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TRANSIENT PERFORMANCE ANALYSIS OF A SINGLE SERVER QUEUING MODEL WITH RETENTION OF RENEGING CUSTOMERS

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Abstract: In this paper, we study a single server queuing model with retention of reneging customers. The transient solution of the model is derived using probability generating function technique. The time-dependent mean and variance of the model are also obtained. Some important special cases of the model are derived and discussed. Finally, based on the numerical example, the transient performance analysis of the model is performed.

Keywords: Transient Performance Analysis, Mean and Variance, Probability Generating Function, Reneging, Probability of Customers' Retention.

MSC: 60K25, 68M20.

1. INTRODUCTION

Queuing systems with customers' impatience are used in modeling and analysis of a wide variety of real situations, for example, computer networks with packet loss, perishable inventory systems, call centres, hospital emergency rooms handling critical patients, and impatient telephone switchboard customers. Customers' impatience in queuing theory is extensively discussed by many researchers in terms of reneging and balking. The tendency of a customer not to join a queue upon arrival is called balking while in reneging, a customer joins the queue, waits for some time and leaves the queue without getting service if the wait is more than his expected wait. The pioneer researchers in the area of queuing with balking and reneging are Haight [6], [7], Ancker and Gaffarian [4], [3], Obert [13], and Subba Rao [18], [19]. They come out with basic queuing models with balking and reneging concepts. Since then, these ideas are exploited to a great extent and a number of generalizations are made. Yechiali [27] studies customers' impatience in Markovian queuing systems with disaster and repair. He performs the steady-state analysis of these models. Sudhesh [20] derives the time-dependent solution of a single server Markovian queue with disaster and repairs. Kumar [9] studies a correlated input queuing system with catastrophic and restorative effects facing customers' impatience. He derives the transient solution of the model. Vijaya Laxmi et al. [22] study a finite buffer multiple vacation queue with balking, reneging and vacation interruption under N-policy. They further carry out the cost analysis of the model using swarm optimization and quadratic fit research methods. Vijaya Laxmi and Jyothsna [23] study a single server queue under variant working vacation policy with reneging and balking. Ammar [1] obtains the time-dependent solution of a two-heterogenous servers queue with impatient customers using probability generating function. Vijaya Laxmi and Jyothsna [24] study a renewal input multiple vacations queuing model with balking, reneging and heterogenous servers. They use supplementary variable and recursive techniques to obtain the steady-state probabilities of the model. Goswami [5] derives the study-state solution of the renewal input finite buffer queuing model with balking reneging and multiple working vacation using supplementary technique. Vijaya Laxmi and Jyothsna [25] consider an $M/M/1$ queue with working vacations, bernoulli schedule vacation interruption, balking and reneging. They obtain the steady-state probabilities using generating function. Ammar [2] studies an $M/M/1$ queue with customers'impatience and multiple vacations. Sudhesh et al. [21] perform the time-dependent analysis of two-heterogenous servers queue with disaster, repair and customers' impatience. They discuss the steady-state results of the model also. Rykov [15] studied several monotonicity properties of optimal policies for a multi-server controllable queuing systems with heterogeneous servers. Koba and Kovalenko [8] studied three retrial queuing systems in terms of aircraft landing process. Rykov [16] generalized the slow server problem for the case of additional cost structure and showed that the optimal policy for the problem has a monotone property.

Customers' impatience leads to the loss of potential customers and therefore, it is a serious problem to any firm. If the firms use certain customer retention strategies, then there is a probability that a reneging customer may be retained for his further service. Kumar and Sharma [11] take this idea into account and study a finite capacity single server Markovian queuing system with retention of reneging customers. They obtain the steady-state solution of the model using iterative method. The sensitivity analysis of the model is also performed. Kumar and Sharma [12] obtain the steady-state solution of a Markovian single server queueing system with discouraged arrivals and retention of reneging customers by using iterative method. Kumar [10] extends this idea to the finite capacity multi-server Markovian queue and performs the cost-profit analysis of the model.

The steady-state results do not reveal the real picture of the system under consideration, because the transient and start up effects are not taken care of, Whitt [26]. In most of the applications of the queuing theory, the modelers need to know how the system will operate up to some time instant t. Furthermore, if the system is empty initially, the fraction of time the server is busy and the initial rate of output will be below the steady-state values. Therefore, the steady-state analysis is not sufficient. Thus, in this paper we undertake the transient analysis of a single server queuing system with reneging and retention of reneging customers.

The queuing system studied in this paper finds its application in modeling the computer communication networks with frame loss, Sharma et. al [17]. In a data communication network, each frame waits for a certain length of time for its transmission at a router. If the transmission does not begin by then, the frame may get lost. The lost frames can be considered as reneged customers in queuing terminology. There are probable chances that a lost frame may be traced by a tracer. The traced frame can be considered as a retained customer.

Rest of the paper is structured as follows: in section 2, the queuing model is described. In section 3, the transient solution of the model is obtained. Section 4 deals with the computation of mean and variance. Special cases of the model are discussed in section 5, and in section 6 the numerical example is presented. Finally, the paper is concluded in section 7.

