The biases of the estimators of the shape parameter α for $n=12$

As shown in Table 1, for $p=1$, and $p=0.9$, all RSS estimators have lower bias than their SRS counterparts. The lowest bias between all estimators has $\hat{\alpha}_{mmle,RSS}$, which is, for some values of parameters, almost unbiased. The bias of the method of moment estimator decreases when the value of a parameter α increases, whereas for all other estimators, this change is not noticeable. As expected, with decreasing parameter p, the bias increases. This change is more prominent when the set size is larger. In some cases when the set size and the value of parameter α are high, biases of RSS estimators for $p=0.7$ are even greater than the corresponding biases of SRS estimators. An exception is estimator $\hat{\alpha}_{mmle,RSS}$ for set size $m=2$; in that case, with the increasing value of parameter p, the bias decreases.

When $n = 24$, relative to the case of smaller sample size, all biases are lower, but the pattern of change is the same. The estimator $\hat{\alpha}_{mmle,RSS}$ is again an exception: when the set size is large, the bias is low, but when $m = 2$, bias of this estimator is even higher than biases of SRS estimators.

We also compared estimators by their relative efficiency, defined as the ratio of mean square errors:

$$RE(\hat{\theta}_1, \hat{\theta}_2) = \frac{MSE(\hat{\theta}_1, \theta)}{MSE(\hat{\theta}_2, \theta)}.$$

The higher the efficiency, the better the estimator $\hat{\theta}_2$ is.

MSE of the estimator $\hat{\alpha}_{mmle,SRS}$ is compared with MSE of all other estimators. Comparisons for $n = 12$ are given in Table 2, and for $n = 24$ in Table 11 in Appendix.

α	m	k	p	$\hat{\alpha}_{mom,SRS}$	$\hat{\alpha}_{mle,SRS}$	$\hat{\alpha}_{mom,ah}$	$\hat{\alpha}_{mle,ah}$	$\hat{\alpha}_{mmle,ah}$	$\hat{\alpha}_{mmle,RSS}$
0.8	2	6	1	1.17	1.12	1.78	1.57	1.64	1.83
			0.9			1.5	1.37	1.32	1.5
			0.8			1.33	1.23	1.16	1.35
0.8	3	4	1	1.17	1.12	2.24	1.9	2.2	2.34
			0.9			1.83	1.62	1.67	1.78
			0.8			1.62	1.46	1.45	1.54
0.8	4	3	1	1.17	1.12	2.76	2.29	2.79	2.84
			0.9			1.87	1.64	1.78	1.82
			0.8			1.46	1.31	1.35	1.38
2	2	6	1	1.09	1.12	1.66	1.5	1.55	1.75
			0.9			1.42	1.34	1.32	1.52
			0.8			1.34	1.32	1.24	1.44
2	3	4	1	1.09	1.12	2.29	1.94	2.18	2.32
			0.9			1.88	1.67	1.76	1.87
			0.8			1.51	1.47	1.4	1.49
2	4	3	1	1.09	1.12	2.65	2.15	2.57	2.62
			0.9			1.84	1.61	1.75	1.79
			0.8			1.45	1.34	1.36	1.4
4	2	6	1	1.05	1.11	1.68	1.57	1.62	1.79
			0.9			1.37	1.36	1.31	1.5
			0.8			1.22	1.25	1.17	1.34
4	3	4	1	1.05	1.11	2.34	1.98	2.27	2.41
			0.9			1.8	1.69	1.74	1.84
			0.8			1.48	1.48	1.41	1.5
4	4	3	1	1.05	1.11	2.8	2.27	2.73	2.79
			0.9			1.79	1.64	1.73	1.78
			0.8			1.3	1.21	1.26	1.29

Table 2: The efficiency of estimators of the shape parameter α with respect to $\hat{\alpha}_{mmle,SRS}$ for $n = 12$

As shown in Table 2, all efficiencies are higher than 1, so $\hat{\alpha}_{mmle,SRS}$ has the highest MSE between all estimators, both SRS and RSS. Therefore, despite it has the smallest bias among SRS estimators, the $\hat{\alpha}_{mmle,SRS}$ is not a good choice.

Efficiencies related to RSS are higher than efficiencies related to SRS, therefore MSE of RSS estimators are lower than MSE of SRS estimators, regardless of the ranking being perfect or not. The decrease of the parameter p is accompanied by the decrease of its efficiency. As in the case of the bias, this change is more drastic when the set size is larger. With the increasing of set size, the efficiency increases. It is interesting that when the sample size is larger, efficiency of SRS estimators concerning $\hat{\alpha}_{mmle,SRS}$ increases, while efficiency of RSS estimators decreases. The pattern of changes of the estimators concerning change of the set size, shape parameter, and parameter p is the same for both $n = 12$ and $n = 24$.

Estimators $\hat{\alpha}_{mom,ah}$ and $\hat{\alpha}_{mmle,RSS}$ have the highest efficiency, but $\hat{\alpha}_{mmle,RSS}$ is the best choice because it also has the lowest bias. A flaw of estimator $\hat{\alpha}_{mmle,RSS}$ is a value of bias when the sample size is high and the set size is low.

