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A PRODUCTION INVENTORY PROBLEM WITH RANDOM POINT DEMAND

Miao-Sheng CHEN

Graduate Institute of Management Sciences, Tamkang University, Taiwan, R.O.C.

Ying-Chieh CHEN

Graduate Institute of Management Sciences, Tamkang University, and Department of Communication Management, The World College of Journalism and Communications, Taiwan, R.O.C

Abstract: This article is to provide a model which controls production and inventory of single-period, single-product under random demand, in an attempt to obtain both production function and the production interval. The following assumptions hold: (1) sales occur at a certain time in the future, (2) product demand is a random variable, and (3) the cost of unit production is a linear function of production quantity in a unit time. The sensitivity analysis on key variables of the optimal solution is discussed.

Keywords: Inventory, newsboy problem, optimal control, production interval, production planning.

1. INTRODUCTION

In inventory theory, the market demand can be divided into two different types [2, 3]. The first type is deterministic demand which assumes that the quantity of final product is deterministic. The objective is to determine appropriate lot-size at each stage to maximize profit or minimize the total costs. The second type is random demand. Classical newsboy problem considers the inventory size to be ordered for the sake of meeting random demand so as to maximize expected profit while balancing holding and shortage costs. The characteristics of this inventory problem is that the selling period is relatively short and goods remaining after the selling period are no longer saleable in the next period. For example: the problems of ordering newspapers, journals, vaccine, fashion wear, etc. However, the classical newsboy problem considers only the costs of inventory at the time of sale and neglects the time necessary for production. Indeed, when to begin production and how to control the production rate are important to the production planner. In this article, we shall construct a general mathematical model of the newsboy problem including factors relevant to production and sale. The main question of this article regarding the production planning period is as follows: How should the decision makers control the starting time for production and the production rate to meet the random demand at the end of the planning period such that the expected profit is optimal?

In other words, this article extends the static classical newsboy problem to become a dynamic decision problem.

2. NOTATION AND ASSUMPTIONS

For the sake of convenience, the following notation and assumptions are used in this article:

(A) Parameters and given functions:

T: The selling time.

$[0, T]$: The available time interval for production.

v: The unit price of goods produced.

h: The holding cost per unit of goods in a unit time. It contains the loss caused by holding goods or accumulating capital.

b: The loss per unit of surplus goods. It occurs when the quantity of inventory on hand at time T is larger than the quantity of goods in demand.

p: The penalty cost per unit of goods lacking. It occurs when the quantity of inventory on hand is less than the quantity of goods in demand.

c: A constant in a production cost function. In this article, we assume that the unit cost of producing q goods in a unit time is cq.

Y: The quantity of goods in demand at time T. Here Y is a random variable, its probability density function is $f(y)$, its cumulative distribution function is $F(y)$.

(B) Decision variables:

t_0: The time to begin production, where $t_0 \geq 0$.

$[t_0, T]$: The time interval during which the decision maker is actually engaged in production.

$x(t)$: The cumulative production at time t i.e., the total production in the time interval $[t_0, t]$, where $x(t_0) = 0$ and $x(T)$ = the total quantity of inventory on hand.

3. MODEL

Using the notation in the previous section, we have

- The total production cost = $\int_{t_0}^{T} c\, x'^{2}(t)\, dt$
- The total holding cost = $\int_{t_0}^{T} h\, x(t)\, dt$

Since the quantity of goods for sale Min{ $x(T)$, Y } is a random variable, we have

- The expected revenue = $\int_{0}^{x(T)} v\, y\, f(y)\, dy + \int_{x(T)}^{\infty} v\, x(T)\, f(y)\, dy$
- The expected cost of surplus goods = $\int_{0}^{x(T)} b\big(x(T)-y\big) f(y)\, dy$
- The expected cost of goods lacking = $\int_{x(T)}^{\infty} p\big(y-x(T)\big) f(y)\, dy$

Our objective is profit optimization, then the mathematical model is as follows:

$$\begin{cases} \text{Max}\left[\int_{0}^{x(T)} v\, y\, f(y)\, dy + \int_{x(T)}^{\infty} v\, x(T)\, f(y)\, dy\right] - \left[\int_{t_0}^{T}\big(c\, x'^{2}(t) + h\, x(t)\big)\, dt\right. \\ \qquad \left. + \int_{0}^{x(T)} b\,\big(x(T)-y\big) f(y)\, dy + \int_{x(T)}^{\infty} p\,\big(y-x(T)\big) f(y)\, dy\right] \\ \text{s.t.}\quad x(t_0)=0, \quad x'(t)\ge 0 \quad t\in[t_0, T], \quad t_0 \ge 0 \\ \qquad \text{where } t_0, \text{ and } x(T) \text{ are free} \end{cases} \tag{1}$$

Let $x^*(t)$, $t\in[t_0^*, T]$ be the optimal solution of (1).

