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SIGNAL GROUP: DEFINITIONS AND ALGORITHMS

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Abstract: The traffic control on an intersection consists of permitting and canceling right-of-way to traffic streams. One traffic stream is controlled by one sequence of traffic light indications. These traffic light indications, for one traffic stream, are activated by one module in electronic device (controller). But, one sequence of traffic light indications can control more traffic streams, which are not in conflict.

This paper deals with the problem of classifying traffic streams into groups (signal groups) which will be controlled by identical indications of traffic lights. Therefore, the relation of identical indications of traffic lights is defined. This relation is defined over the set of traffic streams on one intersection. The graph of this relation is, also, introduced. A signal group is defined as a clique (in the sense of Berge) of this graph, and the complete set of signal groups is introduced.

Computer programs are developed for determination of the collection of complete sets of signal groups, and for determination of all minimal complete sets.

In the paper is, also, defined the relation of partial order over the collection of complete sets of signal groups, and the use of this relation in control system design is considered.

Keywords: Traffic, control, signalized intersection, signal group, integer programming.

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1. INTRODUCTION

One of the most important problems to be solved when introducing an intersection traffic control system, is to define the groups of traffic streams [10] (signal groups) that will be controlled by identical traffic light indications [1], [4], [6]. From the point of view of control theory, it means that each signal group is controlled by one control variable, that is to each signal group will be assigned one control variable.

In practical realization of traffic control system, using modern equipment (so called signal group controller), one module of equipment is assigned to each signal group. This module changes the signal light indications, controlling the movement of traffic streams belonging to one signal group. Therefore, the first question, when someone orders the traffic controller is: how many vehicle, pedestrian and special signal groups will be controlled by that controller. It means, how many modules of different types will be installed in the controller. So, it is clear that the price of traffic control equipment, for each intersection, depends on the number of signal groups. That is the reason why in practice, the common intention is to control the traffic on an intersection by using a small number of signal groups (small number of control variables, i.e. small number of controller modules).

The choice of signal groups affects, also, the value of the chosen traffic performance indices, which can be achieved as the solution of optimization traffic control problems.

It is obvious that each traffic stream can be a signal group. In all other cases the number of signal groups is less than number of traffic streams, because some of signal groups are composed of two or more traffic streams. The fact that two or more traffic streams belong to one signal group, i.e. two or more traffic streams are controlled by identical indications, means that they are controlled by one control variable. When each traffic stream is one signal group then each of them is controlled by one control variable.

It is interesting to analyze the influence of signal group choice to the optimal values of chosen criteria in traffic optimization problems. The way of control when the number of signal groups is smaller than the number of traffic streams, implies implementation of constraints in the optimization problem which cause deterioration of all traffic performance indices. Investigations of the influence of signal groups choice on the maximum traffic volume that can pass through the intersection [1] showed that the maximum traffic volume is significantly affected by that choice.

The introduction of the exact notation of the signal plan structure and the control transition graph in the formulation of the optimization traffic control problems [5] gives possibility for an analysis of that influence.

The first step in that analysis is the creation of all possible choices of signal groups. Because of that in the Section 3., Section 4. and Section 5. are presented: the exact definition of signal group, the notation of complete set of signal groups and the programs for obtaining the list of all signal groups, and all complete sets of signal groups.

In the Section 7. the exact method is given, which can be helpful for make the choice of signal groups by which the traffic control system will be implemented at one intersection.

2. TRAFFIC STREAMS AND RELATIONS BETWEEN THEM

The traffic process on one intersection is determined by traffic flows moving through the intersection. The smallest component of that process, which can be controlled by a separate traffic light is termed *a traffic stream*. One traffic stream represents either:

- flow performing the same manoeuvre through the intersection and belongs to one or more traffic lanes, or
- flow which is composed of the vehicles that make different manoeuvres but they belong to one traffic lane.

The vehicles, belonging to one traffic stream, form the same queue on the intersection approach.

There are different types of traffic streams depending on the type of moving elements. So, there are traffic streams of vehicles, pedestrians, public transport vehicles, bicycles etc.

The paths along which a traffic stream move through the intersection is called *the traffic stream trajectory*.

For intersection traffic control system implementation it is necessary to introduce relations defined over the set of traffic streams.

All traffic streams on one intersection form the set S:

$$S = \{ s_1, s_2, \dots , s_i, \dots , s_n \}$$

where n is the number of traffic streams.

The relation of conflictness C_1

The traffic streams are in the conflict if their trajectories cross each other or merge. The set of all pairs of conflicting traffic streams form *the relation of conflictness* which can be described as follows:

$$C_1 = \{ (s_i, s_j) \in S \times S \mid i \neq j \text{ and trajectories of } s_i \text{ and } s_j \text{ cross or merge} \}$$

In Figure 1 an intersection is shown, with ten traffic streams and their trajectories. The conflict points are marked by little circles.

The relation of conflictness is symmetric, i.e.

$$(s_i, s_j) \in C_1 \Rightarrow (s_j, s_i) \in C_1$$

For that reason this relation can be represented by an undirected graph.

