Yugoslav Journal of Operations Research
27 (2017), Number 4, 439–450
DOI: https://doi.org/10.2298/YJOR160512021S

LIMITATION AND MODIFICATION: ON A COST PIPELINE TRADE-OFF IN A TRANSPORTATION PROBLEM

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Received: May 2016 / Accepted: November 2016

Abstract: The present paper attempts to foreground a modified approach to the algorithm as conceived by Sharma et al. [1] in the paper entitled 'A cost and pipeline trade-off in a transportation problem'. In 2013, Sharma et al. [1] came up with an idea of pivotal time and consequentially evolved a convergent algorithm to determine cost pipeline trade off pairs corresponding to the chosen pivotal time. We have observed one limitation of the proposed algorithm which directly affects the global minimum character of the pipeline. In the present paper, we suggest a modification of the limitation found in the procedural formulation developed by Sharma et al. [1].

Keywords: Transportation Problem, Combinatorial Optimization, Bottleneck Transportation Problem, Bi-criteria Transportation Problem, Efficient Points.

MSC: 90B06, 90C05, 90C08.

1. INTRODUCTION

Note: Definitions, notations and symbols as per the paper [1] given in Appendix.
The authors [1] proposed an algorithm to find the cost pipeline trade off pairs corresponding to the chosen pivotal time.

The mathematical models of the problems considered by authors [1] are as follows:
Note: For the detail description of the models and definitions one can refer the paper [1]

(P_1)

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

subject to the constraints:

$$\sum_{j \in J} x_{ij} = a_i,\ a_i > 0, i \in I,$$
$$\sum_{i \in I} x_{ij} = b_j,\ b_j > 0, j \in J,$$
$$x_{ij} \geq 0, \forall\ (i, j) \in I \times J$$

where I is the index set of sources , J is the index set of destinations , c_{ij} is the per unit cost of transportation from i^{th} source to the j^{th} destination , x_{ij} is the amount of commodity transported from from i^{th} source to the j^{th} destination , a_i is the total availability at the i^{th} source and b_j is the total demand at the j^{th} destination.
Also

$$S = \left\{X = x_{ij} \mid X \text{ satisfies the above set of constraints}\right\}$$

and t_{ij} is the time of transportation from i^{th} source to the j^{th} destination.

For any $X \in S$ and $T = \max\left\{t_{ij} \mid x_{ij} > 0\right\}$.

(P_2)

$$\min_{X \in S} \sum_{i \in I} \sum_{j \in J} c'_{ij} x_{ij}$$

where

$$c'_{ij} = \begin{cases} c_{ij} & \text{if } t_{ij} \leq T \\ \infty & \text{if } t_{ij} > T \end{cases}$$

The optimal value (denoted by Z) of the problem (P_2) gives minimum transportation cost at any time T.

$(RT - P)$

$$\min_{X \in S} \sum_{i \in I} \sum_{j \in J} c^*_{ij} x_{ij}$$

where

$$c_{ij}^* = \begin{cases} 0, & \text{if } t_{ij} < T, \\ 1, & \text{if } t_{ij} = T, \\ \infty, & \text{if } t_{ij} > T. \end{cases}$$

The optimal solution (denoted by p) of the problem $(RP - T)$ gives the minimum pipeline at time T.

The algorithm developed by Sharma et al. [1] provide the efficient pairs $(T : Z, p)$ where Z is the minimum cost of transportation and p is the minimum pipeline at pivotal time T,yielded by the optimal solutions of (P_2) and $(RP - T)$, respectively. The algorithm suggested by the authors [1] begins with optimal cost of transportation at a pivotal time chosen and then with reading the corresponding pipeline. The obtained pair is declared to be the first efficient pair if there is no possibility to reduce the pipeline any further, otherwise, pipeline is reduced until a pair having minimal pipeline at the optimal cost of transportation is obtained. For the next efficient pair, cost (transportation) is compromised by increasing and pipeline is reduced. The algorithm proceeds in such a way so that the first efficient pair gives the global minimum cost and the last pair gives the global minimum pipeline for the chosen pivotal time.
The algorithm proposed by Sharma et al. [1] suggest that if

$$N^{qh} = \emptyset \quad \forall\, h = 1, 2, ..., s_q$$

then, proceed to the terminal step which declares E to be the exhaustive set of efficient pairs corresponding to the pivotal time chosen.
This indicates that if $N^{qh} = \emptyset$ for all $h = 1, 2, ..., s_q$ and for any q, then $\left(T : Z_q, p_q\right)$ will be the last efficient pair.
However , the proposed algorithm [1] has one limitation which affects the global minimum character of the pipeline under certain instances. This limitation could be easily seen under the situation when emptiness of N^{qh} does not guarantee the emptiness of L_q. As a result, the last pair in E obtained by the suggested algorithm fails to give global minimum pipeline at the pivotal time.
To overcome this limitation, we provide a modified approach to the one given in [1].