The size of the SRS needed to achieve the same MSE as its RSS counterpart of size $n = 12$ is shown in Table 3. The data for $n = 24$ are given in Table 12 in Appendix. The pairs are $\hat{\alpha}_{mom,SRS}$ and $\hat{\alpha}_{mom,ah}$; $\hat{\alpha}_{mle,SRS}$ and $\hat{\alpha}_{mle,ah}$; $\hat{\alpha}_{mmle,SRS}$ and $\hat{\alpha}_{mmle,ah}$; $\hat{\alpha}_{mmle,SRS}$ and $\hat{\alpha}_{mmle,RSS}$.

α	m	k	p	mom	mle	mmle,ah	mmle,RSS
0.8	2	6	1	16	15	17	19
			0.9	15	14	16	17
			0.8	13	13	13	15
0.8	3	4	1	20	18	24	23
			0.9	17	16	19	20
			0.8	14	14	16	17
0.8	4	3	1	23	20	28	29
			0.9	17	16	19	19
			0.8	14	13	16	16
2	2	6	1	17	15	17	19
			0.9	15	14	15	16
			0.8	13	13	13	15
2	3	4	1	22	18	23	24
			0.9	17	15	19	19
			0.8	15	14	16	16
2	4	3	1	25	21	27	29
			0.9	18	16	19	20
			0.8	14	13	16	16
4	2	6	1	17	15	17	19
			0.9	15	13	15	17
			0.8	14	13	14	16
4	3	4	1	22	18	24	25
			0.9	18	16	20	19
			0.8	16	14	16	17
4	4	3	1	27	20	27	29
			0.9	18	16	19	19
			0.8	15	13	15	15

Table 3: The size of the SRS needed to achieve the same quality of the estimator of α as estimator from the RSS of size $n = 12$

All the values in Table 3 are greater than 12, thus the RSS estimators are better (have smaller MSE) than their SRS counterparts even for imperfect ranking. When ranking is perfect, a number of required units from the SRS is more than twice the number of units from the RSS. Though the MLE estimator is the worst, it is still better than its counterpart, but it is worse than other estimators. Estimator $\hat{\alpha}_{mmle,RSS}$ is the best, especially when ranking is perfect. The advantage of the RSS estimators is more evident for larger set sizes. This is true even when ranking is imperfect, but in that case, the advantage is not so convincing.

3. ESTIMATION OF λ WHEN α IS KNOWN

Let us now suppose that the shape parameter α is fixed.

3.1. SRS estimators

Let $X_1, ..., X_n$ be a SRS from Nadarajah-Haghighi extension of the exponential distribution with fixed shape parameter and unknown scale parameter.

- *Method of moments.* By equating the theoretical first moment $E(X) = \frac{1}{\lambda}(-1 + e\Gamma\left(1 + \frac{1}{\alpha}, 1\right))$ with sample first moment $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, we obtain

$$\hat{\lambda}_{mom,SRS} = \frac{e\Gamma\left(1 + \frac{1}{\alpha}, 1\right) - 1}{\overline{X}}.$$

- *Maximum likelihood method.* The maximum likelihood estimator is the solution of the equation:

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} + (\alpha - 1)\sum_{i=1}^{n} X_i(1 + \lambda X_i)^{-1} - \alpha \sum_{i=1}^{n} X_i(1 + \lambda X_i)^{\alpha-1} = 0.$$

where L is the maximum likelihood function. This equation does not have its analytical solution, so estimator $\hat{\lambda}_{mle,SRS}$ can be obtained by numerical methods.

3.2. RSS estimators

Let $X_{11}, X_{21}, ..., X_{m1}, X_{12}, ..., X_{mk}$ be a RSS with set size m and number of cycles k, i.e., the sample of total size $n = mk$, from Nadarajah-Haghighi extension of the exponential distribution where α is fixed.

- *Ad hoc estimators.* RSS estimators can be obtained by substituting the SRS statistics with the RSS statistics in estimators from the previous subsection. Method of moment estimator is

$$\hat{\lambda}_{mom,ah} = \frac{e\Gamma\left(1+\frac{1}{\alpha},1\right)-1}{\overline{X}_{RSS}}.$$

where $\overline{X}_{RSS} = \frac{1}{mk}\sum_{i=1}^{m}\sum_{j=1}^{k} X_{ij}$. In the same way, we got estimator $\hat{\lambda}_{mle,ah}$.

3.3. Comparison of the estimators

Monte Carlo simulation with 10,000 iterations is done with the same values of parameters p, m, and k as in the case when the scale parameter is known. We considered the case where shape parameter is fixed, $\alpha = 2$. The considered value of the scale parameter are $\lambda \in \{0.8, 2, 4\}$.

λ	m	k	p	$\hat{\lambda}_{mom,SRS}$	$\hat{\lambda}_{mle,SRS}$	$\hat{\lambda}_{mom,ah}$	$\hat{\lambda}_{mle,ah}$
			1			4.22	5.83
0.8	2	6	0.9	6.37	7.95	5.11	6.68
			0.8			5.53	7.1
			1			3.44	5.02
0.8	3	4	0.9	6.37	7.95	4.06	5.58
			0.8			4.82	6.33
			1			2.75	4.2
0.8	4	3	0.9	6.37	7.95	5.05	6.67
			0.8			7.2	8.97
			1			4.31	5.92
2	2	6	0.9	6.31	7.89	4.92	6.51
			0.8			5.48	7.08
			1			3.36	4.94
2	3	4	0.9	6.31	7.89	3.98	5.53
			0.8			4.75	6.31
			1			2.72	4.17
2	4	3	0.9	6.31	7.89	5.18	6.82
			0.8			6.87	8.65
			1			4.4	5.99
4	2	6	0.9	6.36	7.9	4.94	6.53
			0.8			5.54	7.08
			1			3.32	4.87
4	3	4	0.9	6.36	7.9	4.14	5.67
			0.8			4.93	6.44
			1			2.74	4.18
4	4	3	0.9	6.31	7.9	5.08	6.69
			0.8			6.94	8.68

Table 4: The biases of the estimators of the scale parameter λ for $n = 12$

Relative biases of the estimators of the scale parameter λ, for $n = 12$ and $n = 24$, are given in Table 4 and Table 13 in Appendix, respectively.