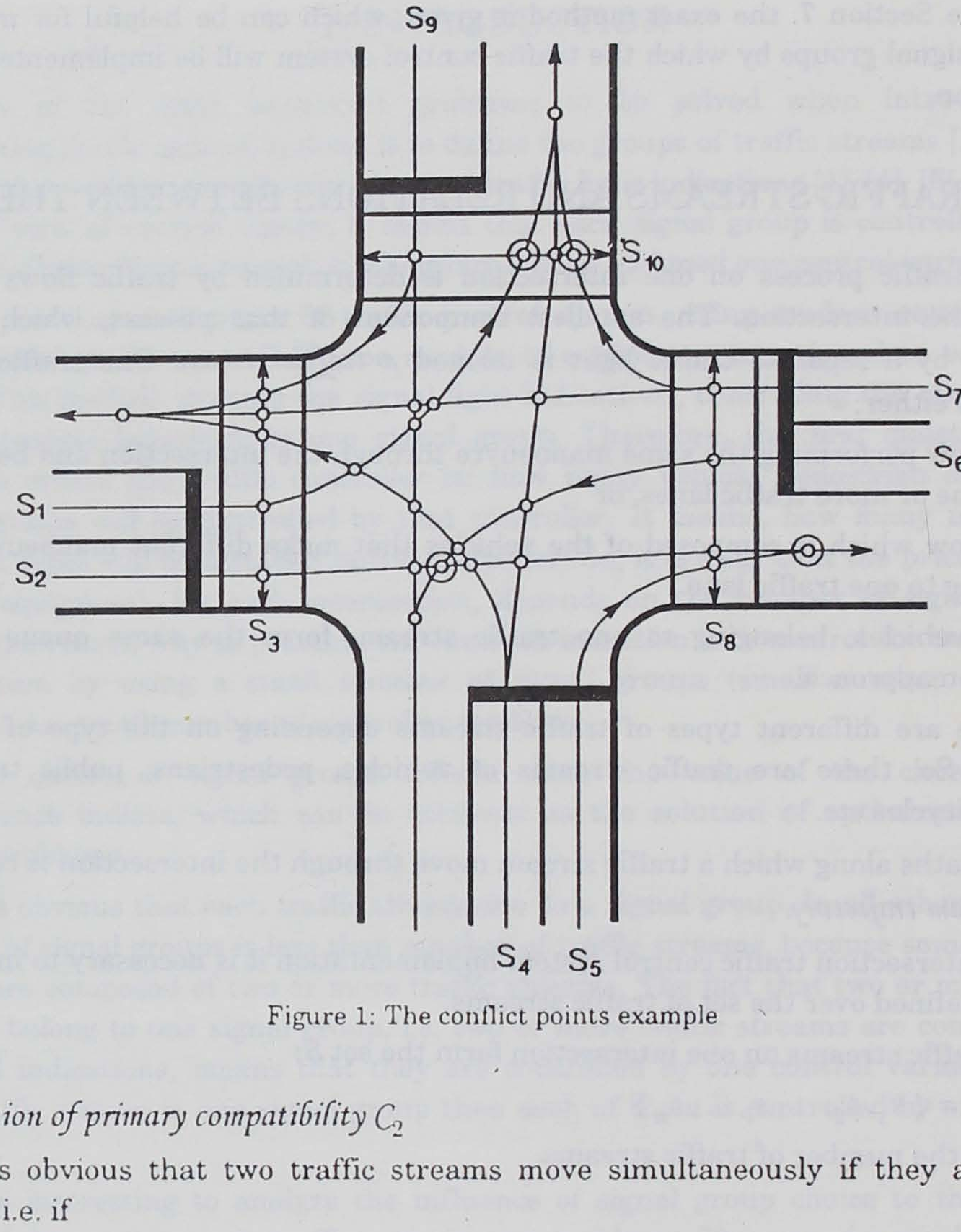


Figure 1: The conflict points example

The relation of primary compatibility C_2

It is obvious that two traffic streams move simultaneously if they are not in conflict, i.e. if

$$(s_i, s_j) \in \overline{C}_1 \qquad \text{where} \quad \overline{C}_1 = (S \times S) \setminus C_1$$

Besides, different conflicts between traffic streams are not equally dangerous. Therefore, the relation should be introduced, by which the pairs of traffic streams, allowed to move simultaneously, will be defined. This relation is called the *relation of primary compatibility.* Two traffic streams satisfy the relation of primary compatibility, if they are not in conflict or their conflicts may be allowed considering their flow volumes, i.e.

$$C_2 = \overline{C}_1 \cup \{ (s_i, s_j) \in C_1 \mid \text{the traffic streams } s_i \text{ an } s_j \text{ may have right–of–way simultaneously considering their volumes} \}$$

In order to determine the relation of primary compatibility, it is necessary to consider both geometrical characteristics of traffic streams movement (sufficient for the relation of conflictness) and information about values of traffic process parameters on an intersection.

The consideration of traffic streams volumes is, in fact, the indirect consideration of safety. Allowance of some conflicts are, usually, made by the judgment of experienced professionals. The only quantitative information, that experts use, while making those decisions, is information about volumes of traffic streams in conflict. Some new research [8] investigates the relation between the volumes of conflict traffic streams and the number of dangerous manoeuvres. This could be the basis of the introduction of exact analytical methods for choosing the allowed conflicts.

The relation of primary compatibility is, also, a symmetric relation.

Consider, again, the intersection at Figure 1. It might be possible to allow the conflicts corresponding to the conflict points marked by larger circles. For example, traffic streams s_2 and s_6 are in conflict, but, at the same time they could in some circumstances be taken as being primary compatible.

By allowing some conflicts the intersection traffic system performance is affected in such a way that the number of cars, passing through the intersection, is increased (because the time interval for right-of-way for some traffic streams is increased) and the safety is decreased. These effects have been discussed in literature [7].

The relation of compatibility C_3

In some countries there are traffic signals with arrows which control protected movements. According to traffic laws, the vehicles to which the right-of-way is given by such signals shall not meet any conflict when passing through the intersection. That caused the introducing the *relation of compatibility.*

$C_3 = \{ (s_i, s_j) \in C_2$ | traffic streams s_i and s_j can have simultaneously right-of-way with regard to traffic safety laws $\}$

In Figure 1 the conflict between traffic streams s_2 and s_6 is conflict which could be allowed, in some countries, if their volumes were small - then, these two traffic streams would satisfy the relation of primary compatibility. But, if traffic streams s_6 is controlled by a direction signal, then the pair of streams (s_2, s_6), following the traffic laws, does not satisfy the compatibility relation.

3. THE SIGNAL GROUP

The traffic control on an intersection consists of permitting and canceling right-of-way to different traffic streams. It is realized by different indications of traffic lights so that green indication has the meaning "movement is permitted", and red indications has the meaning "movement is forbidden".

The change of traffic light indications can be described by changes of the values of one mathematical variable called *control variable.* In order to simplify mathematical description of traffic light indications, it is implied that control variable has only two values:

$$u = \begin{cases} 1, & \text{for effective green} \\ 0, & \text{for effective red} \end{cases}$$

and this values are corresponding to real situation on vehicle traffic light in the manner presented at Figure 2.