2. MODIFIED ALGORITHM

To overcome this limitation, a modification has been done in Step 4 and Step 5 of the algorithm (proposed by Sharma et al. [1]). Rest of the steps (Step 0 to Step 3 and step 6) remain intact.To support our modification, a theorem has also been proposed.

Let the partition of the time routes be given by $t_0 > t_1 > t_2 > ..... > t_k$, where

$t_k = \min\left\{t_{ij} \mid i \in I, j \in J\right\}$.
First start with $r = 0$ and $l = 0$
Step 0. Solve the problem (P_2) for time $T = t_l$ and proceed to step 1.

Step 1. If every optimal basic feasible solution of problem (P_2) yield time T, then declare T as pivotal and jump to step 2, or else go to step 0 for $l = l+1$.

Step 2. Compute the cost Z_1 and pipeline p_1 for the optimal basic feasible solution of problem (P_2) corresponding to the time T and basis B_r.

Step (2.a) Construct the set $N_r = \left\{(i, j) \notin B_r \mid \Delta_{ij} = 0, \Delta'_{ij} > 0\right\}$. If $N_r = \emptyset$,then proceed to step (2.a) , otherwise, go to the next step.

Step (2.b) Choose $(i, j) \in N_r$ for which $\Delta'_{st} = \max\left\{\Delta'_{ij} \mid (i, j) \in N_r\right\}$. Now, set $r = r + 1$ and obtain a new basic feasible solution X_r with basis B_r by entering the cell (s, t) into the basis B_r. Proceed to step (2.a).

Step (2.c) Record $(T : Z_1, p_1)$ as the first efficient pair corresponding to the basic feasible solution X^{11} and basis B^{11}.
Construct $H = \left\{(i, j) \notin B^{11} \mid \Delta_{ij} = 0, \Delta'_{ij} = 0\right\}$
and compute $X^{1h}, h = 2, 3, ...s_1$. Now, set $X = \left\{X^{1h}, h = 1, 2....s_1\right\}$ and proceed to Step 3.

Step 3. Record $E = (T : Z_1, p_1)$ and set $q = 1$.Now, proceed to the next step.

Step 4. Construct sets N^{qh}, for all $h = 1, 2, ..., s_q$., D_q, L'_q and L_q.
If $L_q = \emptyset$, then $\left(T : Z_q, p_q\right)$ is the last efficient pair in E (as proved in Theorem 3.1) proceed to the terminal Step 6, otherwise, go to Step 5.

Step 5. Note $(q + 1)^{th}$ as the efficient pair $\left(T : Z_{q+1}, p_{q+1}\right)$ from the set L_q for which $Z_{q+1} = \min\left\{Z \mid (T : Z, p) \in L_q\right\}$ and set $E = E \cup \left(T : Z_{q+1}, p_{q+1}\right)$.
Now, proceed to the Step 4 for $q = q + 1$.

Step 6 (Terminal step). E is the exhaustive set of efficient pairs corresponding to the pivotal time T.

3. THEORETICAL DEVELOPMENT

The whole theory of the paper [1] will not get affected with these modifications and therefore, does not require any changes and new theoretical developments except the one given below.

Theorem 3.1. *If $Lq = \emptyset$, then $\left(T : Z_q, p_q\right)$ is the last efficient pair in E.*

Proof. Suppose on the contrary that $\left(T : Z_q, p_q\right)$ is not the last efficient pair in E. Then there exists a pair $(T : Z, p) \in E$ such that $Z > Z_q$ and $p < p_q$. Thus, either $(T : Z_q, p_q) \in D_q$ or $(T : Z_q, p_q) \in L_q^{'}$.This implies $(T : Z_q, p_q) \in L_q^{'} \cup D_q = L_q$.Which is a contradiction as $L_q = \emptyset$. □

4. NUMERICAL ILLUSTRATION

The following numerical illustrations not only depict the aforementioned limitation but also exhibits the modification to come out of the limitation.

Numerical 1.

As proposed by Sharma et al. [1]
Consider a transportation problem having three sources and four destinations as given in Table 1.