In general, biases of RSS estimators of scale parameter λ are smaller than biases of SRS estimators. The method of moment estimator has a smaller bias in regard to the maximum likelihood estimator in both SRS and RSS. In case of small set size biases of RSS estimators are smaller than the biases of SRS estimators regardless of the quality of ranking, but this is not the case when $m = 4$. In that case, with decreasing of p, bias of RSS estimator starts from the lowest value among all values but then grows rapidly and becomes greater than the bias of the SRS estimator. If ranking is perfect, greater sample size gives better estimators, but this advantage disappears when ranking becomes worse. The smallest bias in most cases has $\hat{\lambda}_{mom,ah}$.

The efficiency of the estimator $\hat{\lambda}_{mle,SRS}$ compared to all other estimators is given in Table 5 for $n = 12$, and in Table 14 in the Appendix for $n = 24$.

λ	m	k	p	$\hat{\lambda}_{mom,SRS}$	$\hat{\lambda}_{mom,ah}$	$\hat{\lambda}_{mle,ah}$
0.8	2	6	1	1	1.58	1.46
			0.9		1.34	1.28
			0.8		1.15	1.13
0.8	3	4	1	1	2.16	1.88
			0.9		1.7	1.58
			0.8		1.4	1.34
0.8	4	3	1	1	2.73	2.29
			0.9		1.82	1.63
			0.8		1.25	1.15
2	2	6	1	1	1.59	1.46
			0.9		1.3	1.24
			0.8		1.14	1.12
2	3	4	1	1	2.07	1.83
			0.9		1.6	1.47
			0.8		1.5	1.43
2	4	3	1	1	2.77	2.32
			0.9		1.76	1.59
			0.8		1.4	1.28
4	2	6	1	1	1.65	1.5
			0.9		1.29	1.23
			0.8		1.15	1.11
4	3	4	1	1	2.14	1.87
			0.9		1.6	1.48
			0.8		1.37	1.32
4	4	3	1	1	2.49	2.11
			0.9		1.61	1.46
			0.8		1.22	1.14

Table 5: The efficiency of estimators of the scale parameter λ with respect to $\hat{\lambda}_{mle,SRS}$ for $n = 12$

Both estimators from SRS have the same MSE. All other efficiencies are greater than 1, so RSS estimators have smaller MSE compared to SRS estimators, regardless of the quality of ranking, set size or value of scale parameter. As expected, efficiency decreases with ranking becoming worse. Efficiency increases with the increase of the set size especially when ranking is perfect. On the other hand, efficiency is smaller in the case when $n = 24$, compared to $n = 12$.

The size of SRS needed to achieve the same MSE as its RSS counterpart of size $n = 12$ when estimating scale parameter is shown in Table 6. The data for $n = 24$ are given in Table 15 in the Appendix.

α	m	k	p	mom	mle
0.8	2	6	1	17	16
			0.9	15	14
			0.8	14	13
0.8	3	4	1	22	19
			0.9	18	16
			0.8	15	15
0.8	4	3	1	26	22
			0.9	18	17
			0.8	15	14
2	2	6	1	16	16
			0.9	15	14
			0.8	13	13
2	3	4	1	23	20
			0.9	18	16
			0.8	16	15
2	4	3	1	27	21
			0.9	18	17
			0.8	15	14
4	2	6	1	17	16
			0.9	15	14
			0.8	13	13
4	3	4	1	22	20
			0.9	18	17
			0.8	16	15
4	4	3	1	26	22
			0.9	18	16
			0.8	15	14

Table 6: The size of SRS needed to achieve the same quality of the estimator of λ as the estimator from the RSS of size $n = 12$

The number of units in SRS needed to gain the same MSE as the RSS of size 12 is up to 27. The superiority of RSS estimators is especially noticeable for a greater set size, even when ranking is not perfect.

4. JOINT ESTIMATION OF α AND λ

4.1. SRS estimators

Let $X_1, ..., X_n$ be a SRS from Nadarajah-Haghighi extension of the exponential distribution with scale parameter λ and shape parameter α.

- *Method of moments.* Estimators of the parameters by the Method of moments can be found as the solution of the system of equations [18]:

$$\Gamma\left(1+\frac{1}{\alpha},1\right)=\frac{\lambda\overline{X}+1}{e},$$

$$\Gamma\left(1+\frac{2}{\alpha},1\right)=\frac{\lambda^2M_2+2\lambda\overline{X}+1}{e},$$

where $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $M_2=\frac{1}{n}\sum_{i=1}^{n}X_i^2$. This solution has to be found by numerical methods.