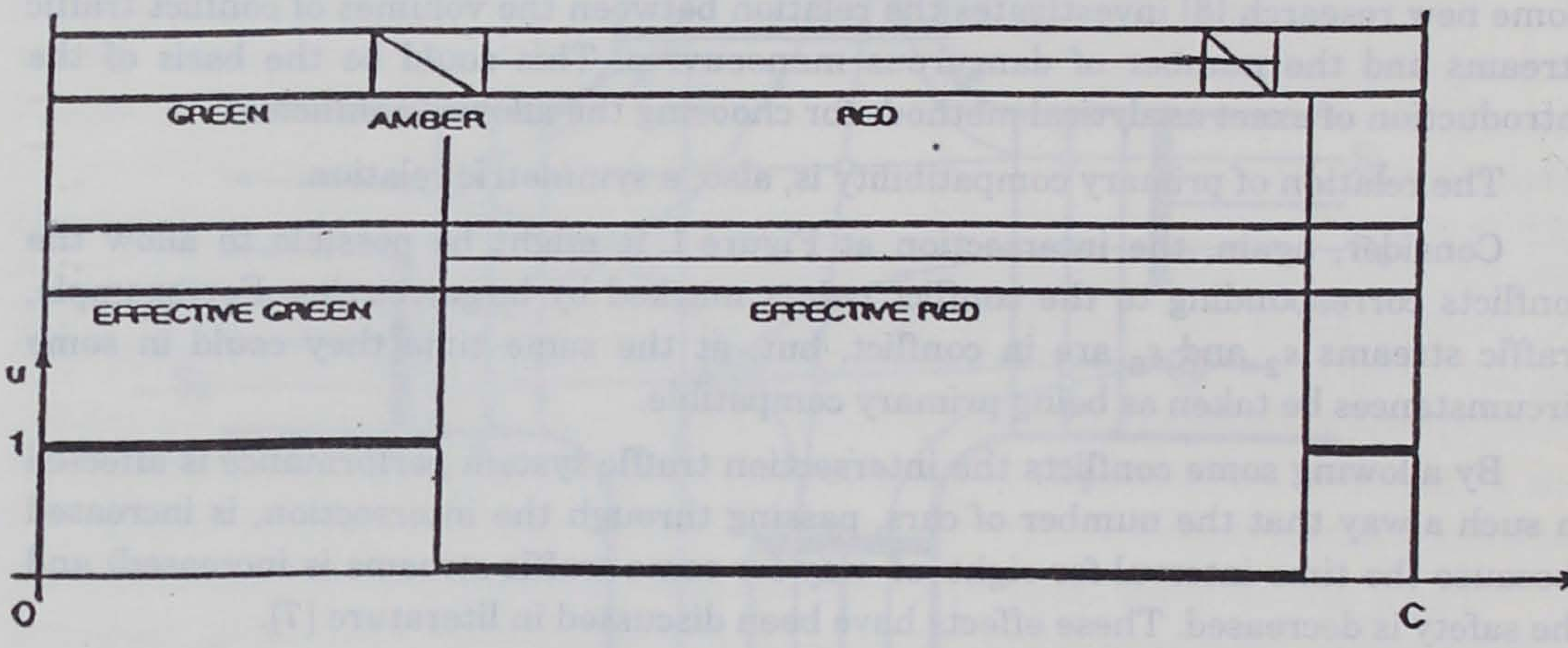


Figure 2: The control variable

Definition of signal group represents assignment of control variables to traffic streams.

The simplest way of that assignment is assignment of one control variable to each traffic stream.

The fact that compatible traffic stream can receive right-of-way simultaneously gives possibility to control more traffic streams by one control variable.

The set of traffic streams controlled by one control variable, i.e. by identical indications of traffic lights is called *a signal group*, denoted by D_i.

$$D_i = \{ s_{i_1}, s_{i_2}, \dots, s_{i_l} \}$$

where l is the number of traffic streams in signal group D_i.

Because of such definition of signal group, the relation of identical indications of traffic lights is introduced, defined over the set of traffic streams.

The relation of identical indications of traffic lights C_4 – The identity relation

This relation is satisfied by two compatible traffic streams if they can be controlled by traffic lights with identical indications. The sequence of indications for different types of traffic streams are not necessarily the same. For example, the sequence of signal indications for vehicles (red, red-amber, green, amber) is not the same as the sequence of signal indications for pedestrians (red, green). So, two compatible traffic streams satisfy the identity relation only if they are of the same type (pedestrians, vehicle, bicycle, etc.)

$$C_4 = \{ (s_i, s_j) \in C_3 \mid \text{traffic streams } s_i \text{ and } s_j \text{ are of the same type} \}$$

It is possible to generate all signal groups using the identity graph (graph of the identity relation). Each complete subgraph of the identity graph defines a signal group. Since the complete subgraph is a clique of the graph (in the sense of Berge) [2], [8], *a signal group* can be redefined as a clique of the identity graph.

All signal groups, i.e., all cliques of identity graph form the collection $\mathbb{D}$.

A simple intersection and the corresponding graphs are given in Figure 3.

4. THE COMPLETE SET OF SIGNAL GROUPS

One of the most important decisions while designing traffic control system on one intersection, is choice of the set of signal groups. These signal groups have to be chosen in such a way that each traffic stream becomes the element of exactly one signal group. It is consequence of the fact that

- one traffic stream has to be element of at least one signal group (because the traffic stream has to get right-of-way for moving through the intersection)
- one traffic stream has to be element of at most one signal group (to one signal group corresponds one control variable, so, if one traffic stream is an element of two signal groups that implicates that traffic stream is controlled by two control variables, which are not necessarily identical through the time. That means that one traffic stream will be controlled by different indications of traffic lights.)

The set of signal groups with the property: each traffic stream is the element of exactly one signal group, is called *the complete set of signal groups*. So, a complete set of signal groups, marked by $\mathcal{d}$, consists of elements from collection $\mathbb{D}$ (collection of all signal groups).

The collection $\mathcal{S}$ represents the set of all complete sets of signal groups.

$$\mathcal{S} = \left\{ \mathcal{d} \mid \left(\left(\forall D', D'' \in \mathcal{d} \right) D' \neq D'' \Rightarrow D' \cap D'' = \varnothing \right) \wedge \left(\bigcup_{D_i \in \mathcal{d}} D_i = S \right) \right\}$$

Such definition of complete sets of signal group, implicates that the complete set of signal groups is a partition of the sets S. Hence, there exist more then one complete set of signal groups for one intersection. This means that it is possible to control the intersection in several different ways - by different choices of complete set of signal groups.