2. QUEUING MODEL

We consider a single server queuing model with retention of reneging customers in which the customers arrive according to a Poisson process with mean rate λ. The service time distribution is negative exponential with parameter μ. The queue discipline is first-come-first-served (FCFS). The capacity of the system is infinite. After joining the queue, each customer will wait for a certain length of time T for his service to begin. If it does not begin by then, he may get renege with probability p and may remain in the queue for his service with probability $q(=1-p)$ if certain customer retention strategy is used. The time T is a random variable which is assumed to follow negative exponential distribution with parameter ξ. It is further assumed that the reneging can only occur if the number of customers in the system are greater than a certain threshold value k. Therefore, the average reneging rate is given by the following function:

$$\xi_n = \begin{cases} 0, & 0 < n \leq k \\ (n-k)\xi, & n \geq k+1 \end{cases}$$

Let $\{X(t), t \geq 0\}$ be the number of customers in the system at time t. Let $P_n(t) = P\{X(t) = n\}, n = 0, 1, ...$ be the probability that there are n customers in the system at time t, and $P(z,t)$ the corresponding probability generating function. We assume that there is no customer in the system at $t=0$.

The queuing model under investigation is governed by the following set of differential-difference equations:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t), \tag{1}$$

$$\frac{dP_n(t)}{dt} = -\left(\lambda+\mu\right)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t); n=1,2,\ldots,k-1, \tag{2}$$

$$\frac{dP_k(t)}{dt} = -\left(\lambda+\mu\right)P_k(t) + \lambda P_{k-1}(t) + (\mu+\xi p)P_{k+1}(t); n=k, \tag{3}$$

$$\begin{aligned}\frac{dP_n(t)}{dt} &= -\left(\lambda+\mu+(n-k)\xi p\right)P_n(t) + \lambda P_{n-1}(t) + \\ &\quad (\mu+(n-k+1)\xi p)P_{n+1}(t); n=k+1,\ldots\end{aligned} \tag{4}$$

3. TRANSIENT SOLUTION OF THE MODEL

In this section, we derive the transient solution of the model. We use the probability generating function technique to obtain the time-dependent system size probabilities.

Theorem 1. *The time-dependent probabilities of the system size of a single server queuing model with retention of reneging customers that is governed by the differential-difference equations* (1) − (4) *are given by:*

$$P_i(t) = b_{i,0}(t) + \mu\int_0^t b_{i,k-2}(u)P_{k-1}(t-u)du, i=0,1,\ldots,k-2,$$

$$\begin{aligned}P_{k-1}(t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\Psi}\left(\frac{\mu}{\Psi}\right)^m\left(\frac{2\Psi}{\alpha}\right)^{n+1}(n+1)\binom{n}{m}\left[\int_0^t D(t-u)\right.\\ &\quad \left.\int_0^u E^{C(m)}(u-v)\exp\{-(\lambda+\mu-\xi p)v\}\frac{I_{n+1}(\alpha v)}{v}dudv\right],\end{aligned}$$

and, for $n=1,2,\ldots$

$$P_{n+k-1}(t) = n\gamma^n\int_0^t \exp\{-(\lambda+\mu-\xi p)(t-u)\}\frac{I_n(\alpha(t-u))}{(t-u)}P_{k-1}(u)du.$$

where $\Psi=\mu-\xi p$, $\alpha=2\sqrt{\lambda(\mu-\xi p)}$, $E^{C(m)}(t)$ *is* m*-fold convolution of* $E(t)$ *with itself with* $E^{C(0)}=\delta(t)$, $\delta(t)$ *is the Dirac delta function,* $I_n(.)$ *is the modified Bessel function of first kind and* $D(t-u)$ *is a function of* u *at a particular value of* t, *obtained from the convolution of two functions* $E(t)$ *and* $D(t)$, *where* $E(t)=L^{-1}(E^*(s))$ *and* $D(t)=L^{-1}(D^*(s))$.

Proof. Define the probability generating function $P(z,t)$ for the transient state probabilities $P_n(t)$ by

$$P(z,t) = \sum_{n=0}^{k-1} P_n(t) + \sum_{n=0}^{\infty} P_{n+k}(t) z^{n+1};\ P(z,0) = 1 \tag{5}$$

with

$$\sum_{n=0}^{k-1} P_n(t) = Q_{k-1}(t). \tag{6}$$

Adding the equations (1) and (2), we get

$$\frac{d(Q_{k-1}(t))}{dt} = -\lambda P_{k-1}(t) + \mu P_k(t). \tag{7}$$

Now, multiplying the equation (3) and (4) by z^n, summing over the respective range of n, we obtain

$$\begin{aligned} \frac{d\left[\sum_{n=0}^{\infty} P_{n+k}(t) z^{n+1}\right]}{dt} &= [(\mu - \xi p)(z^{-1} - 1) + \lambda(z-1)] \sum_{n=0}^{\infty} P_{n+k}(t) z^{n+1} \\ &\quad + \lambda z P_{k-1}(t) - \mu P_k(t) + \xi p(1-z)\frac{\partial P(z,t)}{\partial z}. \end{aligned} \tag{8}$$

Adding the equations (7) and (8) , we obtain the following partial differential equation

$$\begin{aligned} \frac{\partial P(z,t)}{\partial t} - \xi p(1-z)\frac{\partial P(z,t)}{\partial z} &= [(\mu - \xi p)(z^{-1} - 1) + \lambda(z-1)] \\ &\quad [P(z,t) - Q_{k-1}(t)] + \lambda(z-1) P_{k-1}(t). \end{aligned} \tag{9}$$