- *Maximum likelihood method.* Maximum likelihood estimators are numerical solutions of the system of equations [18]:

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\ln(1+\lambda X_i)-\sum_{i=1}^{n}(1+\lambda X_i)^{\alpha}\ln(1+\lambda X_i)=0,$$

$$\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^{n}X_i(1+\lambda X_i)^{-1}-\alpha\sum_{i=1}^{n}X_i(1+\lambda X_i)^{\alpha-1}=0.$$

4.2. RSS estimators

Let $X_{11},X_{21},...,X_{m1},X_{12},...,X_{mk}$ be the RSS of size $n=mk$, where m is the set size, and k is the number of cycles, from Nadarajah-Haghighi extension of the exponential distribution. X_{ij} is i-th unit in j-th cycle.

- *Ad hoc estimators.* As in the previous sections, we can obtain RSS estimators by substituting SRS statistics with RSS statistics. Obtained estimators are $\hat{\alpha}_{mom,ah}$, $\hat{\lambda}_{mom,ah}$, $\hat{\alpha}_{mle,ah}$ and $\hat{\lambda}_{mle,ah}$.

4.3. Comparison of the estimators

Monte Carlo simulation with 10,000 iterations is done with the same values of parameters as in Section 2 and Section 3: $p\in\{1,0.9,0.8\}$; $(m,k)\in\{(2,6),(3,4),(4,3)\}$ for $n=12$ and $(m,k)\in\{(2,12),(3,8),(4,6)\}$ for $n=24$. We carried out this comparison by taking different values of the shape and the scale parameter $(\alpha,\lambda)=\{(0.8,2),(2,3),(5,3)\}$.

Data for $n=24$ are given in the Appendix.

Relative biases of the estimators of the shape parameter α and scale parameter λ for $n=12$ and $n=24$ are given in Tables 7 and 16.

The bias of $\hat{\alpha}_{mom,ah}$ is lower than the bias of $\hat{\alpha}_{mom,SRS}$ regardless of the type of ranking. The situation is opposite when estimating parameter λ, the bias of $\hat{\lambda}_{mom,ah}$ is higher than the bias of $\hat{\lambda}_{mom,SRS}$ regardless of the type of ranking.

α	λ	m	k	p	$\hat{\alpha}_{mom,SRS}$	$\hat{\alpha}_{mle,SRS}$	$\hat{\alpha}_{mom,ah}$	$\hat{\alpha}_{mle,ah}$	$\hat{\lambda}_{mom,SRS}$	$\hat{\lambda}_{mle,SRS}$	$\hat{\lambda}_{mom,ah}$	$\hat{\lambda}_{mle,ah}$
0.8	2	2	6	1 0.9 0.8	7.92	1.57	6.56 6.89 7.43	0.66 0.82 1.14	1.27	5.31	1.33 1.31 1.2	5.08 5.15 5.23
0.8	2	3	4	1 0.9 0.8	7.92	1.57	5.65 6.29 6.87	0.07 0.51 0.85	1.27	5.31	1.35 1.36 1.3	5 5.08 5.14
0.8	2	4	3	1 0.9 0.8	7.92	1.57	5.09 6.79 8.36	0.3 1.23 2.66	1.27	5.31	1.4 1.53 1.7	4.95 5.09 5.24
2	3	2	6	1 0.9 0.8	4.64	0.09	2.57 3.36 3.91	0.45 0.19 0.02	1.3	7.21	2.03 1.76 1.61	6.54 6.75 6.94
2	3	3	4	1 0.9 0.8	4.64	0.09	1.48 2.38 3.11	0.86 0.49 0.3	1.3	7.21	2.39 2.14 1.84	6.19 6.44 6.68
2	3	4	3	1 0.9 0.8	4.64	0.09	0.85 2.8 4.43	1.05 0.37 1.43	1.3	7.21	2.68 2.52 2.32	5.99 6.34 6.72
5	3	2	6	1 0.9 0.8	6.72	2.13	5.99 6.25 6.46	2.27 2.23 2.18	1.97	5.22	2.52 2.29 2.22	3.66 4.23 4.61
5	3	3	4	1 0.9 0.8	6.72	2.13	5.62 5.93 6.14	2.26 2.29 2.24	1.97	5.22	2.96 2.57 2.37	2.72 3.44 3.94
5	3	4	3	1 0.9 0.8	6.72	2.13	5.4 5.79 6.13	2.26 2.39 2.47	1.97	5.22	3.24 1.81 0.62	2.07 3.97 5.69

Table 7: The biases of the estimators of the shape parameter α and the scale parameter λ for $n = 12$

Besides, when parameter p decreases (in which case, RSS is closer to SRS), the bias of $\hat{\alpha}_{mom,ah}$ increases, but the bias of $\hat{\lambda}_{mom,ah}$ decreases. For MOM estimator of shape parameter α, the bias decreases when the set size increases only if ranking is perfect. The estimating scale parameter λ bias increases when the set size increases, even if ranking is perfect.

Estimators obtained by the ML method behave opposite to MOM estimators. $\hat{\lambda}_{mle,ah}$ relative to SRS counterpart has a lower bias, regardless of the ranking type; and $\hat{\alpha}_{mle,ah}$ relative to SRS counterpart has a lower bias for some parameter values, and higher for others. When parameter p decreases, the bias of $\hat{\lambda}_{mle,ah}$ increases, though bias of $\hat{\alpha}_{mle,ah}$ does not follow any pattern. When ranking is perfect, the bias of $\hat{\lambda}_{mle,ah}$ decreases when the set size increases.