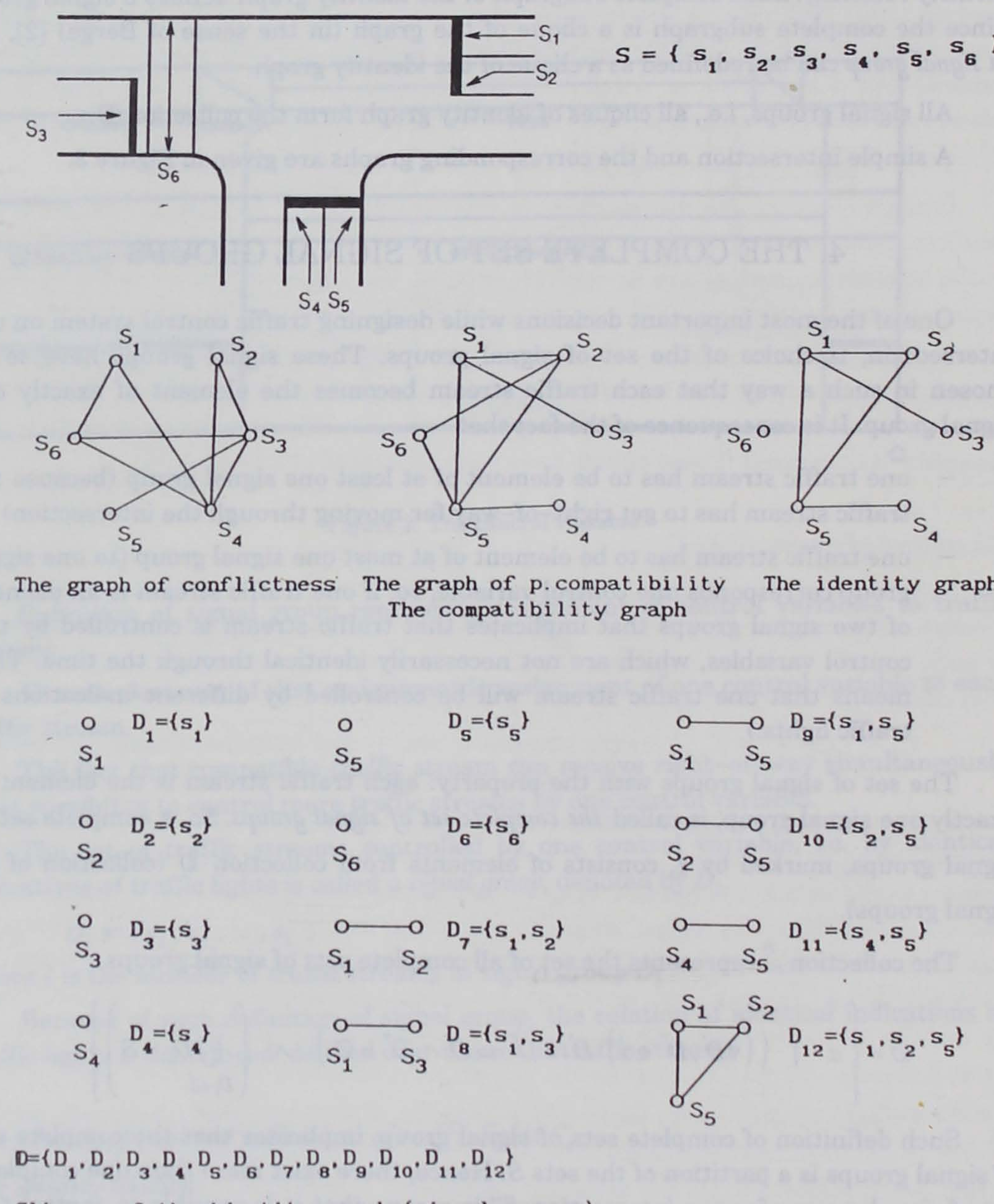
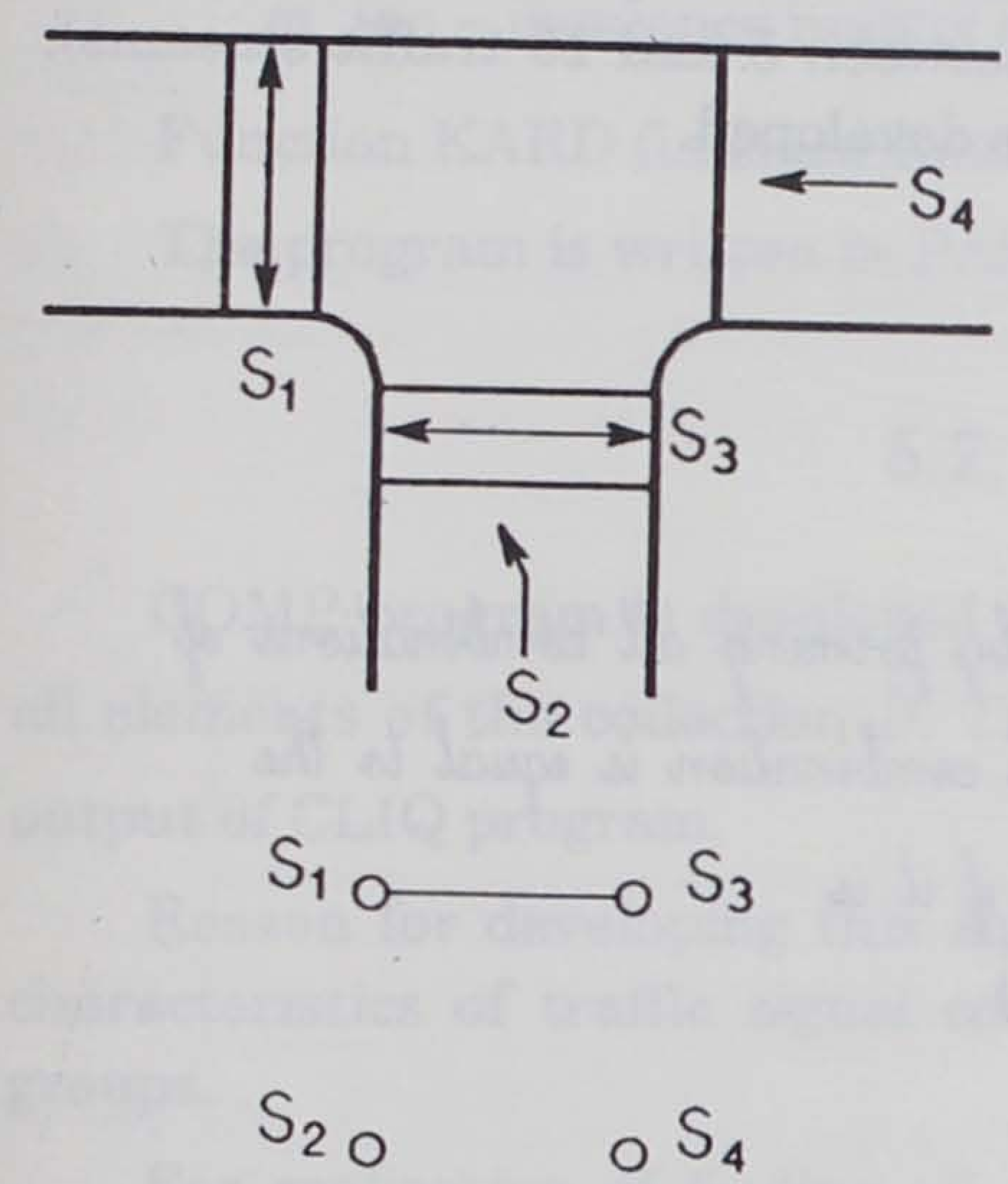


Figure 3: An intersection and its signal groups

The simple intersection (with four traffic streams) shown in Figure 4 has two complete sets of signal groups, namely d_1 and d_2.