Solving the equation (9), we get

$$\begin{aligned} P(z,t) &= \exp\left\{[(\mu - \xi p)(z^{-1} - 1) + \lambda(z-1)]t\right\} + \int_0^t [\lambda(z-1)P_{k-1}(u) - \\ &\quad ((\mu - \xi p)(z^{-1} - 1) + \lambda(z-1))Q_{k-1}(u)] \times \exp\left\{[(\mu - \xi p)(z^{-1} - 1) + \right. \\ &\quad \left. \lambda(z-1)](t-u)\right\} du. \end{aligned} \tag{10}$$

If $\alpha = 2\sqrt{\lambda(\mu - \xi p)}$ and $\gamma = \sqrt{\frac{\lambda}{\mu - \xi p}}$, then using the modified Bessel function of first kind $I_n(.)$ and the Bessel function properties, we get

$$\exp\left\{\left(\lambda z + \frac{\mu - \xi p}{z}\right) t\right\} = \sum_{n=-\infty}^{\infty} (\gamma z)^n I_n(\alpha t), \tag{11}$$

Using (11) in (10), we get

$$
\begin{aligned}
P(z,t) \quad = \quad & \exp\{[-(\lambda+\mu-\xi p)]t\} \sum_{n=-\infty}^{\infty} (\gamma z)^n I_n(\alpha t) \\
& +\lambda \int_0^t P_{k-1}(u) \exp\{[-(\lambda+\mu-\xi p)](t-u)\} \\
& \times \sum_{n=-\infty}^{\infty} (\gamma z)^n [\gamma^{-1} I_{n-1}(\alpha(t-u)) - I_n(\alpha(t-u))] du \\
& + \int_0^t Q_{k-1}(u) \exp\{[-(\lambda+\mu-\xi p)](t-u)\} \\
& \times \sum_{n=-\infty}^{\infty} (\gamma z)^n [-\lambda \gamma^{-1} I_{n-1}(\alpha(t-u)) \\
& +(\lambda+\mu-\xi p) I_n(\alpha(t-u)) - (\mu-\xi p)\gamma I_{n+1}(\alpha(t-u))] du.
\end{aligned} \tag{12}
$$

Now, comparing the coefficients of z^n on either side of (12), we obtain for $n = 1, 2, \ldots$

$$
\begin{aligned}
P_{n+k-1}(t) \quad = \quad & \exp\{-(\lambda+\mu-\xi p)t\}\gamma^n I_n(\alpha t) + \lambda \int_0^t \exp\{-(\lambda+\mu-\xi p)(t-u)\} \\
& \times \left[I_{n-1}(\alpha(t-u))\gamma^{n-1} - I_n(\alpha(t-u))\gamma^n \right] P_{k-1}(u) du \\
& - \int_0^t \exp\{-(\lambda+\mu-\xi p)(t-u)\} Q_{k-1}(u) [\lambda I_{n-1}(\alpha(t-u))\gamma^{n-1} \\
& -(\lambda+\mu-\xi p) I_n(\alpha(t-u))\gamma^n + (\mu-\xi p) I_{n+1}(\alpha(t-u))\gamma^{n+1}] du,
\end{aligned} \tag{13}
$$

Comparing the terms free of z on either side of equation (12), that is, for $n = 0$, we get

$$
\begin{aligned}
Q_{k-1}(t) \quad = \quad & \exp\{-(\lambda+\mu-\xi p)t\} I_0(\alpha t) + \lambda \int_0^t \exp\{-(\lambda+\mu-\xi p)(t-u)\} \\
& P_{k-1}(u) \times \left[I_1(\alpha(t-u))\gamma^{-1} - I_0(\alpha(t-u)) \right] du - \\
& \int_0^t \exp\{-((\lambda+\mu-\xi p))(t-u)\} Q_{k-1}(u) \times [\alpha I_1(\alpha(t-u)) - \\
& (\lambda+\mu-\xi p) I_0(\alpha(t-u))] \, du.
\end{aligned} \tag{14}
$$

As $P(z,t)$ does not contain terms with negative powers of z, the right hand side of (13) with n replaced by $-n$ must be zero. Thus, we obtain

$$
\begin{aligned}
& \int_0^t \exp\{-(\lambda+\mu-\xi p)(t-u)\} Q_{k-1}(u) [\lambda I_{n+1}(\alpha(t-u))\gamma^{n-1} - \\
& \quad (\lambda+\mu-\xi p) I_n(\alpha(t-u))\gamma^n + (\mu-\xi p) I_{n-1}(\alpha(t-u))\gamma^{n+1}] du,
\end{aligned}
$$

$$= \exp\{-(\lambda+\mu-\xi p)t\}I_n(\alpha t)\gamma^n + \lambda\int_0^t \exp\{-(\lambda+\mu-\xi p)(t-u)\}P_{k-1}(u)$$
$$\times[I_{n+1}(\alpha(t-u))\gamma^{n-1} - I_n(\alpha(t-u))\gamma^n]du, \tag{15}$$

The usage of (15) in (13) considerably simplifies the working and results in a elegant expression for $P_n(t)$. This yields, for $n = 1, 2, \ldots$

$$P_{n+k-1}(t) = n\gamma^n \int_0^t \exp\{-(\lambda+\mu-\xi p)(t-u)\}\frac{I_n(\alpha(t-u))}{(t-u)}P_{k-1}(u)du. \tag{16}$$