Table 1

					a_i
	22 19	13 32	34 8	40 12	17
	28 70	18 30	45 40	7 60	27
	11 40	20 8	36 20	21 22	28
b_j	10	18	25	19	

The upper left corner contains the time (t_{ij}) of transportation, and the lower right corner contains the cost (c_{ij}) of transportation.The partition of time routes is given by $t^0(= 45) > t^1(= 40) > t^2(= 36) > t^3(= 34) > t^4(= 28) > t^5(= 22) > t^6(= 21) > t^7(= 20) > t^8(= 18) > t^9(= 13) > t^{10}(= 11) > t^{11}(= 7)$

Step 0. Solve the problem (P_2) for $T = t^0(= 45)$ and go to Step 1.

Step 1. The two alternate optimal feasible solutions obtained by solving problem (P_2) corresponding to the time $T = 45$ (illustrated in Table 2 and Table 3), yield the time 45.

Table 2

					a_i
	22 **10** 19	13 32	34 **7** 8	40 12	17
	28 70	18 **9** 30	45 **18** 40	7 60	27
	11 40	20 **9** 8	36 20	21 **19** 22	28
bj	10	18	25	19	

Table 3

					a_i
	22 **10** 19	13 32	34 8	40 **7** 12	17
	28 70	18 **2** 30	45 **25** 40	7 60	27
	11 40	20 **16** 8	36 20	21 **12** 22	28
b_j	10	18	25	19	

Therefore, $T = 45$ is a pivotal time. Now, go to Step 2.

Step 2. Record the pair $(T : Z, p) = (45 : 1726, 18)$ corresponding to the basis B_0 and go to Step(2.a).

Step (2.a). Construct the set

$$N_0 = \left\{(i, j) \notin B_0 | \Delta_{ij} = 0, \Delta'_{ij} > 0\right\}$$

Since $N_0 = \emptyset$, go to Step(2.c).

Step (2.c). Record (45 : 1726, 18) as the first efficient pair corresponding to the solution $X^1 = X^{11} = \left\{x_{11}^{11} = 10, x_{13}^{11} = 7, x_{22}^{11} = 9, x_{23}^{11} = 18, x_{32}^{11} = 9, x_{34}^{11} = 19\right\}$. Now, go to Step 3.

Step 3. Record $E = (45 : 1726, 18)$ and go to Step 4 for $q = 1$.

Step 4. Construct the set $N^{11} = \{(2,1),(3,1),(3,3)\}$.Since $N^{11} \neq \emptyset$, go to Step 5.

Step 5. Construct the sets
$D_1(= D_{11}) = \{(45 : 1916, 8), (45 : 1825, 9), (45 : 1744, 9)\}$ and
$L_1 = \{(45 : 1744, 9), (45 : 1916, 8)\}$.

Now $(T : Z_2, p_2) = \min\{Z|(T, Z, p) \in L_1\} = (45 : 1744, 9)$
Update $E = \{(45 : 1726, 18), (45 : 1744, 9)\}$ and go to Step 4 for $q = 2$

Step 4. Table 4 illustrates the second efficient pair (45 : 1744, 9)

Table 4

					a_i
	22 **10** 19	13 32	34 **7** 8	40 12	17
	28 70	18 **18** 30	45 **9** 40	7 60	27
	11 40	20 8	36 **9** 20	21 **19** 22	28
b_j	10	18	25	19	

Since $N^{21} = \emptyset$, go to Step 6 which declares $E = \{(45 : 1726, 18), (45 : 1744, 9)\}$ to be the exhaustive set of efficient pairs corresponding to the pivotal time $T = 45$. Therefore, (45 : 1744, 9) is the last efficient pair having global minimum pipeline 9 at the pivotal time $T = 45$.
It may be observed that the efficient pair (45 : 1916, 8) $\in \mathbf{L_1}$ with pipeline 8(< 9) is not recorded in E.

Modified version:
(As proposed in the present paper)

Instead of step 4 and step 5 (as suggested by Sharma et al. [1]), we apply modified step 4 and modified step 5.

Step 4. Construct the sets $N^{21} = \emptyset, D_2 = \emptyset, L_2^{'} = L_1 \setminus \{(45 : 1744, 9)\} = \{(45 : 1916, 8)\}$. and $L_2 = (45 : 1916, 8)$. Since $L_2 \neq \emptyset$, go to step 5.

