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MARTINGALE FOR R–FUZZY VALUED RANDOM VARIABLE

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Abstract: * Fuzziness is discussed in the context of fuzzy random variable and a corresponding view of fuzzy martingale and some their properties are given. Fuzzy random variables are introduced as random variables whose values are not reals but fuzzy numbers.

Keywords: Fuzzy set, fuzzy random variable, martingale.

1. INTRODUCTION

In this paper we expand the work initiated in [1], [14], [13], [15] where the notion of fuzzy random variable, conditional expectation and martingales were introduced and their properties were studied. Some convergence theorems for fuzzy number valued martingales are given.

Fuzzy valued random variable have been studied recently by many authors. We refer to the interesting works of Puri, Ralescu [12], Kwakernaak [9], Kaleva [7], Bán [1], for details. Furthermore, it was illustrated by the recent works of Boswel and Taylor [2], Kruse [8] and the others that the theory of fuzzy random variable has many applications.

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2. PRELIMINARIES

In this paper we restrict our attention to the set of fuzzy random variables on the base space R, adapting in what follows definitions and results from Feron [5] and Puri, Ralescu [12]. A fuzzy set $u \in \mathcal{F}(\mathcal{R})$ is a function $u : \mathcal{R} \to [0, 1]$ for which

1. $u_0 = \overline{co}\,\{x \in \mathcal{R} ; u(x) > 0\}$ is compact,
2. the α–level set u_α of u, defined by

$$u_\alpha = \{x \in \mathcal{R} : u(x) \geq \alpha\}$$

is nonempty, closed and convex subset of $\mathcal{R}$ for all $\alpha \in (0, 1]$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space where P is a probability measure. A fuzzy random variable is a function $X : \Omega \to \mathcal{F}(\mathcal{R})$ such that

$$\{(\omega, x) : x \in (X(\omega))_\alpha\} \in \mathcal{A} \times \mathcal{B} \quad \text{for every } \alpha \in [0, 1],$$

where $\mathcal{B}$ denotes the Borel subsets of $\mathcal{R}$.

It is obvious that the function $X_\alpha : \Omega \to 2^{\mathcal{R}}$ defined by $X_\alpha(\omega) = (X(\omega))_\alpha$ is the $\mathcal{R}$–valued random set. If H is Hausdorff metric defined on $\mathcal{P}(\mathcal{R})$ (the space of all compact and convex subsets of $\mathcal{R}$)

$$H(A, B) = \max\left\{\sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y|\right\}, \quad A, B \in \mathcal{P}(\mathcal{R}),$$

then $(\mathcal{P}(\mathcal{R}), H)$ is a complete metric space.

For any multifunction $F : \Omega \to \mathcal{P}(\mathcal{R})$ we can define the set

$$S_F = \{f \in L(\Omega, \mathcal{A}) : f(\omega) \in F(\omega)\ P - a.e.\}$$

where $L(\Omega, \mathcal{A}) = L$ denotes the set of all functions $h : \Omega \to \mathcal{R}$ which are integrable with respect to the probability measure P.

The set $S_F \subset L$ is closed with respect to a norm in L defined by

$$\|h\| = \int_\Omega |h(\omega)|\, dP, \quad h \in L.$$

Using S_F we can now define an integral for F (first introduced by Aumann [1])

$$\int_\Omega F\, dP = \left\{\int_\Omega f(\omega)\, dP(\omega) : f \in S_F\right\}.$$

The integrals $\int_\Omega f(\omega)\, dP(\omega)$ are defined in the sense of Lebesym. $F : \Omega \to \mathcal{P}(\mathcal{R})$ is called integrably bounded if there exists integrable real valued function $h : \Omega \to \mathcal{R}$ such that $\sup_{x \in F(\omega)} |x| \leq h(\omega)$ $P - a.e.$ The fuzzy random variable $X : \Omega \to \mathcal{F}(\mathcal{R})$ is integrably bounded if X_α is integrably bounded for all $\alpha \in [0, 1]$. Let $\mathcal{L} = \mathcal{L}(\Omega, \mathcal{A})$

denotes the set of all integrably bounded multivalued functions $F:\Omega\to\mathcal{P}(\mathcal{R})$ and let $\Lambda=\Lambda(\Omega,\mathcal{A})$ be the set of all integrably bounded fuzzy random variables $X:\Omega\to\mathcal{F}(\mathcal{R})$.

