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MODELING COOPERATIVE ADVERTISING DECISIONS IN A MANUFACTURER-DISTRIBUTOR-RETAILER SUPPLY CHAIN USING GAME THEORY

Peter E. EZIMADU
Department of Mathematics, Delta State University, Abraka, Nigeria
peterezimadu@yahoo.com

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Abstract: This work considers cooperative advertising decisions in a manufacturer-distributor-retailer supply chain, where the manufacturer is taken as the Stackelberg leader, using differential game theory. The distributor and retailer are the first and the second followers, respectively. We introduce the distributor into the traditional manufacturer-retailer channel through his direct involvement in advertising as being incorporated into the non-stochastic Sethi's sales-advertising dynamics. This is used to model the awareness share dynamics in which the distributor and the retailer directly engage in advertising, while the manufacturer bypasses the distributor to subsidise only the retail advertising effort. We consider a subsidised and unsubsidised channel structures, where each structure results in a system of three nonlinear equations, which cannot be solved analytically, but only numerically. However, we show that the unique solution to each of the systems exists, provided certain conditions are satisfied. The distributor and the retailer's advertising strategies are developed for both when subsidy is provided and when it is not provided. We also obtain the manufacturer's subsidy rate and the market awareness share for both when retail advertising is subsidised and when it is not subsidised. We observe that with the provision of subsidy, the distributor reduces his advertising effort. However, the resulting increase in the retail advertising effort is larger than the reduction in the distributor's advertising commitment, thus making the channel advertising effort larger with subsidy. It further shows that to avoid being shortchanged, each player should adopt only his optimal strategy or strategies as the case may be.

Keywords: Cooperative Advertising, Supply Chain, Sethi Model, Stackelberg Game, Differential Game.

MSC: 49N70, 91A65.

1. INTRODUCTION

Traditionally, cooperative advertising is an arrangement in which the manufacturer pays for a fraction of the cost incurred by the retailer in the process of advertising the manufacturer's product. In such a model setting the manufacturer sells his product through the retailer to the end-users. But in reality, a lot of manufacturers do not deal directly with their retailers. The distributor is usually the link between these two. In this work, we develop models which incorporate the distributor into the traditional cooperative advertising model.

The cooperative advertising literature can be categorized into static and dynamic cooperative advertising models. Berger [3] was the first known static cooperative advertising model. This was followed by a good number of manufacturer(s)-retailer(s) static models (Dant and Berger [7]; Bergen and John [2]; Kali [17]; Huang et al [14]; Xie and Wei [25]; He et al. [13]; Yan and Pei [26]). A major advantage of these static models is that they are relatively quite easy to analyse considering the factors involved (Huang et al. [14]). However, one of their drawbacks is that they are based on a single period.

Dynamic cooperative advertising models use differential game theory to study long-term relationships between various factors comprising various interests of the channel members. Such models have been considered by Chintagunta and Jain [4], Jorgensen et al [15], Jorgensen et al [16]. These papers are based on Nerlove-Arrow model (Nerlove and Arrow [20]). They are generally based on goodwill functions, which are related to the product's brand image, influenced through national and local advertising.

He et al. [12] developed dynamic cooperative advertising models using differential game theory. Their advertising dynamics was based on Sethi's advertising model (Sethi [22]). This Sethi's model is an extension of Vidale-Wolfe model (Vidal and Wolfe [24]). He et al. [12] addressed the issue of retail advertising and participation strategies. Taking a leap from the usual single manufacturer and single retailer traditional setting, He et al. [11] used the Lanchester model (Kimball [18]) to address advertising and subsidy strategies in a retail duopoly where the manufacturer supports his retailer, who is in competition with another retailer. In an extension of He et al. [11], Chutani and Sethi [6] developed cooperative advertising models in a manufacturer-retailers channel where the manufacturer sells his product through two competing retailers. They obtained the optimal retail advertising strategies and the manufacturer's participation strategies. Ezimadu and Nwozo [10] incorporated the manufacturer's advertising effort into He et al. [12]. They showed that with both the manufacturer and retailer involved in advertising, the individual and channel payoffs are larger. The papers discussed above are based on the classical cooperative advertising model where only the manufacturer(s) and retailer(s) are involved. For the first time we use differential game to consider cooperative advertising models where both the retailer and the distributor are directly involved in advertising with the manufacturer bypassing the distributor to support the retail advertising effort through subsidy.

This bypass can arise where there is distrust that if the distributor is given the subsidy, there is the tendency that it may not get to the retailer. Thus the manufacturer may opt for direct provision of advertising support fund, while still engaging the distributor(s) for the purpose of locating potential retailer(s) and engaging in the transfer of goods and services. Another situation where this can arise is where the manufacturer as the channel leader has firm control over the channel such that the products are transferred from the distributor to the retailer at a stipulated margin; and from the retailer to the end-user at a certain margin, a fraction of which can aid his (the retailer's) advertising effort. The essence of this control or strategy is to ensure that advertising support (subsidy) reaches the retailer, who is the actual source of the channel revenue. It is pertinent to note that the retailer is closer to the end-users than any of the other channel members.

We note that in recent times there is general effort towards eliminating middle men from supply channels (where possible). Quite a number of retailers and consumers now prefer ordering for goods directly from producers. Thus under certain conditions and appropriate arrangement or agreement, the manufacturer can bypass the middle man. In this work we note that the distributor's status as the middle man does not necessarily imply that the manufacturer cannot have dealings with the retailer. Further, as we will see later, apart from subsidy, the price margins are very important to the players. The retailer's price margin depends on the distributor's price margin, which depends on that of the manufacturer. Thus, more generally, the retailer's decision depends on the distributor's decision, which in turn depends on the manufacturer's decision.

There exists a number of reasons for the involvement of the retailer and the distributor in advertising. For instance, a new product, recently introduced into the market may require such efforts. Also, if a product has a substitute that has a strong influence on the consumers, there may be the need for such a combined effort if the supply chain members must remain in business. The support for the retailer stems from the fact that the retailer (the player in direct contact with the consumers) is the actual determinant of the sale of the manufacturer's product. Also, the manufacturer can opt for this considering the retailer to be more efficient in influencing would-be consumers' buying behaviour than the other channel members. This is because he has a good knowledge and understanding of the locality. This work determines the retailer and the distributor's optimal advertising strategies for a situation where retail advertising is subsidised, and where it is not subsidised. We also obtain the manufacturer's optimal participation strategy for retail advertising. We compare these strategies and analyse their effects on the payoffs.

2. MATHEMATICAL FORMULATION

2.1. The Game Components

In this subsection we state the components of the game.

The Players

The game-model involves the manufacturer, the distributor, and the retailer.

Players' Strategies

The Retailer's Strategy:	This is the retailer's advertising effort $\alpha_R(t)$, $t > 0$. It is nonnegative.
The Distributor's Strategy:	This is the distributor's advertising effort $\alpha_D(t)$, $t > 0$. It is also nonnegative.
The Manufacturer's Strategy:	This is the advertising support $\phi(t) \in [0, 1]$ from the manufacturer to the retailer. It is also known as subsidy (participation) rate.

The Players' Payoff Functions

The manufacturer, the distributor and the retailer's payoff functions P_M, P_D, and P_M, respectively, are the players' rewards obtained when the game ends.

The Timing of the Game

The game is modelled as an infinite horizon game.

Rules of the Game

The game-model is a hierarchical (Stackelberg) game with the manufacturer as the channel leader. He first unveils his subsidy (participation) rate $\phi(t)$ and margin m_M. Using this, the distributor unveils his advertising effort $\alpha_D(t)$ and his margin m_D. Based on these, the retailer decides on his local advertising effort $\alpha_R(t)$ and his price margin m_R. As a result, the equilibrium is obtained through backward induction.

State of the Game

The state of the game is the proportion of the market aware of the product at any given time. This is the market awareness share $x(t)$.

List of Notations

To aid the formulation of the models and enhance the comprehension of the work, we have the following list of notations:

$t \geq 0$	Time
$x(t) \in [0, 1]$	The proportion (fraction) of the market aware of the given product at time t
$x_0 \in [0, 1]$	The initial proportion (fraction) of the market aware of the given product
$\alpha_R(t) \geq 0$	The retailer's advertising effort at time t
$\alpha_D(t) \geq 0$	The distributor's advertising effort at time t
$\phi(t) \in [0, 1]$	The cooperative advertising participation (subsidy) rate offered by the manufacturer to the retailer
$a_{ef} \in [0, 1]$	The advertising effectiveness parameter
$\delta \in [0, a_{ef}]$	The market awareness share decay parameter
$\rho > 0$	The Discount rate
m_M, m_D, m_R	The margins of the manufacturer, the distributor and the retailer respectively
P_M, P_D, P_R	The value functions of the manufacturer, the distributor and the retailer respectively
$\lambda_M, \lambda_D, \lambda_R$	The intercepts of the value functions of the manufacturer, the distributor and the retailer respectively
$\gamma_M, \gamma_D, \gamma_R$	The rates of increase the value functions of the manufacturer, the distributor and the retailer respectively

2.2. The Players' Expenditures

We consider a supply chain involving one manufacturer, one distributor, and one retailer. The manufacturer sells his product to the retailer through the distributor. The retailer sells to the consumers. The retailer and the distributor's decision variables are their advertising efforts $\alpha_R(t)$ and $\alpha_D(t)$, respectively at time t; and the manufacturer's decision variable is his participation rate ϕ.