Tables 8 and 17 show efficiency of estimators of the shape parameter α with respect to the $\hat{\alpha}_{mle,SRS}$, and the scale parameter λ with respect to the $\hat{\lambda}_{mle,SRS}$ for $n = 12$ and $n = 24$, respectively.

As shown in Table 8, efficiency related to SRS estimators are mostly lower than 1, ML estimators have lower MSE than MOM estimators, so they are a better

α	λ	m	k	p	$\hat{\alpha}_{mom,SRS}$	$\hat{\alpha}_{mom,ah}$	$\hat{\alpha}_{mle,ah}$	$\hat{\lambda}_{mom,SRS}$	$\hat{\lambda}_{mom,ah}$	$\hat{\lambda}_{mle,ah}$
				1		1.12	1.37		0.38	1.12
0.8	2	2	6	0.9	0.88	1.06	1.25	0.36	0.35	1.1
				0.8		1.01	1.17		0.37	1.05
				1		1.39	1.84		0.41	1.13
0.8	2	3	4	0.9	0.88	1.22	1.5	0.36	0.39	1.13
				0.8		1.06	1.26		0.36	1.06
				1		1.59	2.25		0.49	1.28
0.8	2	4	3	0.9	0.88	1.28	1.61	0.36	0.45	1.21
				0.8		1.01	1.24		0.47	1.13
				1		1.25	1.25		1.04	1.5
2	3	2	6	0.9	0.86	1.08	1.17	0.75	0.91	1.24
				0.8		0.96	1.07		0.96	1.35
				1		1.59	1.5		1.36	1.72
2	3	3	4	0.9	0.86	1.3	1.31	0.75	1.15	1.61
				0.8		1.11	1.17		0.95	1.36
				1		1.79	1.66		1.67	1.98
2	3	4	3	0.9	0.86	1.37	1.39	0.75	1.44	1.74
				0.8		1	1.13		1.14	1.27
				1		1.39	1.08		1.51	1.36
5	3	2	6	0.9	0.95	1.14	1.08	1.2	1.44	1.25
				0.8		1.17	1.07		1.3	1.15
				1		1.77	1.36		1.91	1.64
5	3	3	4	0.9	0.95	1.56	1.23	1.2	1.59	1.44
				0.8		1.31	1.2		1.42	1.27
				1		1.86	1.5		2.25	1.91
5	3	4	3	0.9	0.95	1.59	1.38	1.2	1.7	1.42
				0.8		1.26	1.23		1.44	1.18

Table 8: The efficiency of estimators of the shape parameter α and the scale parameter λ concerning $\hat{\alpha}_{mle,SRS}$ and $\hat{\lambda}_{mle,SRS}$ for $n = 12$

choice between SRS estimators. In contrast, RSS estimators $\hat{\alpha}_{mom,ah}$,$\hat{\alpha}_{mle,ah}$ and $\hat{\lambda}_{mle,ah}$ have efficiency higher than 1 for all values of parameters p, m, and k and all values of shape and scale parameters. It is interesting that for the low values of shape and scale parameters, the efficiency of $\hat{\lambda}_{mom,ah}$ is lower than 1; but for high values, efficiency is even higher then the efficiencies of all other estimators. Efficiency is lower for lower values of the parameter p. When ranking is perfect or approximately perfect, with increasing of the set size, efficiency increases.

As shown in Table 17, changes in efficiency for different parameter values are smaller in the case when $n = 24$ then when $n = 12$. When estimating the shape parameter, efficiency of RSS estimators is greater than 1 (with few exceptions), regardless of the ranking type. As far as the scale parameter is considered, for $(\alpha, \lambda) = (0.8, 2)$ efficiency is near to 1 (but lower), and for other values of parameters, efficiency is greater than 1.

The sizes of SRS needed to achieve the same quality of the estimator (the same MSE) as in the case of RSS of size $n = 12$ for α and λ are given in Table 9. The data for $n = 24$ are given in Table 18. All the values in Table 9 are greater or equal to 12. When ranking is perfect, the advantage of RSS estimators is evident. The RSS estimators are better relative to their counterparts (have lower MSE) even when ranking is not perfect. For the low values of parameters α and λ, ML estimators are better, whereas MOM estimators are better for the high values of these parameters.

					α		λ	
α	λ	m	k	p	mom	mle	mom	mle
0.8	2	2	6	1	15	16	13	16
				0.9	14	14	12	14
				0.8	13	13	13	13
0.8	2	3	4	1	17	20	14	24
				0.9	16	17	14	17
				0.8	14	15	14	17
0.8	2	4	3	1	19	24	13	24
				0.9	16	17	13	16
				0.8	13	14	13	13
2	3	2	6	1	17	16	16	16
				0.9	15	14	14	14
				0.8	14	14	13	13
2	3	3	4	1	22	19	18	19
				0.9	17	16	16	16
				0.8	15	15	15	14
2	3	4	3	1	25	22	22	25
				0.9	17	17	19	16
				0.8	15	15	18	14
5	3	2	6	1	17	14	17	15
				0.9	15	14	15	14
				0.8	13	12	13	13
5	3	3	4	1	21	16	21	18
				0.9	19	14	18	16
				0.8	17	14	16	14
5	3	4	3	1	26	19	26	21
				0.9	17	16	19	16
				0.8	15	17	16	14

Table 9: The size of SRS needed to achieve the same quality of the estimator as the estimator from the RSS of size $n = 12$ for α and λ

5. CONCLUSIONS

In this paper, we explored the estimation problem of unknown parameters of the Nadarajah-Haghighi extension of the exponential distribution based on the RSS. We considered three cases, the first one when the scale parameter is known, the second, when the shape parameter is known, and the last, when both shape and scale parameters are unknown. We used the MOM and the ML method, and two modifications of the ML method in the case when the scale parameter is known. Further more, we compared the estimators got from perfect and imperfect RSS data.