The remaining probabilities $P_n(t), n = 0, 1, \ldots, k-1$ can be obtained by solving the equations (1) and (2). In matrix form, the equations (1) and (2) can be written as:

$$\frac{d\mathbf{P(t)}}{dt} = A\mathbf{P(t)} + \mu P_{k-1}(t)\mathbf{e_{k-1}} \tag{17}$$

where the matrix $A = (a_{i,j})_{k-1\times k-1}$ is given as:

$$A = \begin{bmatrix} -(\lambda) & \mu & \cdots & 0 \\ \lambda & -(\lambda+\mu) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -(\lambda+\mu) \end{bmatrix}$$

$\mathbf{P(t)} = (P_0(t) \ \ P_1(t) \ \ \ldots \ \ P_{k-2}(t))^T$, $\mathbf{e_{k-1}} = (0 \ \ 0 \ \ \ldots \ \ 1)^T$ is column vector of order $k-1$.

Let $\mathbf{P^*(s)} = (P_0^*(s) \ \ P_1^*(s) \ \ \ldots \ \ P_{k-2}^*(s))^T$ denote the Laplace transform of $\mathbf{P(t)}$. Taking the Laplace transform of equation (17) and solving for $\mathbf{P^*(s)}$, we get

$$\mathbf{P^*(s)} = (sI-A)^{-1}\{\mu P_{k-1}^*(s)\mathbf{e_{k-1}} + \mathbf{P(0)}\}, \tag{18}$$

with $\mathbf{P(0)} = (1 \ \ 0 \ \ \ldots \ \ 0)^T$. Thus, only $P_{k-1}^*(s)$ remains to be found. We observe that if $\mathbf{e} = (1 \ \ 1 \ \ \ldots \ \ 1)_{k-1\times 1}^T$, then

$$\mathbf{e}^T\mathbf{P^*(s)} + P_{k-1}^*(s) = Q_{k-1}^*(s), \tag{19}$$

where $Q_{k-1}^*(s)$ is the Laplace inverse of $Q_{k-1}(t)$. Define,

$$\phi(s) = \left[(s+\lambda+(\mu-\xi p)) - \sqrt{(s+\lambda+(\mu-\xi p))^2-\alpha^2}\right].$$

Taking Laplace transform of (14) and solving for $Q_{k-1}^*(s)$, we obtain

$$sQ_{k-1}^*(s) = 1 + P_{k-1}^*(s)\left[\frac{1}{2}\{\phi(s)\} - \lambda\right] \tag{20}$$

Using equations (20) in (19) and simplifying, we get

$$P^*_{k-1} = \frac{1 - s\mathbf{e^T}(sI - A)^{-1}\mathbf{P(0)}}{\{(s+\lambda) - \frac{1}{2}[\phi(s)] + \mu s\mathbf{e^T}(sI - A)^{-1}\mathbf{e_{k-1}}\}}. \tag{21}$$

In equations (18) and (21), $(sI - A)^{-1}$ has to be found. Let us assume that

$$(sI - A)^{-1} \quad = \quad (b^*_{ij}(s))_{k-1\times k-1}$$

where $b^*_{ij}(s)$ is the Laplace inverse of $b_{ij}(t)$. We note that $(sI - A)^{-1}$ is almost lower triangular. Following Raju and Bhat [14], we obtain for $i = 0, 1, \ldots, k-2$

$$b^*_{ij}(s) = \begin{cases} \frac{1}{\mu}\left[\frac{u_{k-1,j+1}(s)u_{i,0}(s) - u_{i,j+1}(s)u_{k-1,0}(s)}{u_{k-1,0}(s)}\right], & j = 0, 1, \ldots, k-3, \\ \frac{u_{i,0}(s)}{u_{k-1,0}(s)}, & j = k-2, \end{cases} \tag{22}$$

where $u_{i,j}(s)$ are recursively given as

$$\begin{aligned} &u_{i,i}(s) = 1, && i = 0, 1, \ldots, k-2; \\ &u_{i+1,i}(s) = \tfrac{s+\lambda+\mu}{\mu}, && i = 0, 1, \ldots, k-3; \\ &u_{i+1,i-j}(s) = \tfrac{(s+\lambda+\mu)u_{i,i-j} - \lambda u_{i-1,i-j}}{\mu}, && j \le i, i = 1, 2, 3, \ldots, k-3; \end{aligned}$$

$$u_{k-1,j}(s) = \begin{cases} [s+\lambda+\mu]u_{k-2,j} - \lambda u_{k-3,j}, & j = 0, 1, \ldots, k-3; \\ s+\lambda+\mu, & j = k-2 \end{cases} \tag{23}$$

and
$u_{i,j}(s) = 0$, for other i and j. We have suppressed the argument s to facilitate computation. The advantage in using these relations is that we do not evaluate any determinant. Using these in equation (21), we get

$$P^*_{k-1}(s) = \frac{\{1 - s\sum_{i=0}^{k-2} b^*_{i,0}(s)\}}{\{(s+\lambda) - \frac{1}{2}[\phi(s)] + \mu s\sum_{i=0}^{k-2} b^*_{i,k-2}(s)\}}, \tag{24}$$