Due to the increasing nature of marginal costs associated with advertising, articles on cooperative advertising usually consider the cost function to be quadratic (Deal [8]; Prasad and Sethi [21]; Chutani and Sethi [6]). In line with this view, we let the cost functions to be quadratic in the distributor and the retailer's advertising efforts $\alpha_D(t)$ and $\alpha_R(t)$ respectively. Thus the manufacturer, the distributor, and the retailer's advertising expenditures are $\phi(t)\alpha_R(t)^2$, $\alpha_D(t)^2$, $(1-\phi(t))\alpha_R(t)^2$ respectively.

2.3. Market Share Dynamics

We model the dynamic effect of advertising on sale using Sethi's model (Sethi [22]). This is a modification of Vidale-Wolfe model (Vidale and Wolfe [24]). This (Sethi's) model is given by

$$x'(t) = a_{ef}\alpha(t)\sqrt{1-x(t)} - \delta x(t), \qquad x(0) = x_0 \in [0,1], \quad t \geq 0, \tag{1}$$

where $x(t)$ is the market awareness share; x_0 is the initial proportion of the market awareness share; a_{ef} is the advertising effectiveness parameter; $\alpha(t)$, $t \geq 0$ is the advertising effort, and δ is the decay rate. This model has been modified and extended into different versions; and has been empirically validated (Sogar [23]; Chintagunta and Jain [5]; Prasad and Sethi [21]; Bass et al. [1]; Naik et al. [19]; Erickson [9]). In this work we extend this dynamics by introducing the distributor's advertising effort. Thus we have

$$x'(t) = a_{ef}(\alpha_R(t) + \alpha_D(t))\sqrt{1-x(t)} - \delta x(t), \qquad x(0) = x_0 \in [0,1], \quad t \geq 0. \tag{2}$$

2.4. Players' Decision Sequence

The manufacturer being the channel leader first unveils his participation rate $\phi(x(t)) \in [0,1]$. Next, the distributor decides his advertising effort $\alpha_D(x(t) \mid \phi(t))$. In reaction, the retailer decides his advertising effort $\alpha_R(x(t) \mid \alpha_D(t), \phi(t))$ by solving the optimal control problem

$$P_R(x) = \max_{\alpha_R(x(t)|\alpha_D(t),\phi(t))\geq 0} \int_0^\infty e^{-\rho t}[m_R x(t) - (1-\phi(t))\alpha_R(x(t) \mid \alpha_D(t), \phi(t))^2]dt. \tag{3}$$

subject to (2), with $\alpha_R(t) = \alpha_R(x(t) \mid \alpha_D(t), \phi(t))$ and $\alpha_D(t) = \alpha_D(x(t) \mid \phi(t))$. In (3) $P_R(x)$ is the payoff of the retailer; ρ is the discount rate, and m_R is the retailer's margin.

In anticipation of the retailer's reaction, the distributor incorporates same (the retailer's reaction) into his control problem, and solves for his advertising effort $\alpha_D(x(t) \mid \phi(t))$. As such, the distributor's control problem is given by

$$P_D(x) = \max_{\alpha_D(x(t)|\phi(t))\geq 0} \int_0^\infty e^{-\rho t}[m_D x(t) - \alpha_D(x(t) \mid \phi(t))^2]dt, \tag{4}$$

$$\begin{aligned} x'(t) &= a_{ef}\Bigg(\alpha_R\Big(x(t) \mid \alpha_D(t), \phi(t)\Big) + \alpha_D\Big(x(t) \mid \phi(t)\Big)\Bigg)\sqrt{1-x(t)} - \delta x(t), \\ x(0) &= x_0 \in [0,1], t \geq 0, \end{aligned} \tag{5}$$

where P_D and m_D are the distributor's payoff and margin, respectively.

Further, in anticipation of the distributor and the retailer's reactions, the manufacturer incorporates their reactions into his optimal control problem

$$P_M(x) = \max_{0\leq\phi(t)\leq 1} \int_0^\infty e^{-\rho t}\Big[m_M x(t) - \phi(t)\alpha_R\big(x(t) \mid \alpha_D(t), \phi(t)\big)^2\Big]dt \tag{6}$$

$$\begin{aligned} x'(t) &= a_{ef}\Bigg(\alpha_R\Big(x(t) \mid \alpha_D(t), \phi(t)\Big) + \alpha_D\Big(x(t) \mid \phi(t)\Big)\Bigg)\sqrt{1-x(t)} - \delta x(t), \\ x(0) &= x_0 \in [0,1], t \geq 0. \end{aligned} \tag{7}$$

where P_M and m_M are the manufacturer's payoff and margin, respectively.

From the discussion above, we note that the decision variables are implicit functions of time.

3. THE PLAYERS' DECISIONS

Proposition 3.1

Suppose the players' margins are known, then the retailer and distributor's advertising strategies are

$$\alpha_R\Big(x(t) \mid \alpha_D(t), \phi(t)\Big) = \frac{a_{ef}P_{Rx}\sqrt{1-x}}{2(1-\phi(t))} \tag{8}$$

and

$$\alpha_D\Big(x(t) \mid \phi(t)\Big) = \frac{a_{ef}P_{Dx}\sqrt{1-x}}{2} \tag{9}$$

respectively; and the manufacturer's participation rate is

$$\phi(t) = \begin{cases} \frac{2P_{Mx}-P_{Rx}}{2P_{Mx}+P_{Rx}}, & 2P_{Mx} > P_{Rx} \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Proof:

From (2) and (3), we have the HJB equation

$$\rho P_R = \max_{\alpha_R(x(t)|\alpha_D(t),\phi(t))\geq 0}\Bigg[m_R x(t) - (1-\phi(t))\alpha_R\Big(x(t) \mid \alpha_D(t), \phi(t)\Big)^2$$

$$+P_{Rx}\Bigg[a_{ef}\Big(\alpha_R\big(x(t) \mid \alpha_D(t), \phi(t)\big) + \alpha_D\big(x(t) \mid \phi(t)\big)\Big)\sqrt{1-x(t)} - \delta x(t)\Bigg]\Bigg]. \tag{11}$$

Maximizing with respect to α_R, we have

$$-2(1-\phi(t))\alpha_R\Big(x(t)\mid\alpha_D(t),\phi(t)\Big)+a_{ef}P_{Rx}\sqrt{1-x(t)}=0.$$

Thus we have (8).

From (4) and (5), we have

$$\rho P_D=\max_{\alpha_D(x(t)|\phi(t))\geq 0}\Bigg[m_Dx(t)-\alpha_D\Big(x(t)\mid\phi(t)\Big)^2$$
$$+P_{Dx}\bigg[a_{ef}\Big(\alpha_R\big(x(t)\mid\alpha_D(t),\phi(t)\big)+\alpha_D\big(x(t)\mid\phi(t)\big)\Big)\sqrt{1-x(t)}-\delta x(t)\bigg]\Bigg]. \quad (12)$$

Maximizing with respect to α_D, we have

$$-2\alpha_D\big(x(t)\mid\phi(t)\big)+a_{ef}P_{Dx}\sqrt{1-x(t)}=0$$

which leads to (9).

From (6) and (7), we have

$$\rho P_M=\max_{0\leq\phi(t)\leq 1}\Bigg[m_Mx(t)-\phi(t)\alpha_R\big(x(t)\mid\alpha_D(t),\phi(t)\big)^2$$
$$+P_{Mx}\bigg[a_{ef}\Big(\alpha_R\big(x(t)\mid\alpha_D(t),\phi(t)\big)+\alpha_D\big(x(t)\mid\phi(t)\big)\Big)\sqrt{1-x(t)}-\delta x(t)\bigg]\Bigg]. \quad (13)$$

Putting (8) and (9) into (13), we have

$$\rho P_M=\max_{0\leq\phi(t)\leq 1}\Bigg[m_Mx(t)-\phi(t)\bigg(\frac{a_{ef}P_{Rx}\sqrt{1-x}}{2(1-\phi(t))}\bigg)^2$$
$$+P_{Mx}\bigg[a_{ef}\bigg(\frac{a_{ef}P_{Rx}\sqrt{1-x}}{2(1-\phi(t))}+\frac{a_{ef}P_{Rx}\sqrt{1-x}}{2}\bigg)\sqrt{1-x(t)}-\delta x(t)\bigg]\Bigg]. \quad (14)$$

Maximizing with respect to ϕ, we have

$$-\frac{P_{Rx}}{2}\left[\frac{(1-\phi)^2+2\phi(1-\phi)}{(1-\phi)^2}\right]+P_{Mx}=0$$

which leads to (10). □

Proposition 3.1 presents the general form of the retailer, the distributor, and the manufacturer's strategies α_R, α_D, and ϕ, respectively. From (8), we observe that the advertising effort α_R is directly proportional to the advertising effectiveness parameter a_{ef}; the rate of increase of the retailer's payoff P_{Rx} and the manufacturer's participation rate ϕ. Thus with improved effectiveness, the retailer will be motivated to advertise. Further, with an increasing rate of payoff, he would be motivated to advertise. This is understandable owing to the fact that the central goal of the players is larger payoffs. We also observe that with assistance given to only the retailer, whereas both the retailer and distributor are involved in advertising, the retailer is expected to justify this unilateral assistance. Thus his advertising effort increases with subsidy.