In general, regardless of the ranking type (perfect or imperfect), estimators based on the RSS show better properties as compared to estimators based on the SRS.

When the scale parameter is known, the best estimator in terms of bias and MSE is $\hat{\alpha}_{mmle,RSS}$, the new estimator obtained by the modified ML method.

In the case when the shape parameter is fixed, among the examined estimators, the best choice is $\hat{\lambda}_{mom,ah}$ in most scenarios. This estimator has low performance in terms of bias for greater set sizes and imperfect ranking.

When both parameters are unknown, there is not a single best estimator for estimating α and λ simultaneously. Both RSS estimators, ML and MOM, give satisfying results for certain combinations of parameters. In each of the examined scenarios, the RSS estimators show smaller MSE relative to their counterparts.

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6. Appendix

α	m	k	p	$\hat{\alpha}_{mom,SRS}$	$\hat{\alpha}_{mle,SRS}$	$\hat{\alpha}_{mmle,SRS}$	$\hat{\alpha}_{mom,ah}$	$\hat{\alpha}_{mle,ah}$	$\hat{\alpha}_{mmle,ah}$	$\hat{\alpha}_{mmle,RSS}$
0.8	2	12	1	2.82	3.27	2.19	2.12	2.69	1.45	5.01
			0.9				2.36	2.9	1.69	4.78
			0.8				2.53	3.01	1.88	4.61
0.8	3	8	1	2.82	3.27	2.19	1.77	2.34	1.16	0.75
			0.9				1.93	2.46	1.32	0.59
			0.8				2.24	2.74	1.62	0.3
0.8	4	6	1	2.82	3.27	2.19	1.44	2.01	0.84	0.26
			0.9				2.91	3.48	2.26	1.67
			0.8				4	4.59	3.29	2.69
2	2	12	1	2.41	3.28	2.21	1.65	2.68	1.45	5.01
			0.9				1.9	2.87	1.71	4.76
			0.8				2.14	3.06	1.95	4.54
2	3	8	1	2.41	3.28	2.21	1.3	2.31	1.13	0.78
			0.9				1.56	2.56	1.38	0.53
			0.8				1.78	2.73	1.6	0.32
2	4	6	1	2.41	3.28	2.21	1.06	2.06	0.9	0.32
			0.9				2.4	3.4	2.2	1.62
			0.8				3.5	4.55	3.26	2.67
4	2	12	1	2.32	3.31	2.23	1.51	2.68	1.42	5.04
			0.9				1.87	2.93	1.78	4.7
			0.8				2.01	3.08	1.91	4.57
4	3	8	1	2.32	3.31	2.23	1.17	2.27	1.1	0.81
			0.9				1.43	2.55	1.34	0.57
			0.8				1.7	2.77	1.62	0.3
4	4	6	1	2.32	3.31	2.23	0.97	2.05	0.9	0.31
			0.9				2.3	3.42	2.2	1.62
			0.8				3.43	4.62	3.32	2.72

Table 10: The biases of the estimators of the shape parameter α for $n = 24$

α	m	k	p	$\hat{\alpha}_{mom,SRS}$	$\hat{\alpha}_{mle,SRS}$	$\hat{\alpha}_{mom,ah}$	$\hat{\alpha}_{mle,ah}$	$\hat{\alpha}_{mmle,ah}$	$\hat{\alpha}_{mmle,RSS}$
			1			1.66	1.53	1.48	1.46
0.8	2	12	0.9	1.23	1.22	1.59	1.49	1.36	1.37
			0.8			1.4	1.36	1.16	1.2
			1			2.12	1.86	2.02	2.1
0.8	3	8	0.9	1.23	1.22	1.81	1.66	1.62	1.7
			0.8			1.56	1.47	1.34	1.41
			1			2.63	2.23	2.65	2.7
0.8	4	6	0.9	1.23	1.22	1.84	1.67	1.69	1.73
			0.8			1.37	1.28	1.23	1.26
			1			1.63	1.6	1.5	1.46
2	2	12	0.9	1.12	1.22	1.45	1.45	1.32	1.33
			0.8			1.29	1.35	1.17	1.2
			1			2.18	1.89	2.07	2.17
2	3	8	0.9	1.12	1.22	1.78	1.68	1.65	1.74
			0.8			1.48	1.5	1.35	1.42
			1			2.6	2.16	2.49	2.55
2	4	6	0.9	1.12	1.22	1.8	1.66	1.69	1.73
			0.8			1.4	1.36	1.3	1.34
			1			1.62	1.58	1.55	1.51
4	2	12	0.9	1.06	1.23	1.36	1.43	1.29	1.3
			0.8			1.24	1.41	1.17	1.21
			1			2.14	1.93	2.06	2.15
4	3	8	0.9	1.06	1.23	1.65	1.62	1.58	1.66
			0.8			1.36	1.44	1.3	1.37
			1			2.61	2.19	2.54	2.59
4	4	6	0.9	1.06	1.23	1.66	1.61	1.6	1.64
			0.8			1.29	1.29	1.24	1.28