We shall recall a lemma which we shall use in the sequel.

Lemma 1. ([10]). Let M be a set and let $\{M_\alpha:\alpha\in[0,1]\}$, be a family of subsets of M such that

1. $M_0=M$
2. $\alpha\le\beta\Rightarrow M_\beta\subseteq M_\alpha$
3. $\alpha_1\le\alpha_2\le\ldots\le\alpha_n\le\ldots,\quad \lim_{n\to\infty}\alpha_n=\alpha\Rightarrow M_\alpha=\bigcap_{n=1}^{\infty}M_{\alpha_n}$.

Then, the function $\phi:M\to[0,1]$ defined by $\phi(x)=\sup\{\alpha\in[0,1]:x\in M_\alpha\}$ has the property that $\{x\in M:\phi(x)\ge\alpha\}=M_\alpha$ for every $\alpha\in(0,1]$.

For all $X,Y\in\Lambda$ we can define the function $\mathcal{D}:\Lambda\times\Lambda\to\mathcal{R}$

$$\mathcal{D}(X,Y)=\sup_{\alpha\ge 0}\Delta(X_\alpha,Y_\alpha)=\sup_{\alpha\ge 0}\int_\Omega H(X_\alpha(\omega),Y_\alpha(\omega))\,dP$$

Two fuzzy variables $X,Y\in\Lambda$, are considered to be identical if $\mathcal{D}(X,Y)=0$. It is obvious that $\mathcal{D}$ is a metric in Λ since Δ is metric in $\mathcal{L}$ (Th. 3.3 [6]).

Theorem 1. ([14]). $(\Lambda,\mathcal{D})$ is a complete metric space.

Let $(\Omega,\mathcal{A},P)$ be a probability space and $\mathcal{F}$ a sub–σ–algebra of $\mathcal{A}$ and $F\in\mathcal{L}$. The conditional expectation of F with respect to $\mathcal{F}$, which is in $\mathcal{L}(\Omega,\mathcal{F})$, is determined by setting

$$S_{E(F|\mathcal{F})}=cl\{g\in L(\Omega,\mathcal{F}):g=E(f|\mathcal{F}),\quad f\in S_F\}.$$

Finally if X is a fuzzy random variable we can define the conditional expectation of $X\in\Lambda$ in such a way that the following conditions are satisfied:

$$E(X|\mathcal{F})\in\Lambda(\Omega,\mathcal{F}),$$
$$\{x\in\mathcal{R}:E(X|\mathcal{F})(\omega)(x)\ge\alpha\}=E(X_\alpha|\mathcal{F})(\omega).$$

The next theorem shows that there exists a unique fuzzy random variable satisfying these requirements.

Theorem 2. ([14]). If $X\in\Lambda(\Omega,\mathcal{A})$ then there exists a unique fuzzy random variable $Y\in\Lambda(\Omega,\mathcal{F})$ such that

$$Y_\alpha(\omega)=E(X_\alpha|\mathcal{F})(\omega).$$

Theorem 3. [14]. The fuzzy conditional expectation has the following properties:

1. $\mathcal{D}(E(X_1|\mathcal{F}),E(X_2|\mathcal{F}))\le\mathcal{D}(X_1,X_2)$ for all $X_1,X_2\in\Lambda$.
2. If $\mathcal{F}_1\subset\mathcal{F}\subset\mathcal{A}$ and $X\in\Lambda$, then $E(X|\mathcal{F}_1)$ taken on the base space $(\Omega,\mathcal{A},P)$ is equal to $E(X|\mathcal{F}_1)$ taken on the base space $(\Omega,\mathcal{F},P)$.
3. If $\mathcal{F}_1\subset\mathcal{F}\subset\mathcal{A}$ and $X\in\Lambda$, then $E(E(X|\mathcal{F})\,|\,\mathcal{F}_1)=E(X|\mathcal{F}_1)$.