The identity graph

$$S = \{ s_1, s_2, s_3, s_4 \}$$

$$D_1 = \{ s_1 \}$$
$$D_2 = \{ s_2 \}$$
$$D_3 = \{ s_3 \}$$
$$D_4 = \{ s_4 \}$$
$$D_5 = \{ s_1, s_3 \}$$

$$\mathbb{D} = \{ D_1, D_2, D_3, D_4, D_5 \}$$

$$d_1 = \{ D_1, D_2, D_3, D_4 \}$$
$$d_2 = \{ D_2, D_4, D_5 \}$$

$$\mathcal{D} = \{ d_1, d_2 \}$$

Figure 4: An intersection with complete sets of signal groups

5. THE COMPUTER PROGRAMS

So, problems connected with implementation of traffic control system at one intersection are:

1) finding all signal groups, and then
2) finding all complete sets of signal groups.

That is why following programs are developed:

I) CLIQ – program for determination of all signal groups,

II) COMP – program for determination of all complete sets of signal groups,

III) MIN – program for determination of all minimal complete sets of signal groups – all complete sets with the minimal number of signal groups.

5.1. CLIQ PROGRAM

Because each clique (in the sense of Berge) of the identity graph represents one signal group, the algorithm for determination of all cliques of a given graph will find all signal groups, i.e., all elements of the collection $\mathbb{D}$.

So, CLIQ program is a program for determination of all cliques (in the sense of Berge) of some graph. This program is used to find all cliques of identity graph. The input into the program is incidence matrix of identity graph.

The problem of finding cliques of identity graph, is not complex one, because the identity graph is small dimension graph (it is rare that one intersection has more than 32 traffic streams, and most of intersections have between 5 and 15 traffic streams). Because of that, algorithm for solving that problem is developed.

Pseudo-code of the program CLIQ:

```
begin
class := 1
   { finding all complete subgraphs with two nodes (by forming all combinations of
     the second class when the number of elements of combination is equal to the
     number of nodes of the graph and then asking if it is
   ( m_inc[i,j] = 1 ) AND ( m_inc[j, i] = 1 ) )        }
   { finding all other complete subgraphs  }
while ( n_comb[class+1] <> 0 ) do
      class := class + 1
      for i := 1 to n_comb[class] - class do
            for j := i + 1 to n_comb[class] - class + 1 do
               p = kard( comb[class][j] ∩ comb[class][i] )
               if (p = class - 1) then
                     k := 0
                     for l := j + 1 to n_comb[class] do
                           p = kard( comb[class][i] ∪ comb[class][j] ) ∩ comb[class][l]
                           if ( p = class ) then k := k + 1
                           if ( k = class - 1 ) then
                                 n_comb[class+1] := n_comb[class+1] + 1
                                 comb[class+1][n_comb[class+1]] :=
                                       comb[class][i] ∪ comb[class][j]
                           end if
                     end for
               end if
            end for
      end for
end while
end.
```

Variables in the program are:

comb – complete subgraph with *class* number of nodes. Each subgraph is in the form of set of nodes.

n_comb – number of complete sets with *class* number of nodes

m_inc – incidence matrix of the graph

Function KARD finds cardinal number of a set.

The program is written in PASCAL.

5.2. COMP PROGRAM

COMP program is developed to determine all complete sets of signal groups, i.e., all elements of the collection $\mathcal{D}$. The input is the set of all signal groups, that is, the output of CLIQ program.

Reason for developing this algorithm is that one of our tasks was comparing of characteristics of traffic signal control system for different complete sets of signal groups.

For realization of finding all complete sets of signal groups it is developed one simple search algorithm.

The program is written in PASCAL.

5.3. MIN PROGRAM

The designers solving the practical problems, try to realize the control systems which are simple enough and which are not too expensive. The price of the electronic device - traffic controller directly depends on the number of modules inside the controller. That means that the price for traffic control system at one intersection directly depends of number of signal groups that is chosen for traffic control. So, the usual practice in traffic control design for a signalized intersection is to pay attention to the complete sets with the minimum number of signal groups.

Therefore, the program MIN is developed.

MIN program is made to find all minimal complete sets of signal groups. This program is developed regardless that the COMP program already finds minimal complete sets of signal groups. One of the reasons this program was made, is that the number of complete sets of signal groups and the time of running the COMP program increase to fast when the number of signal groups increases. So, the more elegant way is to establish optimization problem in order to determine a minimal complete set of signal groups.

The optimization problem

From the formulation of the problem of finding the minimal complete set of signal groups, it is easy to see that this is *the set partitioning problem.* This is the combinatorial optimization problem and it is formulated in the following way:

A base set X, collection $\mathbb{X}$ of subsets of a base set, and the prices corresponding to each subset are given. The problem is to find the partition of set X with the minimum cost.

A *set partition* of base set is a set of subsets of a base set – collection of elements from $\mathbb{X}$, in the form of

$$\{ X_{i_1}, X_{i_2}, \dots, X_{i_q} \}$$

such that their union is equal to base set, i.e.

$$\bigcup_{k=1}^{q} X_{i_k} = X$$

and such that they are mutually disjunctive, i.e.

$$X_{i_j} \cap X_{i_k} = \varnothing, \qquad (\forall j \neq k)$$

In the minimal complete set determination problem the base set is the set of traffic streams S; the collection of subsets is the set $\mathbb{D}$ of all signal groups; the cost vector is the vector of 1's. (In the solution it is necessary that signal groups are mutually disjunctive because each traffic stream has to be controlled by only one control variable. It is necessary, also, that union of signal groups has to be equal to set S because each traffic stream has to be controlled.

So, by solving the set-partitioning problem, the minimal partition of set S will be found. This partition is exactly minimal complete set of signal groups.