and for $i = 0, 1, \ldots, k-2$ from equation (18), we get

$$P^*_i(s) = b^*_{i,0}(s) + \mu b^*_{i,k-2}(s)P^*_{k-1}(s). \tag{25}$$

We observe that $b^*_{i,j}(s)$ are all rational algebraic functions in s. The cofactor of the $(i,j)^{th}$ element of $(sI - A)$ is a polynomial of degree $k-2- \mid i-j \mid$. Since the characteristic roots of A are all distinct, the inverse transform $b_{i,j}(t)$ of $b^*_{i,j}(s)$ can be obtained by partial fraction decomposition. Let s_i, $i = 0, 1, \ldots, k-2$, be the characteristic roots of the matrix A. Then after partial fraction decomposition and simplification, $P^*_{k-1}(s)$ equals to

$$\frac{D^*(s)}{\frac{1}{2}[\phi(s)]\left[1-\frac{2(\mu-\xi p)(1-\frac{\mu}{\mu-\xi p}E^*(s))}{(s+\lambda+\mu-\xi p)-\sqrt{(s+\lambda+\mu-\xi p)^2-\alpha^2}}\right]}, \tag{26}$$

where

$$D^*(s)=\sum_{m=0}^{k-2}\frac{D_m}{s-s_m}, \tag{27}$$

$$E^*(s)=\sum_{m=0}^{k-2}\frac{E_m}{s-s_m}. \tag{28}$$

with constants D_m and E_m given by

$$D_m=\lim_{s\to s_m}(s-s_m)\left[1-\sum_{l=0}^{k-2}sb^*_{l,0}(s)\right] \tag{29}$$

$$E_m=\lim_{s\to s_m}(s-s_m)\left[\sum_{l=0}^{k-2}sb^*_{l,k-2}(s)\right]. \tag{30}$$

Hence, (26) simplifies into

$$\begin{aligned}P^*_{k-1}(s) \quad = \quad & \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\Psi}\left(\frac{2\Psi}{\alpha}\right)^{n+1}(n+1)\binom{n}{m}\left(\frac{\mu}{\Psi}\right)^m D^*(s)(E^*(s))^m\\ &\times\frac{[(s+\lambda+\mu-\xi p)-\sqrt{(s+\lambda+\mu-\xi p)^2-\alpha^2}]^{n+1}}{(n+1)\alpha^{n+1}},\end{aligned} \tag{31}$$

where $\Psi=(\mu-\xi p)$.
Taking Laplace inverse of (31), we obtain

$$\begin{aligned}P_{k-1}(t) \quad = \quad & \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\Psi}\left(\frac{2\Psi}{\alpha}\right)^{n+1}\left(\frac{\mu}{\Psi}\right)^m(n+1)\binom{n}{m}\left[\int_0^t D(t-u)\right.\\ &\left.\int_0^u E^{C(m)}(u-v)\exp\{-(\lambda+\mu-\xi p)v\}\frac{I_{n+1}(\alpha v)}{v}dudv\right],\end{aligned} \tag{32}$$

where $E^{C(m)}(t)$ is m-fold convolution of $E(t)$ with itself with $E^{C(0)}=\delta(t)$. Now, the Laplace inverse of equation (25) yields,

$$P_i(t)=b_{i,0}(t)+\mu\int_0^t b_{i,k-2}(u)P_{k-1}(t-u)du, i=0,1,\ldots,k-2, \tag{33}$$

where $P_{k-1}(u)$ is given by (32). Therefore, all the transient-state probabilities are obtained explicitly in (16), (32), and (33).

□

4. MEAN AND VARIANCE

This section deals with the derivation of time-dependent mean and variance of the system.

4.1 MEAN, $M(t)$

The mean number of customers in the system at time t is given by

$$M(t) = E(X(t)) = \sum_{n=1}^{k} nP_n(t) + \sum_{n=k+1}^{\infty} nP_n(t), \tag{34}$$

and

$$M'(t) = \sum_{n=1}^{k} nP'_n(t) + \sum_{n=k+1}^{\infty} nP'_n(t). \tag{35}$$

From equations (2), (3) and (4) after considerable mathematical simplification, the above equation will lead to the following differential equation

$$\begin{aligned} M'(t) = & -(\lambda+\mu)M(t) + \lambda\sum_{n=1}^{\infty} nP_{n-1}(t) + \mu\sum_{n=1}^{\infty} nP_{n+1}(t) + \\ & \xi p\left[\sum_{n=k+1}^{\infty} n(n-k)(P_{n+1}(t) - P_n(t)) + \sum_{n=k}^{\infty} nP_{n+1}(t)\right]. \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned} M(t) = & \lambda\sum_{n=1}^{\infty} n\int_0^t P_{n-1}(u)\exp\{-(\lambda+\mu)(t-u)\}du + \mu\sum_{n=1}^{\infty} n\int_0^t P_{n+1}(u) \\ & \exp\{-(\lambda+\mu)(t-u)\}du + \xi p\sum_{n=k+1}^{\infty} n(n-k)\int_0^t (P_{n+1}(u) - P_n(u)) \\ & \exp\{-(\lambda+\mu)(t-u)\}du + \xi p\sum_{n=k}^{\infty} n\int_0^t P_{n+1}(u) \times \\ & \exp\{-(\lambda+\mu)(t-u)\}du. \end{aligned} \tag{37}$$

where $P_n(t)$ for $n = 0, 1, ..., k-2$; $P_{k-1}(t)$ and $P_{n+k-1}(t)$ for $n = 1, 2, ...$ are given in equations (16), (32), and (33) respectively.