A further look shows that the retailer's effort is inversely proportional to the market awareness $x(t)$. As the market awareness increases, the retailer reduces effort since the number of those to be wooed into buying the product would have reduced. Similar explanations apply to (9), where the distributor's advertising effort is directly proportional to the advertising effort a_{ef} and rate of increase P_{Dx}, and inversely proportional to the awareness share x.

Equation (10) shows that a certain condition must be satisfied before subsidy can be provided. The rate of increase of the manufacturer's payoff P_{Mx} must be twice greater than the rate of increase of the retailer's payoff P_{Rx}.

Now, using (8) and (9) in (11) and (12), we have

$$\rho P_R = m_R x + \frac{a_{ef}^2 P_{Rx}^2 (1-x)}{4(1-\phi(t))} + \frac{a_{ef}^2 P_{Rx} P_{Dx}(1-x)}{2} - P_{Rx}\delta x \tag{15}$$

and

$$\rho P_D = m_D x + \frac{a_{ef}^2 P_{Dx}^2 (1-x)}{4} + \frac{a_{ef}^2 P_{Rx} P_{Dx}(1-x)}{2(1-\phi(t)} - P_{Dx}\delta x \tag{16}$$

respectively.

4. EQUILIBRIUM DECISIONS

The next result presents the retailer and the distributor's advertising efforts when retail advertising is not subsidised. It also gives the players' payoffs.

Proposition 4.1

Suppose the manufacturer does not participate in retail advertising, then the retailer and the distributor's advertising efforts are

$$\alpha_R\Big(x(t) \mid \phi(t)\Big) = \frac{a_{ef}\gamma_R\sqrt{1-x}}{2} \tag{17}$$

and

$$\alpha_D\Big(x(t) \mid \phi(t)\Big) = \frac{a_{ef}\gamma_D\sqrt{1-x}}{2} \tag{18}$$

respectively; and the players' payoffs are

$$P_R(x) = \lambda_R + \gamma_R x, \tag{19}$$

$$P_D(x) = \lambda_D + \gamma_D x, \tag{20}$$

$$P_M(x) = \lambda_M + \gamma_M x, \tag{21}$$

where

$$\gamma_R = \frac{4m_R}{4(\rho+\delta) + a_{ef}^2(\gamma_R + 2\gamma_D)}, \tag{22}$$

$$\gamma_D = \frac{4m_D}{4(\rho+\delta) + a_{ef}^2(\gamma_D + 2\gamma_R)}, \tag{23}$$

$$\gamma_M = \frac{2m_M}{2(\rho+\delta) + a_{ef}^2(\gamma_R + \gamma_D)}, \tag{24}$$

$$\lambda_R = \frac{a_{ef}^2 \gamma_R(\gamma_R + 2\gamma_D)}{4\rho}, \tag{25}$$

$$\lambda_D = \frac{a_{ef}^2 \gamma_D(\gamma_D + 2\gamma_R)}{4\rho}, \tag{26}$$

$$\lambda_M = \frac{a_{ef}^2 \gamma_M(\gamma_R + \gamma_D)}{2\rho}. \tag{27}$$

Proof:

Since $\phi = 0$, we have that (8) and (9) become

$$\alpha_R\Big(x(t) \mid \alpha_D(t)\Big) = \frac{a_{ef} P_{Rx}\sqrt{1-x}}{2} \tag{28}$$

and

$$\alpha_D\Big(x(t)\Big) = \frac{a_{ef} P_{Dx}\sqrt{1-x}}{2} \tag{29}$$

respectively.

Since the manufacturer does not participate in retail advertising, we have that (15) becomes

$$\rho P_R = m_R x + \frac{a_{ef}^2 P_{Rx}^2(1-x)}{4} + \frac{a_{ef}^2 P_{Rx} P_{Dx}(1-x)}{2} - P_{Rx}\delta x; \tag{30}$$

equation (16) becomes

$$\rho P_D = m_D x + \frac{a_{ef}^2 P_{Dx}^2(1-x)}{4} + \frac{a_{ef}^2 P_{Rx} P_{Dx}(1-x)}{2} - P_{Dx}\delta x; \tag{31}$$

and equation (13) becomes

$$\rho P_M = m_M x + \frac{a_{ef}^2 P_{Rx} P_M x(1-x)}{2} + \frac{a_{ef}^2 P_{Dx} P_{Mx}(1-x)}{2} - P_{Mx}\delta x. \tag{32}$$

Let

$$P_R(x) = \lambda_R + \gamma_R x, \tag{33}$$

$$P_D(x) = \lambda_D + \gamma_D x, \tag{34}$$

$$P_M(x) = \lambda_M + \gamma_M x, \tag{35}$$

$$\Rightarrow \qquad P_{Rx} = \gamma_R, \quad P_{Dx} = \gamma_D, \quad P_{Mx} = \gamma_M. \tag{36}$$

Using (36) in (28) and (29), we have (17) and (18) respectively.
Using (33) and (36) in (30), we have

$$\rho(\lambda_R + \gamma_R x) = m_R x + \frac{a_{ef}^2 \gamma_R^2 (1-x)}{4} + \frac{a_{ef}^2 \gamma_R \gamma_D (1-x)}{2} - \gamma_R \delta x. \tag{37}$$

Equating the coefficients of x, we have (22). Equating constants, we have (25).
Using (34) and (36) in (31), we have

$$\rho(\lambda_D + \gamma_D x) = m_D x + \frac{a_{ef}^2 \gamma_D^2 (1-x)}{4} + \frac{a_{ef}^2 \gamma_R \gamma_D (1-x)}{2} - \gamma_D \delta x. \tag{38}$$

Equating the coefficients of x, we have (23). Equating constants, we have (26).
Using (35) and (36) in (32), we have

$$\rho(\lambda_M + \gamma_M x) = m_M x + \frac{a_{ef}^2 \gamma_R \gamma_M (1-x)}{2} + \frac{a_{ef}^2 \gamma_D \gamma_M (1-x)}{2} - \gamma_M \delta x. \tag{39}$$

Equating the coefficients of x, we have (24). Equating constants, we have (27).
□

The next proposition is a version of Proposition 4.1 in which retail advertising is subsidised by the manufacturer.

Proposition 4.2

Suppose the manufacturer bypasses the distributor to subsidise the retail advertising effort, then the retailer and the distributor's advertising efforts are

$$\alpha_R\Big(x(t) \mid \alpha_D(t), \phi(t)\Big) = \frac{a_{ef}(2\gamma_M + \gamma_R)\sqrt{1-x}}{4} \tag{40}$$

and

$$\alpha_D\Big(x(t) \mid \phi(t)\Big) = \frac{a_{ef}\gamma_D\sqrt{1-x}}{2} \tag{41}$$

respectively; and the players' payoffs are

$$P_R(x) = \lambda_R + \gamma_R x, \tag{42}$$

$$P_D(x) = \lambda_D + \gamma_D x, \tag{43}$$

$$P_M(x) = \lambda_M + \gamma_M x, \tag{44}$$

where

$$\gamma_R = \frac{8m_R}{8(\rho+\delta) + a_{ef}^2(\gamma_R + 4\gamma_D + 2\gamma_M)}, \tag{45}$$

$$\gamma_D = \frac{4m_D}{4(\rho+\delta) + a_{ef}^2(\gamma_R + \gamma_D + 2\gamma_M)}, \tag{46}$$

$$\gamma_M = \frac{16m_M - a_{ef}^2\gamma_R^2}{16(\rho+\delta) + 4a_{ef}^2(\gamma_R + 2\gamma_D + \gamma_M)}, \tag{47}$$

$$\lambda_R = \frac{a_{ef}^2\gamma_R(\gamma_R + 4\gamma_D + 2\gamma_M)}{8\rho}, \tag{48}$$

$$\lambda_D = \frac{a_{ef}^2\gamma_D(\gamma_R + \gamma_D + 2\gamma_M)}{4\rho}, \tag{49}$$

$$\lambda_M = \frac{a_{ef}^2[4\gamma_M(\gamma_R + 2\gamma_D + \gamma_M) + \gamma_R^2}{16\rho}. \tag{50}$$

Proof:

Since retail advertising is subsidised, we have that (8) and (9) become

$$\alpha_R\Big(x(t) \mid \alpha_D(t), \phi(t)\Big) = \frac{a_{ef}(2P_{Mx} + P_{Rx})\sqrt{1-x}}{4} \tag{51}$$

and

$$\alpha_D\Big(x(t) \mid \phi(t)\Big) = \frac{a_{ef}P_{Dx}\sqrt{1-x}}{2} \tag{52}$$

respectively.
Using (10) in (15), (16), and (14), we have

$$\rho P_R = m_R x + \frac{a_{ef}^2(2P_{Mx} + P_{Rx})P_{Rx}(1-x)}{8} + \frac{a_{ef}^2 P_{Rx}P_{Dx}(1-x)}{2} - P_{Rx}\delta x, \tag{53}$$

$$\rho P_D = m_D x + \frac{a_{ef}^2 P_{Dx}^2(1-x)}{4} + \frac{a_{ef}^2(2P_{Mx} + P_{Rx})P_{Dx}(1-x)}{4} - P_{Dx}\delta x \tag{54}$$

and

$$\rho P_M = m_M x - \frac{a_{ef}^2(4P_{Mx}^2 - P_{Rx}^2(1-x)}{16} + \frac{a_{ef}^2(2P_{Mx} + P_{Rx})P_{Mx}(1-x)}{4} + \frac{a_{ef}^2 P_{Dx}P_{Mx}(1-x)}{2} - P_{Mx}\delta x \tag{55}$$

respectively.
Let

$$P_R(x) = \lambda_R + \gamma_R x, \tag{56}$$

$$P_D(x) = \lambda_D + \gamma_D x, \tag{57}$$

$$P_M(x) = \lambda_M + \gamma_M x, \tag{58}$$

$$\Rightarrow \qquad P_{Rx} = \gamma_R, \quad P_{Dx} = \gamma_D, \quad P_{Mx} = \gamma_M. \tag{59}$$

Using (59) in (51) and (52), we have (40) and (41) respectively.
Using (56) and (59) in (53), we have

$$\rho(\lambda_R + \gamma_R x) = m_R x + \frac{a_{ef}^2 \gamma_R (2\gamma_M + \gamma_R)(1-x)}{8} + \frac{a_{ef}^2 \gamma_R \gamma_D (1-x)}{2} - \gamma_R \delta x.$$

Equating the coefficients of x, we have (45). Equating constants, we have (48).
Using (57) and (59) in (54), we have

$$\rho(\lambda_D + \gamma_D x) = m_D x + \frac{a_{ef}^2 \gamma_D^2 (1-x)}{4} + \frac{a_{ef}^2 \gamma_D (2\gamma_M + \gamma_R)(1-x)}{4} - \gamma_D \delta x.$$

Equating the coefficients of x, we have (46). Equating constants, we have (49).
Using (58) and (59) in (55), we have

$$\rho(\lambda_M + \gamma_M x) = m_M x - \frac{a_{ef}^2 (4\gamma_M^2 - 2\gamma_R^2)(1-x)}{16} + \frac{a_{ef}^2 \gamma_M (2\gamma_M + \gamma_R)(1-x)}{4} + \frac{a_{ef}^2 \gamma_D \gamma_M (1-x)}{2} - \gamma_M \delta x.$$

Equating the coefficients of x, we have (47). Equating constants, we have (50).
□

Proposition 4.1 gives the feedback Stackelberg equilibrium characterising the retailer and the distributor's advertising efforts for a situation where retail advertising is not subsidised. We observe from (19), (20), and (21) that the rates of increase γ_R, γ_D, and γ_M are very important to the retailer, the distributor, and the manufacturer respectively. Now, we see from (22), (23), and (24) that the players' margins are pivotal to these rates of increase. This is also the case with Proposition 4.2 where γ_R, γ_D, and γ_M depend on the players' respective margins as can be seen in (45), (46), and (47) respectively. Further, with the manufacturer's participation, the retail advertising effort improved from being influenced by only the rate of increase of the retailer's payoff, to being influenced by the rates of increase of both the retailer and manufacturer's payoffs. Clearly, this is a reflection of the subsidy from the manufacturer.

5. EXISTENCE OF UNIQUE SOLUTIONS

Observe that it is appropriate to consider the retailer's payoff to increase with increase in the market awareness. Thus we write $P_{Rx} = \gamma_R > 0$. Considering (22)-(24) and (45)-(47), we observe from (22) that for γ_R to be positive, m_R and a_{ef} must both be positive. Similarly, from (23) and (24), we observe that γ_D and γ_M can only be positive if m_D and m_M are respectively positive with a_{ef} also positive. Similarly, considering (46) and (47), we note that m_R, m_D, and a_{ef} must be positive for γ_R and γ_D to be positive. Now, from (47), we observe that γ_M can only be positive if m_M and a_{ef} are positive, and $16m_M > a_{ef}^2\gamma_R^2$.

In addition to the discussion above, we observe that based on the nature of the payoffs in (19)-(21) and (42)-(44), it is impossible for the rates of increase of the payoffs to be negative, else, we will have a situation where the players are running at a loss. We also note that having $\gamma_R = 0$, $\gamma_D = 0$, $\gamma_R = 0$ will imply $\lambda_R = 0$, $\lambda_D = 0$, $\lambda_M = 0$ (from (25)-(27) and (48)-(50)), consequently leading to zero payoffs.

Also, considering the advertising efforts as given by (8) and (9), we observe that $P_{Rx} \le 0$, $P_{Dx} \le 0$ (that is $\gamma_R \le 0$, $\gamma_D \le 0$) would imply negative or zero advertising efforts, of which the admissible class of decisions have been earlier taken to be non-negative. Thus, it would be unrealistic to have any of the rates γ_R, γ_D, γ_M as negative. As such, the rates γ_R and γ_D, and hence from (24), γ_M are positive.

Further, considering (40), we observe that $\gamma_M \le 0$ ($P_{Mx} \le 0$) would imply $\gamma_R \le 0$ ($P_{Rx} \le 0$) since the condition for the provision of subsidy as given by (10) requires that $2P_{Mx} > P_{Rx}$ (that is $2\gamma_M > \gamma_R$). But $\gamma_R \ge 0$, hence $\gamma_M \ge 0$. Also, from (41), it is clear that $\gamma_D \ge 0$.

5.1 Existence of a Unique Solution when Retail Advertising Is not Subsidised

We rewrite the system (22)-(24) as

$$\gamma_R = \frac{M_R}{2Q + a_{ef}^2(\gamma_R + 2\gamma_D)}, \tag{60}$$

$$\gamma_D = \frac{M_D}{2Q + a_{ef}^2(2\gamma_R + \gamma_D)}, \tag{61}$$

$$\gamma_M = \frac{M_M}{Q + a_{ef}^2(\gamma_R + \gamma_D)}, \tag{62}$$

where $M_R = 4m_R$, $M_D = 4m_D$, $M_M = 2m_M$, $Q = 2(\rho + \delta)$. From (60)

$$\gamma_R = \frac{-(2Q + 2\gamma_D a_{ef}^2) \pm \sqrt{(2Q + 2\gamma_D a_{ef}^2)^2 - 4a_{ef}^2(-M_R)}}{2a_{ef}^2}, \tag{63}$$

and from (61) we have

$$\gamma_R = \frac{\gamma_D^2 a_{ef}^2 + 2Q\gamma_D - M_D}{2\gamma_D a_{ef}^2}. \tag{64}$$

Now from (63) and (64), we have

$$3a_{ef}^2\gamma_D^4 + 8a_{ef}^2 Q\gamma_D^3 + (4M_R a_{ef}^2 + 4Q^2 - 2M_D a_{ef}^2)\gamma_D^2 - M_D^2 = 0.$$

Now, let

$$G(\gamma_D) = 3a_{ef}^2\gamma_D^4 + 8a_{ef}^2 Q\gamma_D^3 + (4M_R a_{ef}^2 + 4Q^2 - 2M_D a_{ef}^2)\gamma_D^2 - M_D^2. \tag{65}$$

We observe that as γ_R tends to $\pm\infty$, $G(\gamma_D)$ tends to $+\infty$. Also, it is clear that the function is negative at $\gamma_R = 0$.
Now, observe that G is continuous and differentiable passing through the γ_D-axis at least two times. Thus we have that
if there is only one negative real root, then there are three positive real roots.
if there are three negative real roots, then there is one positive real root.
if there are two negative real roots, then there are two positive real roots.
if there are only two real roots, then while one will be negative, the other will be positive.

Suppose that the four roots are $\gamma_D(1)$, $\gamma_D(2)$, $\gamma_D(3)$, and $\gamma_D(4)$, then (65) can be expressed as

$$G(\gamma_D) = (\gamma_D - \gamma_D(1))(\gamma_D - \gamma_D(2))(\gamma_D - \gamma_D(3))(\gamma_D - \gamma_D(4))$$

where $\gamma_D(4) > \gamma_D(3) > \gamma_D(2) > \gamma_D(1)$. Observe that the slope at $\gamma_D(3)$ is negative.
Differentiating G, we have that

$$G'(\gamma_D) = 12a_{ef}^2\gamma_D^3 + 24a_{ef}^2 Q\gamma_D^2 + 2(4M_R a_{ef}^2 + 4Q^2 - 2M_D a_{ef}^2)\gamma_D > 0.$$

Hence with

$$4M_R a_{ef}^2 + 4Q^2 \geq 2M_D a_{ef}^2$$

there exists only one positive real root to (65), which by extension implies the existence of a unique positive γ_D. With the existence of unique γ_R and γ_D, it follows from (62) that $\exists$ a unique γ_M. Hence $\exists$ a unique $(\gamma_R, \gamma_D, \gamma_M)$ to the system (22)-(24). This implies that there exists a unique solution to the differential games (2)-(7) when retail advertising is not subsidised.