Step 5. Now the third efficient pair is given by $(T : Z_3, p_3) = (45 : 1916, 8)$ and updated $E = \{(45 : 1726, 18), (45 : 1744, 9), (45 : 1916, 8)\}$. Now, proceed to step 4 for $q = 3$

Step 4. Table 5 illustrates the third efficient pair (45 : 1916, 8)

Table 5

					a_i
	22 / 19	13 / 32	34 / **17** / 8	40 / 12	17
	28 / **10** / 70	18 / **9** / 30	45 / **8** / 40	7 / 60	27
	11 / 40	20 / **9** / 8	36 / 20	21 / **19** / 22	28
b_j	10	18	25	19	

Now, $N^{31} = \emptyset, D_3 = \emptyset, L_3^{'} = \emptyset$, and $L_3 = \emptyset$. Since $L_3 = \emptyset$, therefore go to Step 6 which declares $E = \{(45 : 1726, 18), (45 : 1744, 9), (45 : 1916, 8)\}$ to be the exhaustive set of efficient pairs corresponding to the pivotal time $T = 45$.

Numerical 2.

Consider another transportation problem having three sources and four destinations (given in Table 6),where the upper left corner contains the time (t_{ij}) of transportation and the lower right corner contains the cost (c_{ij}) of transportation.

Table 6

					a_i
	40 10	30 41	16 52	10 25	13
	48 2	5 8	18 21	12 30	20
	38 35	25 20	50 23	45 40	27
b_j	10	20	15	15	

Here the exhaustive set of efficient pairs $\mathbf{E} = \{(\mathbf{50} : \mathbf{970}, \mathbf{15}), (\mathbf{50} : \mathbf{1150}, \mathbf{5})\}$ obtained by the algorithm suggested in [1] fails to record the pair $(\mathbf{50} : \mathbf{1622}, \mathbf{2}) \in \mathbf{L_1}$.

After applying the modified version,
$\mathbf{E} = \{(\mathbf{50} : \mathbf{970}, \mathbf{15}), (\mathbf{50} : \mathbf{1150}, \mathbf{5}), (\mathbf{50} : \mathbf{1622}, \mathbf{2})\}$

Some Observations on the Numericals
There are two scenarios:
First Scenario
When there does not exists any q for which $N^{qh} = \emptyset$, $\forall\, h = 1, ..., s_q$ and $L_q^{'} \neq \emptyset$ (can be seen in the illustration given by Sharma et al. [1])
Second Scenario
When there exits at least one q for which $N^{qh} = \emptyset$, $\forall\, h = 1, .., s_q$ and $L_q^{'} \neq \emptyset$ (can be seen in Numerical 1 for $q = 2$).

5. CONCLUDING REMARKS

1. The present paper provides a modified version of the algorithm proposed by Sharma et al. [1] to obtain the cost pipeline trade-off pairs corresponding to the pivotal time. The algorithm in [1] obtain the collection of efficient pairs in such a way so that the first efficient pair gives the global minimum cost and the last pair gives the global minimum pipeline for the chosen pivotal time. However, the algorithm fails to record the global minimum pipeline corresponding to the pivotal time over the second scenario (reported in the observation of Section 4).
2. The modified version proposed in the present paper is efficient to record the global minimum pipeline over the scenarios reported in the observation of

Section 4. To reinforce the impact of the modified version, two Numerical illustrations are also given. The table (Table 7) given below exhibits the efficiency of the Modified Algorithm more clearly.

Table 7

	Numerical 1		Numerical 2	
	Algorithm by Sharma et al. [1]	Modified Algorithm	Algorithm by Sharma et al. [1]	Modified Algorithm
First efficient pair	(45:1726,18)	(45:1726,18)	(50:970,15)	(50:970,15)
Second efficient pair	(45:1744,9)	(45:1744,9)	(50:1150,5)	(50:1150,5)
Third efficient pair	*	(45:1916,8)	*	(50:1622,2)
Fourth efficient pair		*		*
Global Minimum Pipeline	9	8	5	2

* Algorithm Stops

It is observed from the above table that in Numerical 1, the global minimum of pipeline 8 (recorded by the Modified Algorithm) is less than the global minimum of pipeline 9 (recorded by the algorithm proposed by Sharma et al.[1]). Similarly, in Numerical 2,the global minimum pipeline 2 (recorded by Modified Algorithm) is less than the global minimum pipeline 5 (recorded by the algorithm proposed by Sharma et al.[1]).

3. As $L_q = D_q \cup L_q^{'}$, therefore $L_q = \emptyset \Longleftrightarrow D_q = \emptyset$ and $L_q^{'} = \emptyset$.
   But $D_q = \emptyset \Longleftrightarrow N^{qh} = \emptyset \; \forall h = 1, ..., s_q$.
   Therefore, $L_q = \emptyset$ if and only if $N^{qh} = \emptyset \; \forall h = 1, ..., s_q$ and $L_q^{'} = \emptyset$. This implies that emptiness of L_q not only depends on emptiness of N^{qh} but also on emptiness of $L_q^{'}$.