4.2 VARIANCE, V($X(t)$)

The variance of number of customers in the system at time t is given by:

$$\begin{aligned} V(X(t)) &= E(X^2(t)) - [E(X(t))]^2 \\ &= K(t) - [M(t)]^2 \\ &= \sum_{n=1}^{\infty} n^2 P_n(t) - \left[\sum_{n=1}^{\infty} n P_n(t)\right]^2. \end{aligned} \tag{38}$$

where $K(t) = [E(X^2(t))]$. From equations $(2), (3)$, and (4) after considerable mathematical simplification, the above equation will lead to the following differential equation

$$\begin{aligned} K'(t) &= -(\lambda+\mu)K(t) + \lambda\sum_{n=1}^{\infty} n^2 P_{n-1}(t) + \mu\sum_{n=1}^{\infty} n^2 P_{n+1}(t) + \\ &\quad \xi p\left[\sum_{n=k+1}^{\infty} n^2(n-k)(P_{n+1}(t) - P_n(t)) + \sum_{n=k}^{\infty} n^2 P_{n+1}(t)\right]. \end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned} K(t) &= \lambda\sum_{n=1}^{\infty} n^2 \int_0^t P_{n-1}(u)\exp\{-(\lambda+\mu)(t-u)\}du + \mu\sum_{n=1}^{\infty} n^2 \int_0^t P_{n+1}(u) \\ &\quad \exp\{-(\lambda+\mu)(t-u)\}du + \xi p \sum_{n=k+1}^{\infty} n^2(n-k)\int_0^t (P_{n+1}(u) - P_n(u)) \\ &\quad \exp\{-(\lambda+\mu)(t-u)\}du + \xi p\sum_{n=k}^{\infty} n^2 \int_0^t P_{n+1}(u) \times \\ &\quad \exp\{-(\lambda+\mu)(t-u)\}du. \end{aligned} \tag{40}$$

Substituting the above equation in (38), we get

$$\begin{aligned} V(X(t)) &= \lambda\sum_{n=1}^{\infty} n^2 \int_0^t P_{n-1}(u)\exp\{-(\lambda+\mu)(t-u)\}du + \mu\sum_{n=1}^{\infty} n^2 \int_0^t P_{n+1}(u) \\ &\quad \exp\{-(\lambda+\mu)(t-u)\}du + \xi p \sum_{n=k+1}^{\infty} n^2(n-k)\int_0^t (P_{n+1}(u) - P_n(u)) \\ &\quad \exp\{-(\lambda+\mu)(t-u)\}du + \xi p\sum_{n=k}^{\infty} n^2 \int_0^t P_{n+1}(u)\exp\left[-(\lambda+\mu)\right. \\ &\quad \left.(t-u)\right]du - [M(t)]^2. \end{aligned} \tag{41}$$

where $P_n(t)$ for $n = 0, 1, ..., k-2$; $P_{k-1}(t)$; $P_{n+k-1}(t)$ for $n = 1, 2, ...$ and $M(t)$ are given in equations $(16), (32)$, (33), and (37), respectively.

5. SPECIAL CASES OF THE MODEL

Case 1 When $q = 0$, i.e. when the retention mechanism is absent. The model reduces to a single server queuing model with reneging having system size probabilities as given below:

$$\begin{aligned} P_{n+k-1}(t) &= n\gamma_1^n \int_0^t \exp\{-((\lambda+\mu-\xi))(t-u)\} \frac{I_n(\alpha_1(t-u))}{(t-u)} P_{k-1}(u)du, \\ &\qquad n = 1, 2, \ldots \end{aligned} \tag{42}$$

$$\begin{aligned} P_{k-1}(t) &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{\Psi} \left(\frac{2\Psi}{\alpha_1}\right)^{n+1} \left(\frac{\mu}{\Psi}\right)^m (n+1)\binom{n}{m} \left[\int_0^t D(t-u)\right. \\ &\quad \left. \int_0^u E^{C(m)}(u-v) \exp\{-(\lambda+\mu-\xi)v\} \frac{I_{n+1}(\alpha_1 v)}{v} dudv \right], \end{aligned} \tag{43}$$

$$P_i(t) = b_{i,0}(t) + \mu \int_0^t b_{i,k-2}(u) P_{k-1}(t-u)du, i = 0, 1, \ldots, k-2. \tag{44}$$

where $\alpha_1 = 2\sqrt{\lambda(\mu-\xi)}$,$\gamma_1 = \sqrt{\frac{\lambda}{\mu-\xi}}$, and $\Psi = (\mu - \xi)$.

Case 2 When there is no reneging, the queuing system reduces to the one having probabilities as under:

$$P_{n+k-1}(t) = n\gamma_2^n \int_0^t \exp\{-(\lambda+\mu)(t-u)\} \frac{I_n(\alpha_2(t-u))}{(t-u)} P_{k-1}(u)du, \quad n = 1, 2, \ldots$$

$$\begin{aligned} P_{k-1}(t) &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{\mu} \left(\frac{2\mu}{\alpha_2}\right)^{n+1} (n+1)\binom{n}{m} \left[\int_0^t D(t-u)\right. \\ &\quad \left. \int_0^u E^{C(m)}(u-v) \exp\{-(\lambda+\mu)v\} \frac{I_{n+1}(\alpha_2 v)}{v} dudv \right], \end{aligned} \tag{45}$$

$$P_i(t) = b_{i,0}(t) + \mu \int_0^t b_{i,k-2}(u) P_{k-1}(t-u)du, i = 0, 1, \ldots, k-2. \tag{46}$$

where $\alpha_2 = 2\sqrt{\lambda\mu}$ and $\gamma_2 = \sqrt{\frac{\lambda}{\mu}}$.