5.2 Existence of a Unique Solution when Retail Advertising Is Subsidised

We rewrite (45)-(47) as follows:

$$\gamma_R = \frac{N_R}{2N + a_{ef}^2(\gamma_R + 4\gamma_D + 2\gamma_M)}, \tag{66}$$

$$\gamma_D = \frac{N_D}{N + a_{ef}^2(\gamma_R + \gamma_D + 2\gamma_M)}, \quad (67)$$

$$\gamma_M = \frac{N_M - a_{ef}^2\gamma_R^2}{4N + 4a_{ef}^2(\gamma_R + 2\gamma_D + \gamma_M)}, \quad (68)$$

where $N_R = 8m_R, N_D = 4m_D, N = (\rho + \delta)$.
From (66), we have that

$$\gamma_D = \frac{N_R - 2N\gamma_R - a_{ef}^2{\gamma_R}^2 - 2a_{ef}^2\gamma_M\gamma_R}{4a_{ef}^2\gamma_R}. \quad (69)$$

From (67) and (69), we have that

$$\frac{N_R - 2N\gamma_R - a_{ef}^2{\gamma_R}^2 - 2a_{ef}^2\gamma_M\gamma_R}{4a_{ef}^2\gamma_R}$$

$$= \frac{4a_{ef}^2N_D\gamma_R}{4a_{ef}^2N\gamma_R + 4a_{ef}^4{\gamma_R}^2 + a_{ef}^2(N_R - 2N\gamma_R - a_{ef}^2{\gamma_R}^2 - 2a_{ef}^2\gamma_M\gamma_R) + 8a_{ef}^4\gamma_M\gamma_R}. \quad (70)$$

Thus

$$\gamma_M = \frac{\pm\sqrt{U^2 + V} + U}{W}, \quad (71)$$

where

$$\begin{aligned} U &= -12{\gamma_R}^3a_{ef}^6 - 16N{\gamma_R}^2a_{ef}^4 + 4N_R\gamma_Ra_{ef}^4, \\ V &= -144a_{ef}^{12}{\gamma_R}^6 - 384Na_{ef}^{10}{\gamma_R}^5 + 96N_Ra_{ef}^{10}{\gamma_R}^4 - 768N_Da_{ef}^{10}{\gamma_R}^4 \\ &\quad -192N^2a_{ef}^8{\gamma_R}^4 + 48{N_R}^2a_{ef}^8{\gamma_R}^2, \\ W &= 24a_{ef}^6{\gamma_R}^2. \end{aligned}$$

From (68) and (69), we have that

$$\gamma_M = \frac{(N_M - a_{ef}^2{\gamma_R}^2)(4a_{ef}^2\gamma_R)}{16Na_{ef}^2\gamma_R + 4a_{ef}^2[4a_{ef}^2{\gamma_R}^2 + 2(N_R - 2N\gamma_R - a_{ef}^2{\gamma_R}^2 - 2a_{ef}^2\gamma_M\gamma_R) + 4a_{ef}^2\gamma_R\gamma_M]}. \quad (72)$$

From (71) and (72), we have that

$$9437184N^2a_{ef}^{10}{\gamma_R}^8 + (9437184NN_Ra_{ef}^{10} + 14155776NN_Ma_{ef}^{10}){\gamma_R}^7$$

$$+(7077888N_MN_Ra_{ef}^{10} + 2359296N_R^2a_{ef}^{10} + 5308416N_M^2a_{ef}^{10} + 18874368N_RN^2a_{ef}^8){\gamma_R}^6$$

$$+(14155776NN_MN_Ra_{ef}^8 + 4718592NN_R^2a_{ef}^8){\gamma_R}^5$$

$$+(9437184N^2{N_R}^2a_{ef}^6 - 3538944N_M{N_R}^2a_{ef}^8 - 2359296{N_R}^3a_{ef}^8$$

$$+147456N_Da_{ef}^4 - 12288N^2a_{ef}^2){\gamma_R}^4$$

$$+(24576NN_Ra_{ef}^2 - 4718592NN_R^3a_{ef}^6){\gamma_R}^3$$

$$+(147456N_DN_Ra_{ef}^2 - 12288a_{ef}^2N_R^2 - 12288N_RN^2 + 589824N_R^4a_{ef}^6){\gamma_R}^2$$
$$+24576NN_R2\gamma_R - 12288N_R^3 = 0. \quad (73)$$

Letting

$$\begin{aligned}
k_0 &= 12288N_R^3,\\
k_1 &= 24576NN_R2,\\
k_2 &= 147456N_DN_Ra_{ef}^2 - 12288a_{ef}^2N_R^2 - 12288N_RN^2 + 589824N_R^4a_{ef}^6,\\
k_3 &= 24576NN_Ra_{ef}^2 - 4718592NN_R^3a_{ef}^6\\
k_4 &= 9437184N^2{N_R}^2a_{ef}^6 - 3538944N_M{N_R}^2a_{ef}^8 - 2359296{N_R}^3a_{ef}^8\\
&\quad +147456N_Da_{ef}^4 - 12288N^2a_{ef}^2,\\
k_5 &= 9437184N^2{N_R}^2a_{ef}^6 - 3538944N_M{N_R}^2a_{ef}^8 - 2359296{N_R}^3a_{ef}^8,\\
&\quad +147456N_Da_{ef}^4 - 12288N^2a_{ef}^2,\\
k_6 &= 7077888N_MN_Ra_{ef}^{10} + 2359296N_R^2a_{ef}^{10} + 5308416N_M^2a_{ef}^{10} + 18874368N_RN^2a_{ef}^8,\\
k_7 &= 9437184NN_Ra_{ef}^{10} + 14155776NN_Ma_{ef}^{10}\\
&\quad \textit{and}\\
k_8 &= 9437184N^2a_{ef}^{10},
\end{aligned}$$

then (73) can be expressed as

$$G(\gamma_R) = k_8{\gamma_R}^8+k_7{\gamma_R}^7+k_6{\gamma_R}^6+k_5{\gamma_R}^5+k_4{\gamma_R}^4+k_3{\gamma_R}^3+k_2{\gamma_R}^2+k_1\gamma_R-k_0. \quad (74)$$

Suppose $k_0, k_1, ..., k_8$ are positive, then as γ_R approaches $\pm\infty$, $G(\gamma_R)$ approaches $+\infty$. Further, we observe that at $\gamma_R = 0$, $G(\gamma_R) < 0$.

Now, observe that G is continuous with its graph passing through the γ_R-axis at least two times.

Now, if all eight roots are real, then there are seven positive roots and one negative root; six positive and two negative roots; five positive and three negative roots; four positive and four negative roots; three positive and five negative roots; two positive and six negative roots; or one positive and seven negative roots. Further, suppose that there are just two real roots, then one is positive, while the other is negative.

If the roots are $\gamma_R(1)$, $\gamma_R(2)$, ... , $\gamma_R(8)$, such that $\gamma_R(8) > \gamma_R(7) > ... > \gamma_R(1)$, then G can be expressed as

$$G(\gamma_R) = (\gamma_R - \gamma_R(1))(\gamma_R - \gamma_R(2))...(\gamma_R - \gamma_R(8)).$$

We note that the slope at $\gamma_R(2)$, $\gamma_R(4)$, $\gamma_R(6)$, $\gamma_R(8)$ are negative. That is $G'(\gamma_R)\mid_{\gamma_R=\gamma_R(2)}$, $G'(\gamma_R)\mid_{\gamma_R=\gamma_R(4)}$, $G'(\gamma_R)\mid_{\gamma_R=\gamma_R(6)}$, $G'(\gamma_R)\mid_{\gamma_R=\gamma_R(8)} < 0$.

But with $k_1, k_2, ..., k_8 > 0$, we have that

$$G'(\gamma_R) = 8k_8{\gamma_R}^7 + 7k_7{\gamma_R}^6 + ... + 2k_2\gamma_R + k_1 > 0.$$

Thus we conclude that with

$$9437184N^2{N_R}^2a_{ef}^6+147456N_Da_{ef}^4 \geq 3538944N_M{N_R}^2a_{ef}^8+2359296{N_R}^3a_{ef}^8+12288N^2a_{ef}^2,$$

$$24576NN_Ra_{ef}^2 \geq 4718592NN_R^3a_{ef}^6$$

and

$$147456N_DN_Ra_{ef}^2 + 589824N_R^4a_{ef}^6 \geq 12288a_{ef}^2N_R^2 + 12288N_RN^2,$$

there is only one positive real root, and hence ∃ a unique solution to (74), implying the existence of a unique solution to the system (45)-(47).