4. If $X_n : \Omega \to \mathcal{F}(\mathcal{R})$ are uniformly integrable bounded and $X_n \xrightarrow{\mathcal{D}} X$, then $E(X_n|\mathcal{F}) \xrightarrow{\mathcal{D}} E(X|\mathcal{F})$.

3. FUZZY MARTINGALES

Let $\{\mathcal{F}^n\}_{n\in N}$ be an increasing sequence of sub–σ–fields of $\mathcal{A}$ and let $\{X^n\}_{n\in N}$ be a sequence of integrably bounded fuzzy random variables adapted to $\{\mathcal{F}^n\}_{n\in N}$. Then, in analogy to the single valued and multivalued cases, we can introduce the following notations.

The system $\{X^n, \mathcal{F}^n\}_{n\in N}$ is said to be a fuzzy valued martingale if and only if for all $n \geq 1$

$$E(X^{n+1}|\mathcal{F}^n)(\omega) = X^n(\omega)\ P - a.e.$$

By $\mathcal{F}^\infty$ we shall denote the σ–algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{F}^n$. In applications it is usually assumed that $\mathcal{F}^\infty = \mathcal{A}$. If $X \in \Lambda$ then $\{E(X|\mathcal{F}^n), \mathcal{F}^n\}_{n\in N}$, forms a fuzzy valued martingale by Theorem 3.

Theorem 4. If $\{X^n, \mathcal{F}^n\}_{n\in N}$ is a fuzzy martingale and the sequence $\{X^n\}_{n\in N}$ is uniformly integrably bounded, then there exists $X \in \Lambda$ such that

$$E(X|\mathcal{F}^n) = X^n$$

Proof. We know that $X_\alpha^n : \Omega \to 2^{\mathcal{R}}$ defined by

$$X_\alpha^n(\omega) = \left\{x \in \mathcal{R} : X^n(\omega)(x) \geq \alpha\right\}$$

is a random set and $X_\alpha^n \in \mathcal{L}$. We shall denote the set $S_{X_\alpha^n} \subset L(\mathcal{R})$ by S_α^n and the set of all function $f_\alpha \in L(\mathcal{R})$ such that $E(f_\alpha|\mathcal{F}^n) \in S_\alpha^n$ for all $n \in N$ by S_α.

Now, we can define the family of random sets $\{X_\alpha\}_{\alpha\in(0,1]} : X_\alpha : \Omega \to 2^{\mathcal{R}}$, $X_\alpha(\omega) = cl\left\{f_\alpha(\omega) : f_\alpha \in S_\alpha\right\}$. Using results from [6] and [11] we get that $X_\alpha \subset \mathcal{L}_c$.

On the other hand, according Theorem 2 [14]

$$[E(X^n|\mathcal{F}^n)]_\alpha = E(X_\alpha^n|\mathcal{F}^n)$$

which means that $\{X^n, \mathcal{F}^n\}_{n\in N}$ is a set valued martingale for all $\alpha \in (0, 1]$. From [11] for every $\alpha \in (0, 1]$ there exists a sequence of functions $\{f_\alpha^{n,k}\} \subset L(\mathcal{R})$ such that $\{f_\alpha^{n,k}, \mathcal{F}^n\}_{n\in N}$ is a martingale, $f_\alpha^{n,k}$ is a selection of X_α^n for every $k \in N$ and

$$S_\alpha^n = cl\left\{f_\alpha^{n,k},\quad k \in N\right\}.$$

If $f_\alpha \in S$ then $E(f_\alpha|\mathcal{F}^n) \in S_\alpha^n$. This implies that

$$cl\left\{E(f_\alpha|\mathcal{F}^n), f_\alpha \in S_\alpha\right\} = S_{E(X_\alpha|\mathcal{F}^n)} \subseteq S_\alpha^n.$$