4. OPTIMAL SOLUTION

Since problem (1) is not the standard form of calculus of variation, we neglect the constraint $x'(t)\ge 0$ in order to transform it into the standard form of calculus of variation and state it as follows:

$$\begin{cases} \text{Max}\left[\int_{0}^{x(T)}\big[vy - b\big(x(T)-y\big)\big] f(y)\, dy\right] + \int_{x(T)}^{\infty}\big[v\, x(T) - p\big(y-x(T)\big)\big] f(y)\, dy \\ \qquad - \int_{t_0}^{T}\big[c\, x'^{2}(t) + hx(t)\big]\, dt \\ \text{s.t.}\quad x(t_0)=0 \\ \qquad \text{where } x(T) \text{ and } t_0 \ge 0 \text{ are free} \end{cases} \tag{1'}$$

Suppose that $\bar{x}(t)$, $t\in[\bar{t}_0, T]$ is the optimal solution of (1') which is different from $x^*(t)$, unless $\bar{x}'(t)\ge 0$. Now, let $x(t)$, $t\in[\bar{t}_0, T]$ be a feasible solution of (1').

And, with the function x held fixed, consider a family of feasible solution of (1')

$$Z_a(t) = \bar{x}(t) + a\big(x(t)-\bar{x}(t)\big), \quad \bar{t}_0 \le t \le T \tag{2}$$

where a is a real number.

Let $J(a)$ be the objective value of $Z_a(t)$, i.e.,

$$J(a) = \int_0^{Z_a(T)} \left[vy - b\left(Z_a(T) - y\right)\right] f(y)\, dy + \int_{Z_a(T)}^{\infty} \left[v Z_a(T) - p\left(y - Z_a(T)\right)\right] f(y)\, dy$$
$$- \int_{t_0}^{T} \left[c\, Z_a{}'^{2}(t) + h Z_a(t)\right] dt \tag{3}$$

Since $Z_0(t) = \bar{x}(t)$ is the optimal solution of (1'), by (2), the function J attains its maximum at $a = 0$.

So, we must have

$$J'(0) = 0 \tag{4}$$

CASE 1. If $\bar{t}_0 = 0$, by (3), (4), then the optimal solution $\bar{x}(t)$ must satisfy the following equations ([1], pp. 67–68):

$$h = 2c\,\bar{x}''(t) \tag{5}$$

$$-2c\,\bar{x}'(T) + p + v - (p + v + b) F(\bar{x}(T)) = 0 \tag{6}$$

Integrating equation (5) and using $\bar{x}(0) = 0$ yields that

$$\bar{x}(t) = \frac{h}{4c} t^2 + \left[\frac{\bar{x}(T)}{T} - \frac{h}{4c} T \right] t \tag{7}$$

Together with (6) and (7), it leads to

$$hT + 2c\left[\frac{\bar{x}(T)}{T} - \frac{h}{4c} T \right] = 2c\,\bar{x}'(T) = p + v - (p + v + b) F(\bar{x}(T))$$

That is,

$$\bar{x}(T) + \frac{T}{2c}(p + v + b) F(\bar{x}(T)) = \frac{T}{2c}(p + v - hT) + \frac{h}{4c} T^2 \tag{8}$$

This shows that the value of $\bar{x}(T)$ is determined by (8).

From (7), we know that $\bar{x}''(t) = h/2c > 0$ and

$$\bar{x}'(0) = \frac{\bar{x}(T)}{T} - \frac{hT}{4c}$$

So, $\bar{x}'(t) \geq 0$ if and only if

$$\bar{x}'(0) = \frac{\bar{x}(T)}{T} - \frac{h}{4c} T \geq 0 \tag{9}$$

Together with (8) and (9), we know that $\bar{x}'(t) \geq 0$ if

$$p + v \geq hT + (p + v + b) F(\frac{h}{4c} T^2) \tag{10}$$

Hence, if the inequality (1) holds, then $\bar{x}(t)$ in (7) is also the optimal solution of (1). (11)

CASE 2. If $\bar{t}_0 > 0$, from (3), (4), we know that $\bar{x}(t)$ must satisfy the following ([1], pp. 53–68):

$$h = 2c\,\bar{x}''(t) \tag{12}$$

$$-c\,\bar{x}'^{\,2}(\bar{t}_0) + h\,\bar{x}(\bar{t}_0) - \bar{x}'(\bar{t}_0)\left(-2c\,\bar{x}'(\bar{t}_0)\right) = 0 \tag{13}$$

$$-2c\,\bar{x}'(T) + p + v - (p+v+b)\,F(\bar{x}(T)) = 0 \tag{14}$$

From (12), we have

$$\bar{x}(t) = \frac{h}{4c}t^2 - \frac{k_1}{2c}t + k_2 \tag{15}$$

where k_1, k_2 are constants of integration. From (13) and using $\bar{x}(\bar{t}_0) = 0$, we have $\bar{x}'(\bar{t}_0) = 0$. (16)

From (15), we know that $\bar{x}''(t) = h/2c > 0$.