6. A NUMERICAL EXAMPLE FOR TRANSIENT PERFORMANCE ANALYSIS

In this section, the transient performance analysis of the model is carried out on a numerical example. We present and analyze some important measures of performance, such as average reneging rate at time t (denoted by $R_r(t)$), and average retention rate at time t (denoted by $R_R(t)$). The expressions for $R_r(t)$ and $R_R(t)$ are as follows:

6.1. ***Average Reneging Rate***

$$R_r(t) = \sum_{n=k}^{\infty}(n-k)\xi p P_n(t)$$

6.2. ***Average Retention Rate***

$$R_R(t) = \sum_{n=k}^{\infty}(n-k)\xi q P_n(t)$$

where $P_n(t)$ is provided in equations (16), (32), and (33).

We use MATLAB software to compute the numerical results. All numerical results are presented in Figures 1-4. From these figures, following observations can be made:
1. Fig. 1 shows the effect of the probability of retaining a reneging customer on the expected system size in transient state. One can observe that as the probability of retaining a reneging customer increases, the expected system size also increases. This establishes the role of probability of retention associated with any customer retention strategy.
2. In Fig. 2 the change in average reneging rate with the change in probability of retention is shown in transient state. We can observe that there is a proportional decrease in average reneging rate with the increase in probability of retention, q.
3. The average retention rate increases as the probability of retaining a reneging customer (q) increases. This is evident from Fig. 3 which justifies the functioning of the model.
4. Fig. 4 shows a relative change in time-dependent probabilities of system size with time. One can see that with the passage of time the system attains a steady-state.

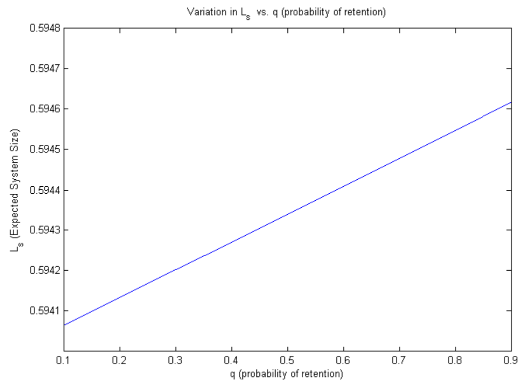

Figure 1: The expected system sizes versus probability of retention (q) are plotted for the case $\lambda = 2, \mu = 3, t = 0.5, k = 2, \xi = 0.1$ and $q = 0.1, 0.2, \ldots, 0.9$

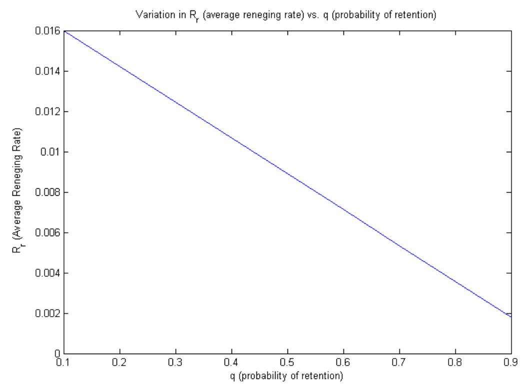

Figure 2: Variation of average reneging rate with the variation in probability of retention for the case $\lambda = 2, \mu = 3, t = 0.5, k = 2, \xi = 0.1$ and $q = 0.1, 0.2, \ldots, 0.9$

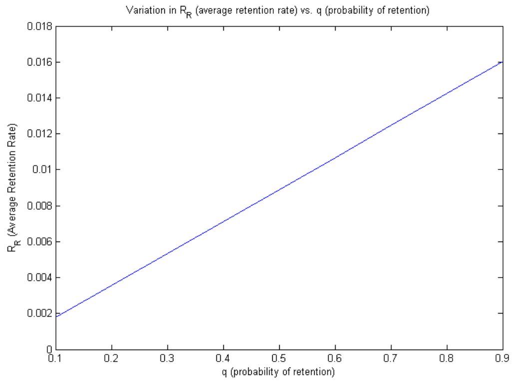

Figure 3: Variation of average retention rate with the variation in probability of retention for the case $\lambda = 2, \mu = 3, t = 0.5, k = 2, \xi = 0.1$ and $q = 0.1, 0.2, \ldots, 0.9$

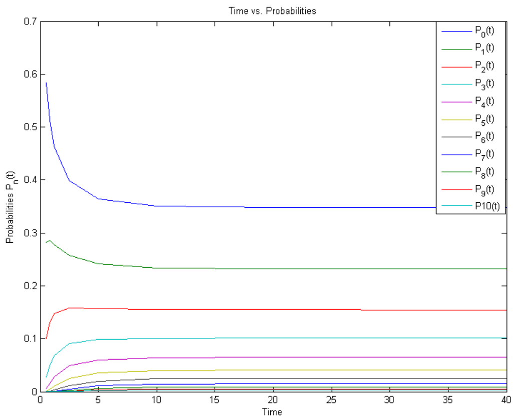

Figure 4: The probabilities versus time are plotted for the case $\lambda = 2, \mu = 3, \xi = 0.1, k = 2$ and $q = 0.4$

7. CONCLUSIONS

The transient performance analysis of a single server queuing model with reneging and retention of reneging customers is carried out. The explicit expressions for mean and variance are obtained. The analysis carried out in this paper may be useful in the study of computer communication networks with loss of frames/packets.