In order to prove that $S_\alpha^n \subseteq S_{E(X_\alpha|\mathcal{F}^n)}$ we shall use the Radom–Nikodym property of space $\mathcal{R}$. Since $\mathcal{R}$ has $\mathcal{R}$NP, for every martingale $\{f_\alpha^{n,k}, \mathcal{F}^n\}_{n\in N}$ there exists $f_\alpha^k \in L$ such that

$$E(f_\alpha^k|\mathcal{F}^n) = f_\alpha^{n,k} \quad \text{for all } n \in N.$$

which means that $f_\alpha^k \in S_\alpha$, that is $S_\alpha^n \subseteq S_{E(X_\alpha|\mathcal{F}^n)}$. So we have proved that

$$S_\alpha^n = S_{E(X_\alpha|\mathcal{F}^n)} \quad \text{for all } n \in N.$$

Now, we shall show that $S_\alpha \subseteq S_\beta$ for all $\alpha, \beta \in (0, 1]$, $\beta < \alpha$.

If $\beta < \alpha$, then $X_\alpha^n(\omega) \subseteq X_\beta^n(\omega)$ for all $\omega \in \Omega$ and all $n \in N$, and $S_\alpha^n \subseteq S_\beta^n$ for all $n \in N$.

In order to prove that $S_\alpha \subseteq S_\beta$ we suppose the opposite, i.e. $S_\beta \subset S_\alpha$. This means that there exists $f \in L$ such that

$$f \in S_\alpha \quad \text{and} \quad f \notin S_\beta.$$

Knowing the structure of S_α and S_β, the last statement implies that there exists $k \in N$ such that

$$E(f|\mathcal{F}^k) \in S_\alpha^k \quad \text{and} \quad E(f|\mathcal{F}^k) \notin S_\beta^k$$

which is contradiction with the statement that $S_\alpha^n \subseteq S_\beta^n$ for all $n \in N$.

If $\{\alpha_i\}_{i\in N} \subset (0, 1]$ is a nondecreasing sequence converging to $\alpha \in (0, 1]$ we have to prove that

$$S_\alpha = \bigcap_{i=1}^{\infty} S_{\alpha_i}.$$

Since $\{S_{\alpha_i}\}_{i\in N}$ is the sequence of closed, bounded subsets of L and $S_{\alpha_1} \supseteq S_{\alpha_2} \supseteq \ldots \supseteq S_{\alpha_i} \supseteq \ldots S_\alpha$, we get that

$$S_\alpha \subseteq \bigcap_{i=1}^{\infty} S_{\alpha_i}.$$

From the supposition that there exists $f \in L$ such that

$$f \in \bigcap_{i=1}^{\infty} S_{\alpha_i} \quad \text{and} \quad f \notin S_\alpha$$

it follows that

$$E(f|\mathcal{F}^n) \in S^n_{\alpha_i} \text{ for all } i \in N \text{ and all } n \in N$$

$$\Rightarrow E(f|\mathcal{F}^n) \in \bigcap_{i=1}^{\infty} S^n_{\alpha_i} \text{ for all } n \in N$$

and there exists $k \in N : E(f|\mathcal{F}^k) \notin S^k_\alpha$

$$\Rightarrow S^k_\alpha \neq \bigcap_{i=1}^{\infty} S^k_{\alpha_i}.$$

But, from [14] we know that

$$S^n_\alpha = \bigcap_{i=1}^{\infty} S^n_{\alpha_i} \text{ for all } n \in N.$$

Using the family $\{X_\alpha\}_{\alpha\in(0,1]}$ of random sets defined above, we can form the mapping $X : \Omega \to \mathcal{F}(\mathcal{R})$

$$X(\omega)(x) = \sup\{\alpha \in (0,1],\ \ x \in X_\alpha(\omega)\}$$

but first we must show that definition is correct, that is that for every $\omega \in \Omega$ the family of sets $\{X_\alpha\}_{\alpha\in(0,1]}$ define one and only one fuzzy set $X(\omega)$. First, we shall prove that the conditions of Lemma 1 are satisfied.