It is important to note that differential game problems generalize optimal control problems involving two or more controllers or players. Conceptually, differential game models are more complicated than optimal control models. This is because unlike optimal control problems, determining what makes up a solution is not apparent in the case of differential game, but is based on the admissible class of decisions, which are based on the model requirements and assumptions.

Thus, these conditions (having the rates of increase to be positive) are not merely imposed, but are obvious requirements for the feasibility/implementation of the model.

6. AWARENESS SHARE

Let $(\phi = 0)$ and $(\phi > 0)$ be subscripts denoting situations where retail advertising is not subsidised and where it is subsidised, respectively.
Now, using (17) and (18) in (2), we have

$$x'(t) = \frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)})}{2} - \frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}x. \tag{75}$$

Using the integrating factor

$$\exp\left[\int\left(\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}\right)dt\right] = \exp\left[\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}t\right], \tag{76}$$

then from (75) and (76) we have that

$$\exp\left[\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}x'\right] + \exp\left[\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}t\right]$$
$$\times\left[\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}x\right]$$
$$= \exp\left[\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}t\right]\left[\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)})}{2}x\right].$$

Thus we have that

$$x = \frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)})}{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta} + \frac{C}{\exp\left[\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}t\right]}. \tag{77}$$

When $t = 0$, $x = x_0$. As such we have that

$$C = x_0 - \frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)})}{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}. \tag{78}$$

Using (78) in (77), we have that

$$x_{(\phi=0)}(t) = \frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)})}{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}$$
$$+\frac{(a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta)x_0 - a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)})}{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}$$
$$\times \exp\left[-\frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}{2}t\right]. \tag{79}$$

As $t \to \infty$, we have that

$$x_{(\phi=0)\infty}(t) = \frac{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)})}{a_{ef}^2(\gamma_{R(\phi=0)} + \gamma_{D(\phi=0)}) + 2\delta}. \tag{80}$$

By similarly substituting (51) and (52) into (1) and following a similar argument, we have that

$$x_{(\phi>0)}(t) = \frac{a_{ef}^2(2(\gamma_{M(\phi>0)} + \gamma_{D(\phi>0)}) + \gamma_{R(\phi>0)})}{a_{ef}^2(2(\gamma_{M(\phi>0)} + \gamma_{D(\phi>0)}) + \gamma_{R(\phi>0)}) + 4\delta}$$
$$+\frac{a_{ef}^2(2(\gamma_{M(\phi>0)} + \gamma_{D(\phi>0)}) + \gamma_{R(\phi>0)}) + 4\delta)x_0 - a_{ef}^2(2(\gamma_{M(\phi>0)} + \gamma_{D(\phi>0)}) + \gamma_{R(\phi>0)})}{a_{ef}^2(\gamma_{R(\phi>0)} + \gamma_{D(\phi>0)}) + 2\delta}$$
$$\times \exp\left[-\frac{a_{ef}^2(2(\gamma_{M(\phi>0)} + \gamma_{D(\phi>0)}) + \gamma_{R(\phi>0)}) + 4\delta}{4}t\right]. \tag{81}$$

and

$$x_{(\phi>0)\infty}(t) = \frac{a_{ef}^2(2(\gamma_{M(\phi>0)} + \gamma_{D(\phi>0)}) + \gamma_{R(\phi>0)})}{a_{ef}^2(2(\gamma_{M(\phi>0)} + \gamma_{D(\phi>0)}) + \gamma_{R(\phi>0)}) + 4\delta}. \tag{82}$$

Equation (79) is the awareness share for a situation where the manufacturer does not subsidise the retail advertising effort, and (80) is it's long-run value. This means that irrespective of the improvement on the retailer and/or the distributor's advertising effort, the market share has a limit, which cannot be exceeded. Also (81) is the subsidised awareness share while (82) is the long-run subsidised awareness share.

7. NUMERICAL ILLUSTRATION

7.1. Parameter Values

This work involves a manufacturer, a distributor, and a retailer playing a Stackelberg game. The manufacture is the channel leader, while the distributor and retailer are the first and the second (last) followers respectively. Since the manufacturer enjoys the first mover's advantage, it follows that $m_M > m_D > m_R$. Thus we let $m_M = 0.6$, $m_D = 0.4$, $m_R = 0.2$. The advertising effectiveness $a_{ef} \in [0, 1]$. Thus we let $a_{ef} = 0.4$. The awareness share decay parameter $\delta \in [0, a_{ef}]$. Thus we take $\delta = 0.2$. The game is played on an infinite horizon in which the players are considered to be foresighted. Thus we let $\rho = 0.04$.

7.2. The Effect of Advertising Efforts on the Payoffs

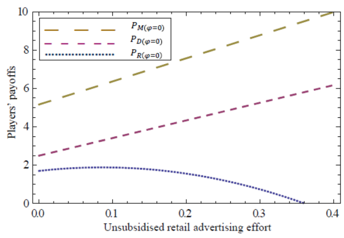

Figure 1: The Effects of retail advertising on the players' payoffs when retail advertising is unsubsidised

We note that each of the games (2)-(3), (4)-(5), and (6)-(7) is an optimal control problem. The control obtained by solving such a problem is not a single fixed value, but a function which gives the time path of the control variable. That is why $\alpha_R(x(t))$ (in (17) and (40)) and $\alpha_D(x(t))$ (in (18) and (41)) are not just single values but functions of the awareness share x, which is a function of time t. Thus $\alpha_R(x(t))$ is an implicit function of time. Observe that these advertising efforts vary as the awareness share changes over time. The varying effects of $\alpha_R(x(t))$ and $\alpha_D(x(t))$ affect the payoffs $P_R(x)$, $P_D(x)$, and $P_M(x)$ as can be seen in (3), (4) and (6).

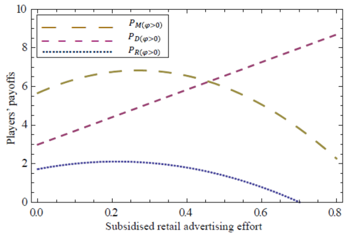

Figure 2: The effects of retail advertising on the players' payoffs when retail advertising is subsidised

From the parameter values, we have that when retail advertising is not subsidised $\gamma_{R(\phi=0)} = 0.5579$, $\gamma_{D(\phi=0)} = 1.2022$ and $\gamma_{M(\phi=0)} = 1.5756$. With the provision of subsidy, we have that $\gamma_{R(\phi>0)} = 0.5265$, $\gamma_{D(\phi>0)} = 1.0008$ and $\gamma_{M(\phi>0)} = 1.2326$. Using the parameter values in (79) and (80), we have that the long-run unsubsidised awareness share $x_{(\phi=0)\infty} = 0.4132$. Similarly, the lung-run subsidised awareness share $x_{(\phi>0)\infty} = 0.4997$. Thus expressing the payoffs in (11), (12), and (13) in terms of the retail advertising effort α_R, we have that

$$P_{R(\phi=0)} = \frac{106282548823}{62500000000} + \frac{16737\sqrt{163}\alpha_R}{50000} - 25\alpha_R^2, \tag{83}$$

$$P_{D(\phi=0)} = \frac{18033\sqrt{163}\alpha_R}{25000} + \frac{156021746507}{62500000000} \tag{84}$$

and

$$P_{M(\phi=0)} = \frac{11817\sqrt{163}\alpha_R}{12500} + \frac{80716054143}{15625000000} \tag{85}$$

respectively, which leads to the plots in Figure 1.
Thus, we allowed α_R to vary while using the parameter values. Using a similar approach, we obtain the plots in Figure 2 to Figure 4.

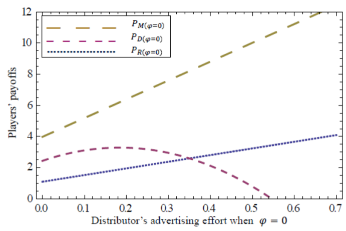

Figure 3: The effects of the distributor's advertising effort on the players' payoffs when retail advertising is unsubsidised

Figure 1 shows how the retail advertising effort affects the players' payoffs. For a situation where retail advertising is not subsidised, both the manufacturer and the distributor enjoy ever-increasing payoffs as the retail advertising effort increases, while the retailer experiences reduction in his payoff after a relatively slight increase. The retailer should focus his advertising effort on the optimal value. At this optimal value his payoff is maximized and both the manufacturer and the distributor's payoffs are put in check. He should not engage in advertising expenditure beyond this level since this will certainly lead to a lower payoff for him. We observe a similar trend in Figure 2, which illustrates the effect of retail advertising on all the players' payoffs in a situation where subsidy is provided. In this case the distributor's payoff increases continuously as the retail advertising effort increases. On the other hand, both the manufacturer and the retailer's payoffs increase with retail effort to a point, and thereafter reduce continuously. It is thus clear that the retailer should resort to his optimal advertising effort if he is not to be short-changed. As for the manufacturer, we recall that his subsidy to the retailer is contributory to the increase in the advertising effort. However, we observe that the manufacturer's payoff reduces with increasing effort (resulting from increasing subsidy spending). This suggests that the manufacturer's subsidy to the retailer must not exceed the optimal value. This will restrain the retailer from unnecessary advertising spending.