To signal groups

$$D_1, D_2, \dots, D_m$$

the following variables are assigned:

$$x_1, x_2, \dots, x_m$$

where m is the total number of signal groups.

A variable x_i takes value 1, if a control variable is associated to signal group D_i, and 0 otherwise.

Now, the mathematical formulation of the optimization problem is

$$\text{minimize} \qquad z = \sum_{i=1}^{m} x_i$$

$$\text{subject to} \qquad Ax = \mathbf{1}, \quad x_i \in \{0,1\}$$

where $\mathbf{1}$ is the column-vector of 1's, and

where the coefficient matrix $A = [\, a_{ij} \,]_{n \times m}$ is such that:

$$a_{ij} = \begin{cases} 1, & \text{if traffic stream } s_i \text{ belongs to signal group } D_j \\ 0, & \text{otherwise} \end{cases}$$

where n is the number of traffic streams, and m is the number of signal groups.

The set of feasible solution of the problem is the collection of all complete sets of signal groups $\mathcal{D}$. The total number of complete sets of signal groups grows too fast with the increasing of traffic streams number, as shown in the Table 1. This table presents the number of signal groups for intersections with different number of traffic streams (from 5 to 11 traffic streams). The data are obtained from COMP program.

Table 1. Complete sets number depending on signal groups number

	NUMBER OF TRAFFIC STREAMS					
	5	6	7	8	9	11
NUMBER OF SIGNAL GROUPS	10	12	14	13	30	48
NUMBER OF COMPLETE SETS OF SIG. GROUPS	10	10	20	16	235	2490

The computer program

The problem of finding the minimal complete set of signal groups is solved in the scope of determining all complete sets of signal groups (by COMP program) because it makes a special case of complete sets. However, the run-time of COMP program rapidly increases with the number of traffic streams: for nine traffic streams (thirty signal groups), it is twenty seconds (on PC-AT 386 computer) and for eleven traffic streams (forty eight signal groups) the program needs twelve minutes of run-time to find 2490 complete sets of signal groups. This is one reason, among others, why the minimal complete set of signal groups problem is established and solved as a combinatorial optimization problem.

Two programs are developed for solving the optimization problem of determining the minimal complete set of signal groups: MINA [3], MINS [11], [9]. First one, MINA, finds all minimal complete sets (finds all optimal solutions), and the other, MINS, finds only one complete set (this program stops when one optimal solution is found).

The input into the program are as follows: the number of traffic streams and the collection of all signal groups.

The program MINA is written in PASCAL.

The program MINS is written in C.

6. THE PROGRAM RESULTS

All programs are realized on PC–AT 386 computer.

The results obtained by program COMP are presented in Table 2. $NCSSG_i$ represents the number of complete sets of signal groups containing i signal groups, while TCSSG represents the total number of complete sets of signal groups.

Table 2. Results obtained from program

NUMBER OF TRAF. STREAMS	5	6	7	8	9	11
NUMBER OF SIG. GROUPS	10	12	14	13	30	48
$NCSSG_{11}$	"	"	"	"	"	1
$NCSSG_{10}$	"	"	"	"	"	20
$NCSSG_9$	"	"	"	"	1	147
$NCSSG_8$	"	"	"	1	13	506
$NCSSG_7$	"	"	1	5	55	855
$NCSSG_6$	"	1	6	7	94	692
$NCSSG_5$	1	5	9	3	60	238
$NCSSG_4$	5	4	4	–	12	30
$NCSSG_3$	4	–	–	–	–	1
TCSSG	10	10	20	16	235	2490

Table 3. Times of program running

NUMBER OF TRAF. STREAMS	5	6	7	8	9	11	13	16
NUMBER OF SIG. GROUPS	10	12	14	13	30	48	72	198
MIN COMPLETE SET	3	4	4	5	4	3	5	3
RUN-TIME 1	t	t	t	2	20	700	MORE THAN 2000	
RUN-TIME 2	t	t	t	t	5	7	70	180
RUN-TIME 3	t	t	t	t	t	t	3	8

In Table 3. the solution of optimization problem are shown. Run–times are given in seconds, while 't' means that the run–time is less than one second. 'Run–time 1' is the time needed for COMP program execution. 'Run–time 2' is the time needed for MINA program execution (this program finds all optimum solutions). 'Run–time 3' is the time needed for MINS program execution (this program finds only one optimum solution).

7. THE RELATION OF PARTIAL ORDER OVER THE COLLECTION OF COMPLETE SETS OF SIGNAL GROUPS

The combinatorial nature of the complete sets determination problem can be seen from the rapid increase of the complete sets number with the increasing of traffic stream number. This can be seen from Table 2, showing that the considered problem is rather complex.

By using *the partial order relation* (defined over the collection of complete sets of signal groups) this problem can be simplified.

The partial order relation is defined in the following way:

$$R_p = \{ (d_r, d_q) \in \mathcal{D} \times \mathcal{D} \quad | \quad (\forall D' \in d_r)(\exists D'' \in d_q)\ D' \subseteq D'' \}$$

where:

$\mathcal{D}$ – collection of all complete sets of signal groups,

d_r, d_q – complete sets of signal groups,

D', D'' – signal groups.

Application of this relation for choosing the complete set of signal groups, is explained on an intersection example, shown in Figure 5.

In Figure 6 the graph of relation R_p is shown, defined over collection $\mathcal{D}$. In the circles, which represent graph nodes, the denotations of appropriate complete sets of signal groups are entered. Next to each node, the maximum volume values are indicated, too.

Each of these values is the solution of the optimization problem of finding signal plan while maximizing volume through the intersection presented at Figure 5. Difference between values is the consequence of different choice of the complete set of signal groups. The problems are solved in the occasions of congestion at the intersection (the flow passing one stop line during whole green interval is equal to the saturation flow) [5].