Table 11: The efficiency of estimators of the shape parameter α with respect to $\hat{\alpha}_{mmle,SRS}$ for $n = 24$

α	m	k	p	mom	mle	mmle,ah	mmle,RSS
			1	32	30	35	34
0.8	2	12	0.9	29	28	30	30
			0.8	27	26	27	30
			1	39	35	47	46
0.8	3	8	0.9	32	31	36	37
			0.8	29	28	32	34
			1	45	40	58	57
0.8	4	6	0.9	33	30	38	39
			0.8	26	25	29	31
			1	33	28	35	35
2	2	12	0.9	29	27	30	30
			0.8	27	27	27	28
			1	44	34	47	48
2	3	8	0.9	35	31	37	39
			0.8	31	28	32	31
			1	51	40	56	57
2	4	6	0.9	36	30	38	39
			0.8	30	26	29	30
			1	34	30	35	34
4	2	12	0.9	30	27	31	31
			0.8	28	25	27	27
			1	44	34	48	46
4	3	8	0.9	37	31	37	38
			0.8	30	29	31	33
			1	54	40	56	59
4	4	6	0.9	36	30	38	39
			0.8	29	26	28	30

Table 12: The size of SRS needed to achieve the same quality of the estimator of α as estimator from RSS of size $n = 24$

λ	m	k	p	$\hat{\lambda}_{mom,SRS}$	$\hat{\lambda}_{mle,SRS}$	$\hat{\lambda}_{mom,ah}$	$\hat{\lambda}_{mle,ah}$
0.8	2	12	1 0.9 0.8	2.85	3.63	2.02 2.38 2.63	2.83 3.16 3.39
0.8	3	8	1 0.9 0.8	2.85	3.63	1.67 2 2.36	2.44 2.8 3.13
0.8	4	6	1 0.9 0.8	2.85	3.63	1.4 3.08 4.49	2.13 3.9 5.41
2	2	12	1 0.9 0.8	2.98	3.73	2.12 2.5 2.64	2.92 3.28 3.43
2	3	8	1 0.9 0.8	2.98	3.73	1.7 2.03 2.44	2.5 2.8 3.2
2	4	6	1 0.9 0.8	2.98	3.73	1.36 3.09 4.44	2.1 3.91 5.37
4	2	12	1 0.9 0.8	3.04	3.81	2.06 2.45 2.68	2.83 3.21 3.46
4	3	8	1 0.9 0.8	3.04	3.81	1.64 2.03 2.35	2.43 2.79 3.11
4	4	6	1 0.9 0.8	3.04	3.81	1.34 3.1 4.53	2.06 3.92 5.46

Table 13: The biases of estimators of the scale parameter λ for $n = 24$

λ	m	k	p	$\hat{\lambda}_{mom,SRS}$	$\hat{\lambda}_{mom,ah}$	$\hat{\lambda}_{mle,ah}$
0.8	2	12	1 0.9 0.8	0.98	1.51 1.24 1.12	1.41 1.22 1.11
0.8	3	8	1 0.9 0.8	0.98	1.88 1.47 1.36	1.67 1.39 1.34
0.8	4	6	1 0.9 0.8	0.98	2.32 1.63 1.21	2 1.5 1.15
2	2	12	1 0.9 0.8	0.97	1.48 1.27 1.11	1.4 1.25 1.1
2	3	8	1 0.9 0.8	0.97	1.92 1.47 1.3	1.72 1.39 1.27
2	4	6	1 0.9 0.8	0.97	2.43 1.51 1.14	2.1 1.38 1.09
4	2	12	1 0.9 0.8	0.97	1.48 1.2 1.11	1.39 1.17 1.13
4	3	8	1 0.9 0.8	0.97	1.9 1.49 1.27	1.7 1.4 1.25
4	4	6	1 0.9 0.8	0.97	2.39 1.52 1.2	2.03 1.4 1.13

Table 14: The efficiency of estimators of the scale parameter λ with respect to $\hat{\lambda}_{mle,SRS}$ for $n = 24$

α	m	k	p	mom	mle
			1	33	31
0.8	2	12	0.9	29	29
			0.8	27	26
			1	44	36
0.8	3	8	0.9	36	31
			0.8	30	29
			1	51	43
0.8	4	6	0.9	35	32
			0.8	30	26
			1	33	30
2	2	12	0.9	31	28
			0.8	28	26
			1	43	37
2	3	8	0.9	36	32
			0.8	31	29
			1	53	44
2	4	6	0.9	37	32
			0.8	29	27
			1	34	32
4	2	12	0.9	29	28
			0.8	27	27
			1	44	38
4	3	8	0.9	35	33
			0.8	31	30
			1	52	43
4	4	6	0.9	36	32
			0.8	29	27

Table 15: The size of SRS needed to achieve the same quality of the estimator of λ as estimator from the RSS of size $n = 24$