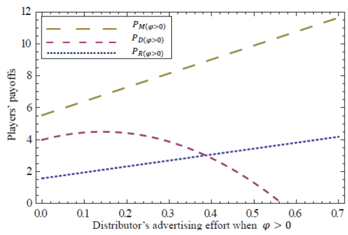

Figure 4: The effects of the distributor's advertising effort on the players' payoffs when retail advertising is subsidised

In Figure 3 and Figure 4 we see how the distributor's advertising effort affects the players' payoffs in a situation where retail advertising is subsidised, and where it is not subsidised, respectively. In both cases both the manufacturer and the retailer's payoffs increase continuously as the distributor's advertising effort increases, while the distributor's payoff increases to a certain point, and thereafter starts declining. Thus a continuous increase in the distributor's advertising effort would lead to a continuous increase of both the manufacturer and the retailer's payoffs, but to the detriment of the distributor except at his optimal advertising level. Further, it is interesting to note that the distributor's optimal payoff for a situation where retail advertising is subsidised is larger than that of the situation where retail advertising is not subsidised.

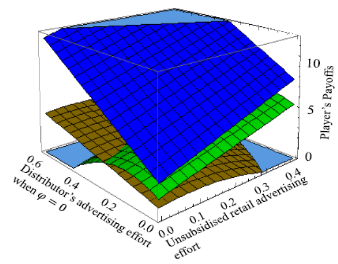

Figure 5: The effects of retail advertising and distributor's advertising on the payoffs when retail advertising is not subsidised

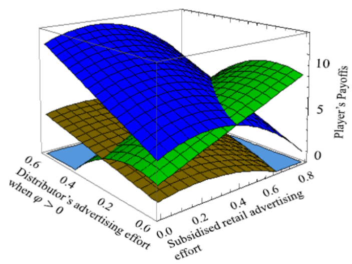

Figure 6: The effects of retail advertising and distributor's advertising on the payoffs when retail advertising is subsidised

7.3. The Effect of Subsidy on the Payoffs

Now, considering (14), (15), and (16) we observe that $P_R(x)$, $P_D(x)$, and $P_M(x)$ depend on $\phi \in [0, 1]$. Each of these functions has a specific value corresponding to the specific value $\phi = 0.6480$ (obtained using equation (10)). These values are $P_R = 2.1043$, $P_D = 4.5027$, $P_M = 6.7710$, and correspond with the result given by the plots in Figure 7, which show how various possible values of ϕ can affect each of the player's payoff. Further, Figure 7 shows the difference in the payoffs of the players for various possible values of ϕ.

As was earlier observed, the manufacturer must not totally subsidise retail advertising. In Figure 7 we see that with the provision of subsidy, both the retailer and the distributor's payoffs increase continuously, while the manufacturer runs at a loss. Particularly, the manufacturer will run at a total loss if he totally subsidises retail advertising.

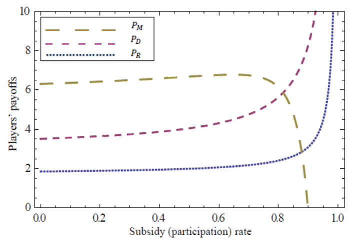

Figure 7: The effects of the manufacturer's subsidy on the players' payoffs

In spite of the fact that the manufacturer enjoys the first mover's advantage, there is the tendency for the distributor and the retailer's payoffs to eventually become larger than that of the manufacturer. That is $P_R > P_M$. Now, from (11) and (13), we observe that if $P_R = P_M$, then

$$\frac{1}{\rho}\Bigg[m_R x(t) - (1-\phi(t))\alpha_R\Big(x(t) \mid \alpha_D(t), \phi(t)\Big)^2$$

$$+P_{Rx}\Bigg[a_{ef}\Bigg(\alpha_R\big(x(t) \mid \alpha_D(t), \phi(t)\big) + \alpha_D\big(x(t) \mid \phi(t)\big)\Bigg)\sqrt{1-x(t)} - \delta x(t)\Bigg]\Bigg]$$

$$= \frac{1}{\rho}\Bigg[m_M x(t) - \phi(t)\alpha_R\big(x(t) \mid \alpha_D(t), \phi(t)\big)^2$$

$$+P_{Mx}\Bigg[a_{ef}\Bigg(\alpha_R\big(x(t) \mid \alpha_D(t), \phi(t)\big) + \alpha_D\big(x(t) \mid \phi(t)\big)\Bigg)\sqrt{1-x(t)} - \delta x(t)\Bigg]\Bigg].$$

Substituting for $\alpha_R, \alpha_D, P_{Rx}, P_{Mx}$, we have that

$$A\phi^2 + B\phi + C = 0,$$

so that

$$\phi = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

where

$$\begin{aligned}
A &= 2x\gamma_D\gamma_M - 2xa_{ef}^2\gamma_D\gamma_R + 2a_{ef}^2\gamma_D\gamma_R - 2a_{ef}^2\gamma_D\gamma_M + 4m_R x + 4\delta x\gamma_M \\
&\quad -4\delta x\gamma_R - 4m_M x \\
B &= 4xa_{ef}^2\gamma_D\gamma_R - 2xa_{ef}^2\gamma_M\gamma_R - 4xa_{ef}^2\gamma_D\gamma_M + 2a_{ef}^2\gamma_M\gamma_R + 4a_{ef}^2\gamma_D\gamma_M \\
&\quad -4a_{ef}^2\gamma_D\gamma_R + 8\delta x\gamma_R - 8\delta x\gamma_M + 8m_M x - 8m_R x \\
C &= 2a_{ef}^2\gamma_D\gamma_R - 2a_{ef}^2\gamma_D\gamma_M - 2a_{ef}^2\gamma_M\gamma_R + 4\delta x\gamma_M - 4\delta x\gamma_R + 4m_R x \\
&\quad -4m_M x - xa_{ef}^2\gamma_M^2 + a_{ef}^2\gamma_M 2 + 2xa_{ef}^2\gamma_D\gamma_M + 2xa_{ef}^2\gamma_M\gamma_R \\
&\quad -2xa_{ef}^2\gamma_D\gamma_R
\end{aligned}$$

and

$$B < \sqrt{B^2 - 4AC}.$$

We recall that $\phi \in [0,1]$, as such it follows that

$$\phi = \frac{-B+\sqrt{B^2-4AC}}{2A}. \tag{86}$$

Similarly, from (12) and (13), we have that if $P_D = P_M$, then

$$\frac{1}{\rho}\Bigg[m_D x(t) - \alpha_D\Big(x(t)\mid\phi(t)\Big)^2$$

$$+P_{Dx}\Bigg[a_{ef}\Big(\alpha_R\big(x(t)\mid\alpha_D(t),\phi(t)\big)+\alpha_D\big(x(t)\mid\phi(t)\big)\Big)\sqrt{1-x(t)}-\delta x(t)\Bigg]\Bigg]$$

$$=\frac{1}{\rho}\Bigg[m_M x(t) - \phi(t)\alpha_R\big(x(t)\mid\alpha_D(t),\phi(t)\big)^2$$

$$+P_{Mx}\Bigg[a_{ef}\Big(\alpha_R\big(x(t)\mid\alpha_D(t),\phi(t)\big)+\alpha_D\big(x(t)\mid\phi(t)\big)\Big)\sqrt{1-x(t)}-\delta x(t)\Bigg]\Bigg].$$

Substituting for $\alpha_R, \alpha_D, P_{Rx}, P_{Mx}$ we have that

$$U\phi^2 + V\phi + W = 0,$$

so that

$$\phi = \frac{-V \pm \sqrt{V^2-4UW}}{2U},$$

where

$$\begin{aligned}
U &= xa_{ef}^2\gamma_R^2 - a_{ef}^2\gamma_R^2 + 2xa_{ef}^2\gamma_D\gamma_M - 2xa_{ef}^2\gamma_D^2 + 2a_{ef}^2\gamma_D^2\\
&\quad -2a_{ef}^2\gamma_D\gamma_M + 4\delta x\gamma_M - 4\delta x\gamma_D + 4m_D x - 4m_M x,\\
V &= 2xa_{ef}^2\gamma_D\gamma_R - 2xa_{ef}^2\gamma_M\gamma_R - 4xa_{ef}^2\gamma_D\gamma_M + 4xa_{ef}^2\gamma_D^2 - 3xa_{ef}^2\gamma_R^2\\
&\quad +2a_{ef}^2\gamma_R\gamma_M + 4a_{ef}^2\gamma_D\gamma_M - 2a_{ef}^2\gamma_D\gamma_R + 3a_{ef}^2\gamma_R^2 - 4a_{ef}^2\gamma_D^2\\
&\quad +8\delta x\gamma_D - 8\delta x\gamma_M + 8xm_M - 8xm_D,\\
W &= 4m_D x - 4m_M x + xa_{ef}^2\gamma_R^2 - a_{ef}^2\gamma_R^2 + 2xa_{ef}^2\gamma_D\gamma_M + 2xa_{ef}^2\gamma_R\gamma_M\\
&\quad -2xa_{ef}^2\gamma_D\gamma_R - 2xa_{ef}^2\gamma_D^2 + 2a_{ef}^2\gamma_D\gamma_R - 2a_{ef}^2\gamma_D\gamma_M - 2a_{ef}^2\gamma_R\gamma_M\\
&\quad +2a_{ef}^2\gamma_D^2 + 4\delta x\gamma_M - 4\delta x\gamma_D\\
&\textit{and}\\
V &< \sqrt{V^2-4UW}.
\end{aligned}$$