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REFERENCES

[1] Ammar, S. I., "Transient analysis of a two-heterogeneous servers queue with impatient behavior", *Journal of the Egyptian Mathematical Society*, 22 (2014) 90-95.

[2] Ammar, S. I., "Transient analysis of M/M/1 queue with impatient behavior and multiple vacations", *Applied Mathematics and Computation*, 260 (2015) 97-105.

[3] Ancker, C.J. and Gafarian, A.V., "Some queuing problems with balking and reneging I", *Operations Research*, 11 (1963a) 88-100.

[4] Ancker, C.J. and Gafarian, A.V., "Some queuing problems with balking and reneging II", *Operations Research*, 11 (1963b) 928-937.

[5] Goswami, V., "Study of customers' impatience in a GI/M/1/N queue with working vacations", *International Journal of Management Science and Engineering Management*, 10 (2015) 144-154.

[6] Haight, F. A., "Queuing with balking I", *Biometrika*, 44 (1957) 360-369.

[7] Haight, F. A., "Queuing with reneging" *Metrika*, 2 (1959) 186-197.

[8] Koba E. V., Kovalenko I. N., "Three Retrial Queueing Systems Representing Some Special Features of Aircraft Landing", *Journal of Automation and Information Sciences*, 34 (4) (2002) DOI 10.1615/JAutomatInfScien.v34.i4.10 4 pages.

[9] Kumar, R., "A catastrophic-cum-restorative queuing problem with correlated input and impatient customers", *International Journal of Agile Systems and Management* , 5 (2012) 122-131.

[10] Kumar, R., "Economic analysis of an M/M/c/N queuing model with balking, reneging and retention of reneged customers", *Opsearch*, 50 (2013) 383-403.

[11] Kumar, R. and Sharma, S. K., "An M/M/1/N queuing system with retention of reneged customers", *Pakistan Journal of Statistics and Operation Research*, 8 (2012) 859-866.

[12] Kumar, R. and Sharma, S. K., "A single-server Markovian queueing system with discouraged arrivals and retention of reneged customers", *Yugoslav Journal of Operations Research*, 24 (2014) 119-216.

[13] Obert, E.R., "Reneging phenomenon of single channel queues", *Mathematics of Operations Research*, 4 (1979) 162-178.

[14] Raju, S.N., and Bhat, U.N., "A computationally oriented analysis of the G/M/1 queue", *Opsearch*, 19 (1982) 67-83.

[15] Rykov V., "Monotone Control of Queueing Systems with Heterogeneous Servers", *Queueing Systems*, 37 (2001) 391-403.

[16] Rykov V., "On a Slow Server Problem", In: Li H., Li X. (eds) *Stochastic Orders in Reliability and Risk. Lecture Notes in Statistics*, Springer, New York, 2013.

[17] Sharma, S. K., Sharma, J. and Sharma, R. "Retransmission of corrupted and lost frames in a computer communication network: A Queuing Theoretic Approach", *AMO - Advanced Modeling and Optimization*, 18 (2016) 97-108.

[18] Subba Rao, S., "Queueing models with balking, reneging and interruptions", *Operations Research*, 13 (1965) 596-608.

[19] Subba Rao, S., "Queueing models with balking and reneging in M/G/1 system", *Metrika*, 2 (1967) 173-188.

[20] Sudhesh, R., "Transient analysis of a queue with system disasters and customer impatience", *Queuing Systems*, 66 (2010) 95-105.

[21] Sudhesh, R., Savitha, P. and Dharmaraja, S., "Transient analysis of a two-heterogeneous servers queue with system disaster, server repair and customers impatience", *TOP*, 25 (2017) 179-205.

[22] Vijaya Laxmi, P., Goswami, V. and Jyothsna K., "Optimization of Balking and Reneging Queue with Vacation Interruption under N-Policy", *Journal of Optimisation*, 2013 (2013) 1-9.

[23] Vijaya Laxmi, P. and Jyothsna, K., "Performance analysis of variant working vacation queue with balking and reneging", *International Journal of Mathematics in Operational Research*, 6 (2014) 505-521.

[24] Vijaya Laxmi, P. and Jyothsna, K., "Balking and Reneging Multiple Working Vacations Queue with Heterogeneous Servers", *Journal of Mathematical Modelling and Algorithms*, 14 (2015) 267-285.

[25] Vijaya Laxmi, P. and Jyothsna, K., "Impatient customer queue with Bernoulli schedule vacation interruption", *Computers & Operations Research*, 56 (2015) 1-7.

[26] Whitt, W., "Untold horrors of the waiting room: what the equilibrium distribution will never tell about the queue length process", *Management Sciences*, 29 (1983) 395-408.

[27] Yechiali, U., "Queues with system disasters and impatient customers when system is down", *Queuing Systems*, 56 (2007) 195-202.