It follows that

$$\bar{x}'(t) \geq 0, \quad t \in [\bar{t}_0, \mathrm{T}] \tag{17}$$

Together with (15) and (16), it leads to

$$\bar{x}(t) = \frac{h}{4c}\left(t - \bar{t}_0\right)^2, \quad t \in [\bar{t}_0, \mathrm{T}] \tag{18}$$

From (14) and (18), we have

$$p + v = h\left(T - \bar{t}_0\right) + (p+v+b)\,F\left(\frac{h}{4c}(T - \bar{t}_0)^2\right) \tag{19}$$

This shows that the value of $\bar{t}_0$ is determined by (19).

Since $\bar{t}_0 > 0$, by (19) we have

$$p + v = hT + (p+v+b)\,F\left(\frac{h}{4c}T^2\right) \tag{20}$$

Therefore, if inequality (20) holds, then, $\bar{x}(t)$ in (18) is also an optimal solution of (1). (21)

Together with (11) and (21), it leads that the optimal solution $x^*(t)$, $t_0^* \leq t \leq T$, of (1) is given by the following.

CASE 1. If $(p + v - hT)\left(1 - F\left(\frac{h}{4c}T^2\right)\right) \geq (hT + b)\,F\left(\frac{h}{4c}T^2\right)$, then

$$x^*(t) = \frac{h}{4c}t^2 + \left[\frac{x^*(T)}{T} - \frac{h}{4c}T\right]t, \quad 0 \leq t \leq T$$

where the value of $x^*(T)$ is determined by the following equation

$$(p+v-hT)\left(1-F(x^*(T))\right)-(hT+b)F(x^*(T))=\frac{2c}{T}\left(x^*(T)-\frac{h}{4c}T^2\right) \tag{22}$$

CASE 2. If $(p+v-hT)\left(1-F(\frac{h}{4c}T^2)\right)<(hT+b)F(\frac{h}{4c}T^2)$, then

$$x^*(t)=\frac{h}{4c}\left(t-t_0^*\right)^2,\quad t\in[t_0^*,T] \tag{23}$$

where the value of t_0^* is determined by the following equation

$$\left(p+v-h\left(T-t_0^*\right)\right)\left(1-F(\frac{h}{4c}(T-t_0^*)^2\right)=\left(h\left(T-t_0^*\right)+b\right)F(\frac{h}{4c}(T-t_0^*)^2) \tag{24}$$

5. SENSITIVITY ANALYSIS

At first, define the quantity A and B as follows

$$\begin{cases} A=\dfrac{p+v-hT}{hT+b} \\ B=\dfrac{F(\frac{h}{4c}T^2)}{1-F(\frac{h}{4c}T^2)} \end{cases} \tag{25}$$

where

- $F(\frac{h}{4c}T^2)$: the probability that market demand is less than the inventory size $\frac{h}{4c}T^2$.
- $1-F(\frac{h}{4c}T^2)$: the probability that market demand is greater than the inventory size $\frac{h}{4c}T^2$
- $p+v-hT$: overall loss of unit' s surplus goods.
- $b+hT$: overall loss of unit' s shortage (including the loss in storage costs).

By (25), the analysis of optimal solution of Case 1 and Case 2 are as follows:

1. If $A=B$, then the optimal solutions of Case 1 and Case 2 mentioned above are the same. Thus, the optimal inventory size $x^*(T)$ is equal to $h/_{4c}\,T^2$. This implies that $x^{*\prime}(T)=0$.

2. If $A > B$, then the optimal solution lies in Case 1. The larger the value of $p + v - hT$ is, from (22), the larger the optimal inventory size $x^{*\prime}(T)$. Moreover, $x^{*\prime}(T)$ also increases if $p + v - hT$ increases. In fact, $x^{*\prime}(0)$ is always greater than zero in this case.
3. If $A < B$, then the optimal solution $x^*(t)$ lies in Case 2. If the value of $b + hT$ increases, from (24), then the value of t_0^* increases. Therefore, from (23), the optimal inventory size $x^*(T)$ decreases. In fact, (24) also determines the optimal production time t_0^*. On the other hand, $x^{*\prime}(T)$ decreases as t_0^* increases and $x^{*\prime}(t_0^*)$ is always zero in this case.

From the analyses above, we can tell that the optimal production plans are determined by the relations between the two quantities A and B appearing in (25).

With this quantitative model, the sensitivity analysis of optimal solution with respect to the parameters could be concretely discussed and presented as in Table 1:

Table 1.

Decision variable	Parameter						Reference equation
	T	b	p	h	c	v	
t_0^*	+	+	–	+	–	–	24
$x^*(T)$	+	–	+	–	–	+	22

"+": Decision variable is an increasing function of parameter

"–": Decision variable is a decreasing function of parameter

6. CONCLUSION

In this article we proposed a dynamic version to the classical newsboy problem in which a production function $x(t)$ substitutes the stock order. Finally, we derive the optimal production plan in two cases: immediate production and postpone production which lies in Case 1 and Case 2.

We believe that time is an important factor for the production planner. However, determination of the length of production time interval depends on the price, the related costs and the situation of market demand which is apparent in the constraint of Case 1 and Case 2.

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