Since $\phi \in [0,1]$, it follows that

$$\phi = \frac{-V+\sqrt{V^2-4UW}}{2U}. \tag{87}$$

As stated earlier, this is as a result of excessive subsidy by the manufacturer. Obviously, any additional subsidy above the level in (86) will lead to a decline in the manufacturer's payoff, thereby making it smaller than the retailer's payoff. Similarly, any additional subsidy above the level in (87) will lead to a reduction in the manufacturer's payoff that will lead to the distributor's payoff becoming larger than that of the manufacturer.

Now consider a situation where the manufacturer gives subsidy rate above 0.8153 (based on our choice of parameter). This will make the manufacturer's payoff to be lower than that of the distributor. Similarly, a subsidy rate above 0.8776 will make the manufacturer's payoff to be lower than both the distributor and retailer's payoffs. Thus the first mover's advantage, which is supposed to be enjoyed by manufacturer in decision making, and expected to lead to a better payoff, will become unachievable since these situations will lead to $P_D > P_M$ or even $P_D > P_R > P_M$ (for subsidy above 0.8776). Thus, it is very important that the manufacturer determines the optimal subsidy rate and ensures not to provide subsidy above this level.

The unilateral changes in the advertising efforts stated for instance in (83), (84), and (85), which were used to obtain the plots in Figure 1 to Figure 4, are intended for easy understanding of the effect of a particular advertising effort on the payoffs.

Basically, it is a fact that to effectively study and appreciate the effect of a given variable on a system, it is appropriate to keep every other variable fixed while allowing the variable of interest to vary. This leads to two dimensional plot which is easy to comprehend. However, it is pertinent to note that these plots are achievable without unilaterally changing the advertising efforts, thus leading to plots in three dimensions. Three dimensional plots can be achieved by plotting the payoffs on the retailer's advertising γ_R and the distributor's advertising effort γ_D on separate axis. This is achieved in Figure 5 and Figure 6, which illustrate the combined effect of the advertising efforts on the payoffs. Actually, the effect of each advertising effort will be well appreciated from this three dimensional plots. However, two dimensional plots provide quick interpretations of the effects of both advertising efforts, which is obvious when viewed from the plane of a given advertising effort.

7.4. The Effect of Subsidy on Advertising Effort

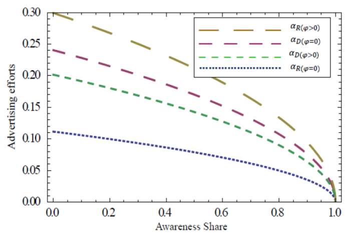

Figure 8: A comparison of the retailer and the distributor's advertising efforts through the market awareness share

We note that the advertising efforts α_R and α_D are functions of the awareness share x, which is a function of time t. Thus using x as given in (79) and (81), which in the long-run becomes (80) and (82), respectively, we have that (17), (18), (40), and (41) would become functions of time, leading to the plots in Figure 9 and Figure 11.

Apart from showing that the channel advertising effort increases with the provision of subsidy in spite of the distributor's reduced advertising effort, Figure 9 and Figure 11 further show that each follower as well as the channel initial advertising efforts should be large enough to create enough awareness. Thereafter, it should eventually reduce to a certain level (not 0 (zero)). This advertising level should be maintained to keep the firms afloat with a certain proportion of the market share. Observe that the awareness share x increases over time. This is clear from (79) and (81), which show that x increases with time and eventually becomes (80) and (82), respectively, and becomes constants in the long-run. Thus irrespective of the advertising expenditure, these long-run awareness levels cannot be exceeded. Hence, it would amount to a waste of resources to spend above this long-run advertising level in the hope of increasing awareness above this level.

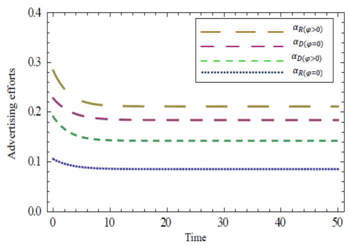

Figure 9: A comparison of the retailer and the distributor's advertising efforts over time

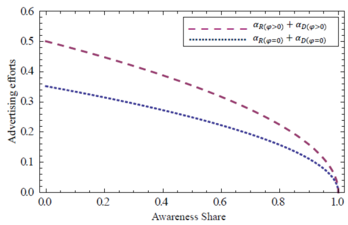

Figure 10: A comparison of the channel's advertising efforts through the market awareness share

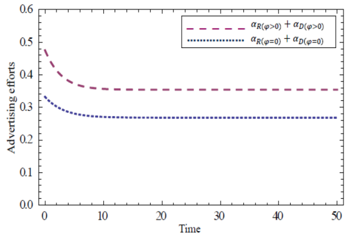

Figure 11: A comparison of the channel's advertising efforts over time

Figure 8 and Figure 9 show that with the provision of subsidy, the retail advertising effort increases while the distributor's advertising effort reduces. Now recall that the essence of both the retailer and the distributor's involvement in advertising is to increase the market awareness. Thus, with the provision of subsidy leading to more commitment towards local advertising, the distributor will become less committed knowing that the retailer's effort covers-up his lapses. This can be explained in Figure 10 and Figure 11, which show that the increase in retail advertising due to subsidy actually covers the reduction in the distributor's advertising effort. Clearly, when the manufacturer subsidises retail advertising, the channel's total advertising effort $\alpha_{R(\phi>0)} + \alpha_{D(\phi>0)}$ is larger than the effort $\alpha_{R(\phi=0)} + \alpha_{D(\phi=0)}$, which is the channel effort for a situation where retail advertising is unsubsidised. That is

$$\alpha_{R(\phi>0)} + \alpha_{D(\phi>0)} > \alpha_{R(\phi=0)} + \alpha_{D(\phi=0)}, \forall x, t \geq 0.$$

Thus the distributor's reduction in advertising spending resulting from the provision of subsidy does not lead to a reduction in total channel advertising effort. This is clear from Figure 8 and Figure 9. Rather the channel advertising effort increases with subsidy as shown in Figure 10 and Figure 11. This implies that the subsidy from the manufacturer has much influence on the entire channel effort. It is pertinent to note that with increase in advertising, there is the tendency for the awareness share to increase, thus leading to larger payoffs. Hence, irrespective of the distributor's advertising effort, the manufacturer should ensure that the retailer is provided with subsidy.

8. CONCLUDING REMARKS

The work considered cooperative advertising in a manufacturer-distributor-retailer channel. It involves a situation where both the distributor and the retailer directly engage in advertising with the manufacturer bypassing the distributor to subsidise retail advertising.

We developed three differential game models involving these players. The sales dynamics was based on Sethi's advertising model. This new dynamics was obtained by incorporating the distributor's advertising effort into this (Sethi's) sales dynamics. Thus the distributor was introduced through his involvement in advertising.

The work considered an unsubsidised channel structure and subsidised channel structure. In both cases, we obtained the retailer and distributor's advertising efforts, the market awareness share, and the players' payoffs. We also obtained the subsidy rate that the manufacturer should use to subsidise the retail advertising effort.

We observed that with the provision of subsidy, the distributor reduces his advertising effort, thus enjoying a high payoff with reduced effort to the detriment of the other channel members. As such, it is necessary for the players to come to a compromise on his (the distributor's) advertising effort/expenditure or a bargain in which he (the distributor) will either share part of his payoff with the other players or re-invest it into the channel. This can be an extension of the work.

Considering the fact that the long-run awareness share is bounded, the retailer and the distributor should avoid the temptation of unnecessary advertising spending in the hope of increasing the market awareness beyond the possible upper bound. Similarly, the manufacturer should provide only optimal subsidy. They (the retailer and the distributor) should determine their long-run advertising efforts which correspond with the long-run awareness. The advertising effort should be focused on achieving and retaining (maintaining) this awareness level.

This work considered the manufacturer as the Stackelberg leader, and the distributor and retailer as followers. An extension can consider a situation where any of the followers is the channel leader. This is possible where such a follower is powerful enough to dictate business terms to the manufacturer. We can consider the players involvement in a Nash game, a situation where all the players are powerful and influential enough to be considered as equals. This can be another insightful extension.

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