If $\alpha > \beta$ then $S_\alpha \subset S_\beta$ which implies that

$$X_\alpha(\omega) = cl\{f_\alpha(\omega) : f_\alpha \in S_\alpha\}$$
$$\subset cl\{f_\beta(\omega) : f_\beta \in S_\beta\} = X_\beta(\omega).$$

Now, we have to prove that $X_\alpha(\omega) = \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega)$ for all $\omega \in \Omega$. We know that $X_\beta(\omega)$ is compact for all $\beta \in (0,1]$ and all $\omega \in \Omega$. Let $Z(\omega) = \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega)$, $\omega \in \Omega$. Since the family of sets $X_{\alpha_i}(\omega)$ has the finite intersection property, we know $Z(\omega) \neq 0$ and $Z(\omega)$ is a compact set. Further, we have the next implication

$$\alpha > \alpha_i \Rightarrow X_\alpha(\omega) \subseteq X_{\alpha_i}(\omega)$$

$$\Rightarrow X_\alpha(\omega) \subseteq \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega)$$

that is

$$X_\alpha(\omega) \subseteq Z(\omega) \text{ for all } \omega \in \Omega.$$

Since $Z \subset \mathcal{L}$ it follows that there exists the sequence $\{f_n\} \subset L$ such that $Z(\omega) = cl\{f_n(\omega)\}$ for all $\omega \in \Omega$ and we get

$$Z(\omega) \subseteq X_{\alpha_i}(\omega) \Rightarrow \{f_n\} \subseteq S_{\alpha_i}$$

$$\Rightarrow \{f_n\} \subseteq \bigcap_{i=1}^{\infty} S_{\alpha_i} = S_\alpha \Rightarrow$$

$$cl\{f_n(\omega)\} \subseteq cl\{g(\omega), \quad g \in S_\alpha\} = X_\alpha(\omega).$$

Now, Lemma 1 is applicable and there exists fuzzy random variable X with $[X(\omega)]_\alpha = X_\alpha(\omega)$ for every $\alpha \in (0, 1]$, and it is obvious that

$$E(X|\mathcal{F}^n) = X^n \text{ for all } n \in N.$$

The proofs of the next two theorem is quite similar to the proof of Theorem 4 so they are omitted.

THEOREM 5. If $\{X^n, \mathcal{F}^n\}_{n \in N}$ is a fuzzy martingale and the sequence $\{X^n\}_{n \in N}$ is uniformly integrably bounded, then

$$\lim_{n \to \infty} X^n \xrightarrow{\mathcal{D}} X.$$

THEOREM 6. If $\{X^n, \mathcal{F}^n\}$ is a fuzzy martingale such that $\sup_{n \in N} \left\| X_\alpha^n \right\| < \infty$, then there exists family $\{f_\alpha\}_{\alpha \in (0,1]} \subset L$ such that

$$\lim_{n \to \infty} \left| X_\alpha^n(\omega) \right| = f_\alpha(\omega)\, P.a.e., \quad \alpha \in (0,1].$$

4. CONCLUSION

In this paper a short survey of results from the theory of fuzzy random variable and some theorems related to this topics are given. This research can be continued in following direction: investigation of properties of fuzzy random variable and martingale defined on $\mathcal{R}$ or on Banach space, definition and theorems related to fuzzy amarts as a simple form of fuzzy random process.

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