α	λ	m	k	p	$\hat{\alpha}_{mom,SRS}$	$\hat{\alpha}_{mle,SRS}$	$\hat{\alpha}_{mom,ah}$	$\hat{\alpha}_{mle,ah}$	$\hat{\lambda}_{mom,SRS}$	$\hat{\lambda}_{mle,SRS}$	$\hat{\lambda}_{mom,ah}$	$\hat{\lambda}_{mle,ah}$
				1			2.78	0.99			1.84	4.91
0.8	2	2	12	0.9	3.31	0.59	2.98	0.84	1.83	0.85	1.87	4.93
				0.8			3.16	0.71			1.83	4.92
				1			2.37	1.35			1.83	4.94
0.8	2	3	8	0.9	3.31	0.59	2.64	1.12	1.83	0.85	1.84	4.94
				0.8			2.89	0.91			1.82	4.93
				1			2.15	1.55			1.85	5.01
0.8	2	4	6	0.9	3.31	0.59	3.4	0.28	1.83	0.85	1.96	5.02
				0.8			4.39	0.7			2.04	5.03
				1			0.29	1.89			3.19	5.73
2	3	2	12	0.9	0.55	1.59	0.06	1.83	2.92	5.95	3.13	5.8
				0.8			0.24	1.71			3.01	5.87
				1			0.84	2.22			3.33	5.62
2	3	3	8	0.9	0.55	1.59	0.45	2.03	2.92	5.95	3.21	5.71
				0.8			0.09	1.83			3.11	5.76
				1			1.1	2.33			3.41	5.52
2	3	4	6	0.9	0.55	1.59	0.27	1.15	2.92	5.95	3.34	5.64
				0.8			1.47	0.1			3.29	5.73
				1			5.19	2.62			3.49	0.27
5	3	2	12	0.9	5.5	2.68	5.32	2.62	3.08	0.85	3.4	0.43
				0.8			5.39	2.66			3.33	0.52
				1			5.04	2.54			3.77	0.13
5	3	3	8	0.9	5.5	2.68	5.16	2.57	3.08	0.85	3.62	0.11
				0.8			5.28	2.6			3.46	0.31
				1			4.92	2.49			3.88	0.34
5	3	4	6	0.9	5.5	2.68	5.09	2.56	3.08	0.85	2.67	1.06
				0.8			5.16	2.68			1.56	2.2

Table 16: The biases of estimators of the shape parameter α and the scale parameter λ for $n = 24$

α	λ	m	k	p	$\hat{\alpha}_{mom,SRS}$	$\hat{\alpha}_{mom,ah}$	$\hat{\alpha}_{mle,ah}$	$\hat{\lambda}_{mom,SRS}$	$\hat{\lambda}_{mom,ah}$	$\hat{\lambda}_{mle,ah}$
0.8	2	2	12	1 0.9 0.8	1.01	1.23 1.14 1.02	1.42 1.21 1.07	0.83	0.84 0.91 0.88	1.01 0.99 0.97
0.8	2	3	8	1 0.9 0.8	1.01	1.46 1.24 1.16	1.79 1.43 1.31	0.83	0.93 0.87 0.81	0.99 0.99 0.98
0.8	2	4	6	1 0.9 0.8	1.01	1.56 1.27 1.03	2.11 1.55 1.2	0.83	0.96 0.99 0.99	0.98 0.97 0.96
2	3	2	12	1 0.9 0.8	0.98	1.29 1.15 1.05	1.26 1.14 1.08	0.96	1.22 1.13 1.02	1.08 1.09 1.03
2	3	3	8	1 0.9 0.8	0.98	1.55 1.36 1.18	1.49 1.31 1.19	0.96	1.38 1.33 1.19	1.12 1.11 1.04
2	3	4	6	1 0.9 0.8	0.98	1.71 1.35 1.16	1.62 1.38 1.21	0.96	1.45 1.33 1.27	1.16 1.11 1.11
5	3	2	12	1 0.9 0.8	0.84	1.03 0.9 0.88	1.23 1.07 1.05	0.98	1.27 1.14 1.04	1.24 1.15 1.06
5	3	3	8	1 0.9 0.8	0.84	1.04 1.01 0.99	1.25 1.23 1.1	0.98	1.64 1.34 1.23	1.52 1.33 1.25
5	3	4	6	1 0.9 0.8	0.84	1.02 1.08 1.01	1.28 1.36 1.24	0.98	1.82 1.48 1.22	1.61 1.3 1.14

Table 17: The efficiency of estimators of the shape parameter α and the scale parameter λ concerning $\hat{\alpha}_{mle,SRS}$ and $\hat{\lambda}_{mle,SRS}$ for $n = 24$

					α		λ	
α	λ	m	k	p	mom	mle	mom	mle
0.8	2	2	12	1 0.9 0.8	28 28 25	33 29 27	28 26 25	46 48 24
0.8	2	3	8	1 0.9 0.8	31 30 26	39 34 30	25 23 25	58 26 30
0.8	2	4	6	1 0.9 0.8	36 29 26	47 35 30	28 27 32	52 46 19
2	3	2	12	1 0.9 0.8	34 29 26	30 27 26	31 29 27	32 30 27
2	3	3	8	1 0.9 0.8	42 34 31	37 32 28	40 31 28	43 32 29
2	3	4	6	1 0.9 0.8	50 36 29	43 34 30	45 36 31	40 37 27
5	3	2	12	1 0.9 0.8	36 33 28	28 30 26	34 28 26	30 28 26
5	3	3	8	1 0.9 0.8	39 37 29	28 29 27	44 35 31	37 31 29
5	3	4	6	1 0.9 0.8	52 37 32	34 32 33	54 40 34	41 35 28

Table 18: The size of SRS needed to achieve the same quality of the estimator as estimator from the RSS of size $n = 24$ for α and λ