For the sake of exact statement of the optimization problem the signal plan x will be described by the ordered pair:

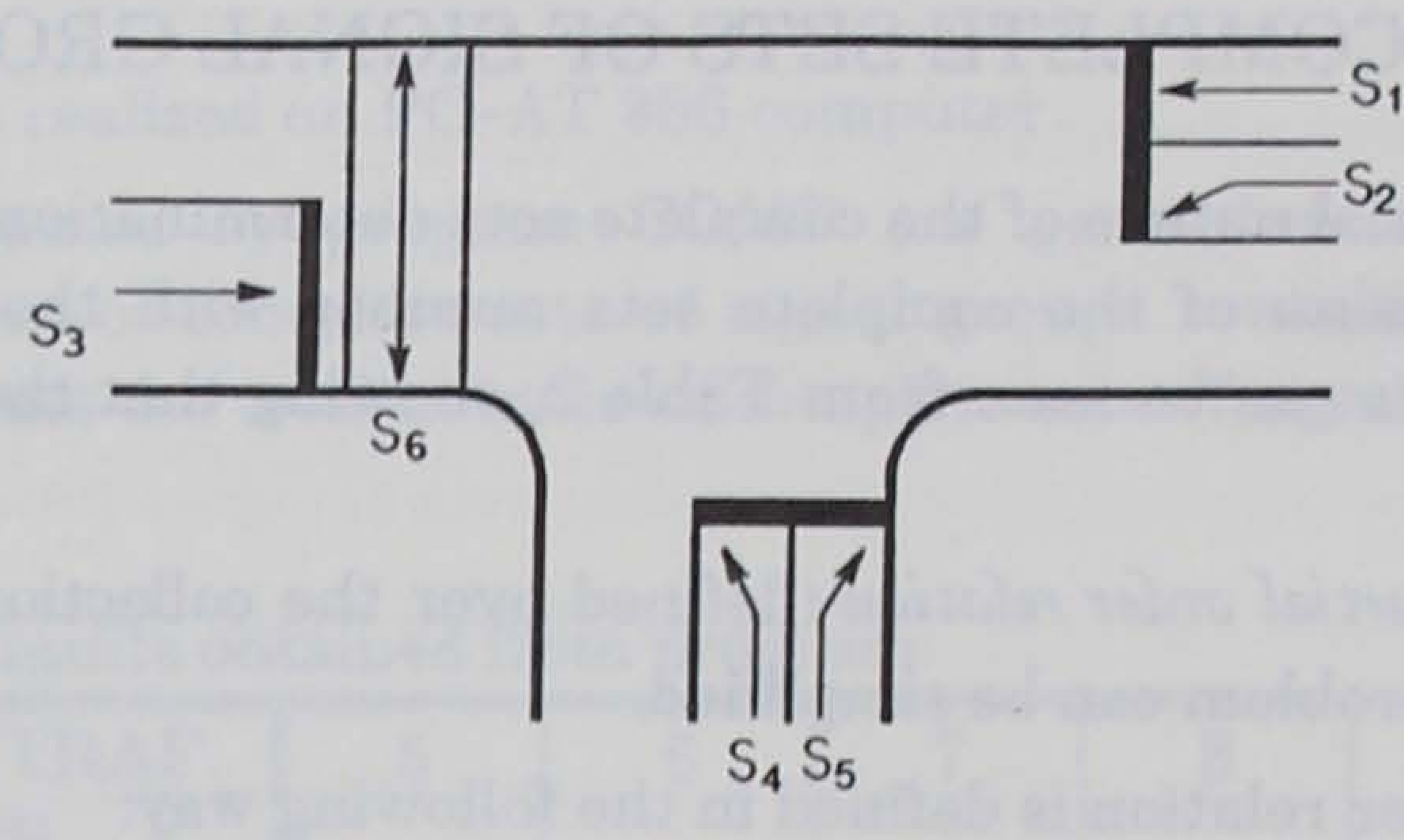
$$x = (\sigma, \tau)$$

where σ is signal plan structure, i.e.

$$\sigma = (u^1, u^2, \dots, u^k, \dots, u^K)$$

and τ is cycle split i.e.

$$\tau = (t^1, t^2, \dots, t^k, \dots, t^K)$$



$$S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$$

$D_1 = \{s_1\}$ $\quad D_7 = \{s_1, s_2\}$ $\quad D_{12} = \{s_1, s_2, s_5\}$

$D_2 = \{s_2\}$ $\quad D_8 = \{s_1, s_3\}$

$D_3 = \{s_3\}$ $\quad D_9 = \{s_1, s_5\}$

$D_4 = \{s_4\}$ $\quad D_{10} = \{s_2, s_5\}$

$D_5 = \{s_5\}$ $\quad D_{11} = \{s_4, s_5\}$

$D_6 = \{s_6\}$

$d_1 = \{D_1, D_2, D_3, D_4, D_5, D_6\}$ $\quad d_2 = \{D_3, D_4, D_5, D_6, D_7\}$ $\quad d_7 = \{D_3, D_4, D_6, D_{12}\}$

$d_3 = \{D_2, D_4, D_5, D_6, D_8\}$ $\quad d_8 = \{D_3, D_6, D_7, D_{11}\}$

$d_4 = \{D_2, D_3, D_4, D_6, D_9\}$ $\quad d_9 = \{D_4, D_6, D_8, D_{10}\}$

$d_5 = \{D_1, D_3, D_4, D_6, D_{10}\}$ $\quad d_{10} = \{D_2, D_6, D_8, D_{11}\}$