4. $L_q^{'} = \emptyset \Longleftrightarrow L_{q-1}$ is a singleton set.
   Cases under which $L_q^{'}$ become empty, or L_{q-1} is a singleton set:

   (a) if there exists a pair in $D_{q-1} \cup L_{q-1}^{'}$ which will dominate all the other pairs in $D_{q-1} \cup L_{q-1}^{'}$ (can be seen in the illustration given by Sharma et al. [1] for $q = 3$).

   (b) if $D_{q-1} \subseteq L_{q-1}^{'}$ and $L_{q-1}^{'}$ is a singleton set (can be seen in Numerical 1 for $q = 3$).

   (c) if $L_{q-1}^{'} \subseteq D_{q-1}$ and D_{q-1} is a singleton set.

5. The problem of finding trade-off pairs of transportation and deterioration costs corresponding to the pivotal time can be explored in future.

Acknowledgment: Authors are thankful to Ms.Branka Mladenovic (Publishing Manager)and honorable reviewer for their valuable suggestions and comments incorporating which has improved our paper substantially.

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APPENDIX

Definitions and Notations:

Pivotal time: Time T is called pivotal time if for any other time of transportation $T^{'} > T \Rightarrow Z^{'} < Z$ and $T^{'} < T \Rightarrow Z^{'} > Z$, where Z and $Z^{'}$ are minimum transportation costs given by (P_2) at times T and $T^{'}$, respectively.Which means T is pivotal if for each optimal basic feasible solution of the problem (P_2) there exists a cell (i, j) with $t_{ij} = T, x_{ij} > 0$.

Dominated pair: A pair (T, Z, p) is called dominated pair at pivotal time T if there exists a pair $(T, Z^{'}, p^{'})$ such that $(Z, p) \geq (Z^{'}, p^{'})$ i.e $Z \geq Z^{'}$ and $p \geq p^{'}$ with strict inequality holding least at one place.

Non-dominated pair: A pair which is not dominated is called non-dominated pair.

Efficient point: A solution yielding a non-dominated pair is called an efficient point.

$\mathbf{q^{th}}$ **efficient pair** $(\mathbf{T : Z_q, p_q})$, $(q \geq 2)$ is an element of L_{q-1} for which $Z_q = \min\left\{Z \mid (T : Z, p) \in L_{q-1}\right\}$.

For $q \geq 1$,
X^q : Set of all basic feasible solutions yielding the q^{th} efficient pair $= \left\{X^{qh} \mid h = 1 \text{ to } s_q\right\}$.
B^{qh} : Set of all basic cells of the solution X^{qh}.
Δ_{ij} : Relative cost co-efficient for the cell (i, j) corresponding to the problem (P_2).
$\Delta^{'}_{ij}$: Relative cost co-efficient for the cell (i, j) corresponding to the problem $(RP - T)$.
$\hat{X}^{qh}$: Basic feasible solution derived from X^{qh} by a single pivot operation such

that the corresponding time of transportation remains T, i.e

$$\max_{\left\{(i,j)\,|\,\hat{x}_{ij}^{qh}>0\right\}} t_{ij} = T$$

For each $h = 1, 2, ..s_q$ and $q \geq 1$

$N^{qh} = \left\{(i, j) \notin B^{qh} \mid \Delta_{ij} < 0, \Delta'_{ij} > 0, \hat{x}_{ij}^{qh} = x_{lm}^{qh}, \hat{x}_{lm}^{qh} = 0, (l, m) \in B^{qh} \text{and} \max_{\left\{(r,w)\,|\,\hat{x}_{rw}^{qh}>0\right\}} t_{rw} = T\right\}$

$D_{qh} = \left\{(T : Z, p) \mid Z = Z_q - \Delta_{ij} x_{lm}^{qh}, p = p_q - \Delta'_{ij} x_{lm}^{qh}, (l, m) \in B^{qh}, (i, j) \in N^{qh}\right\}$

$D_q = \cup_{h=1}^{s_q} D_{qh}$

$L'_q = L_{q-1} - \left\{(T : Z_q, p_q)\right\}$ for $q \geq 2$, where $L'_1 = \emptyset$

$L_q = L'_q \cup D_q - \left\{(T : Z, p) \mid (T : Z, p) \text{ is a dominated pair in } L'_q \cup D_q\right\}$

E : Set of efficient pairs $(T : Z_q, p_q), q = 1, 2...M$.