$d_6 = \{D_1, D_2, D_3, D_6, D_{11}\}$

Figure 5: An intersection with complete sets of signal groups

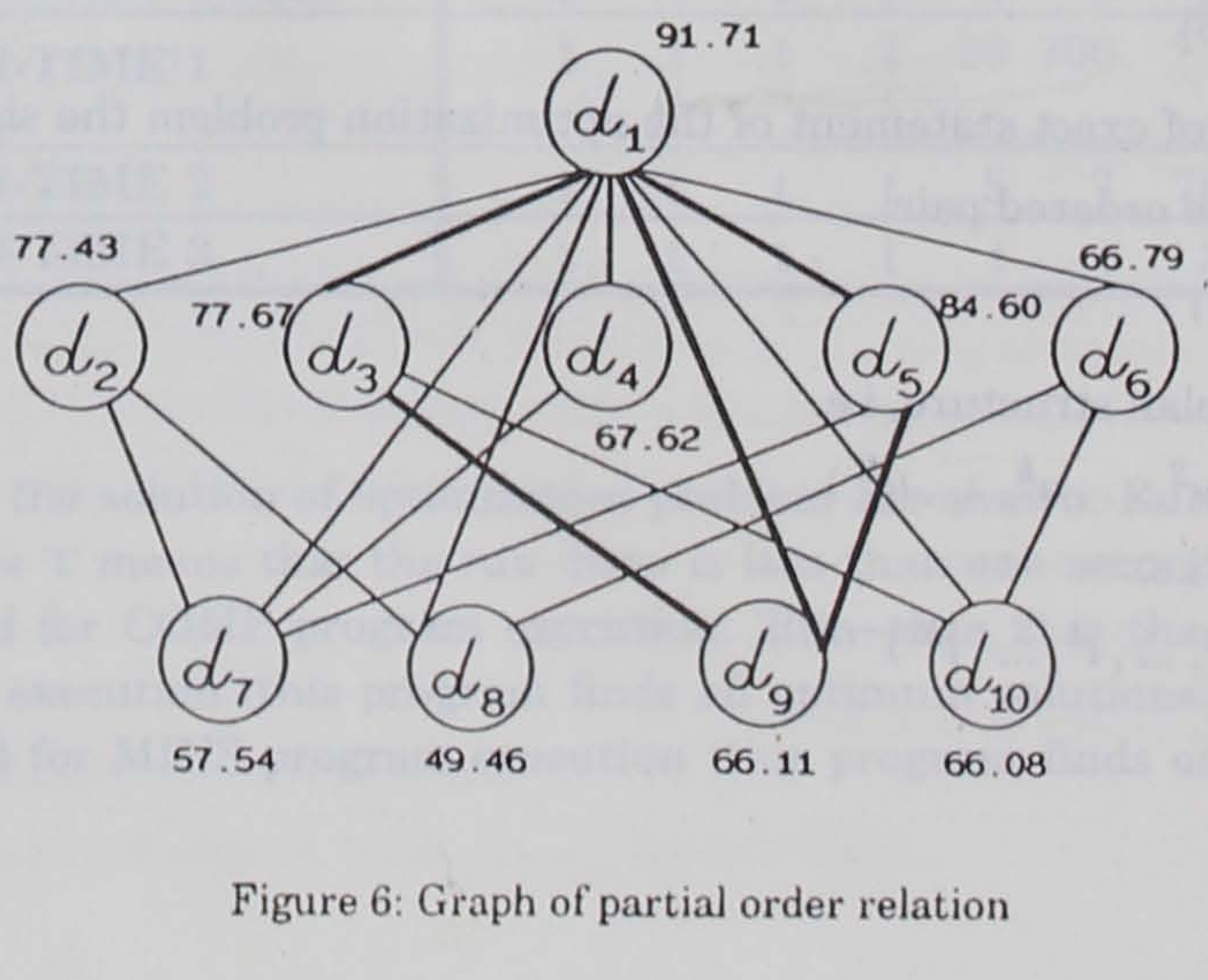


Figure 6: Graph of partial order relation

One component of signal plan structure is a control vector (vector of control variable values for some time interval in the cycle) that represents signal lights indications for each signal group in one time interval in the cycle:

$$u^k = \left(u_1^k, u_2^k, \dots, u_i^k, \dots, u_m^k \right), \qquad k \in \{1, 2, \dots, K\}$$

where m is total number of signal groups, and K is the number of control vectors applied during the cycle and

$$u_i^k = \begin{cases} 1, & \text{for effective green for signal group } D_i \\ & \text{in the } k\text{-th interval of cycle} \\ 0, & \text{for effective red for signal group } D_i \\ & \text{in the } k\text{-th interval of cycle} \end{cases}$$

The exact statement of the optimization problem for volume maximization is [5]:

$$\text{maximize} \quad \sum_{k=1}^{K} c^k t^k$$

subject to

(1) control transition constraints:

$$u^{k(\mathrm{mod}\, K)+1} \in \Gamma_s\, u^k, \quad \forall k \in T$$

(2) the constraints of one green interval during the cycle:

$$\sum_{k=1}^{K} \left[\left(u_i^k + u_i^{k(\mathrm{mod}\, K)+1} \right) (\mathrm{mod}\, 2) \right] = 2, \quad \forall i \in V$$

(3) minimum green time constraints:

$$\sum_{k=1}^{K} u_i^k\, t^k \ge g_i^m, \quad \forall i \in V$$

(4) minimum intergreen time constraints;

$$\begin{aligned} &\sum_{q=0}^{p-1} t^{K-(K+q-k)(\mathrm{mod}\, K)} \ge \\ &\ge \max\Big\{ z_{ij} \;\Big|\; \left[u_i^{K-(K+p-k)(\mathrm{mod}\, K)} \cdot u_j^{k(\mathrm{mod}\, K)+1} \right] \wedge \\ &\qquad \wedge \left[u_i^{K-(K+p-k-1)(\mathrm{mod}\, K)} = u_j^k = 0 \right] \wedge \\ &\qquad \wedge \left[u^{k(\mathrm{mod}\, K)+1} \notin \Gamma_s \left(u^{K-(K+p-k)(\mathrm{mod}\, K)} \right) \right], \\ &\qquad \forall i, j \in V \Big\} \\ &\forall k \in T, \quad \forall p \in \mathcal{D}_V \left(u^k, u^{k(\mathrm{mod} K)+1} \right) \end{aligned}$$

(5) negative intergreen time constraints:

$$\sum_{k=1}^{K} \left(u_i^k\, u_j^k \right) t^k \le \left| z_{ij} \right|, \quad \forall z_{ij} < 0, \quad \forall i, j \in V$$

(6) cycle duration constraint:

$$\sum_{k=1}^{K} t^k = C$$

where K – the number of control vectors applied during the cycle

$$c_k = \sum_{i=1}^{I} s_i\left(u^k\right) u_i^k$$

I – the number of traffic streams in signal group D_i

$s_i(u^k)$ – is the sum of the saturation flows of the traffic streams belonging to the signal group D_i

$T = \{ 1, 2, \ldots ,K \}$

V – the set of signal group indices

g_i^m – the minimal green time for signal group D_i

z_{ij} – the minimal intergreen time between traffic streams s_i and s_j

C – cycle

$$\mathcal{D}_v\left(u^k, u^{k(\mathrm{mod}\,K)+1}\right) = \left\{ 1, 2, \ldots, P_v\left(u^k, u^{k(\mathrm{mod}\,K)+1}\right)\right\}$$

$$P_v\left(u^k, u^{k(\mathrm{mod}\,K)+1}\right) = \min\left\{ \max\left\{ z_{ij} \mid \left[u_j^{k(\mathrm{mod}\,K)+1} = 1 \right] \wedge \left[u_j^k = 0\right]; i, j \in V\right\}, K-2 \right\}$$

The constraints (1) are defined by the *control transition graph*:

$G_s = (U_f, \Gamma_s)$

where U_f is the set of feasible control vectors and Γ_s is mapping

$$\Gamma_s:\ U_f \rightarrow \wp(U_f)$$

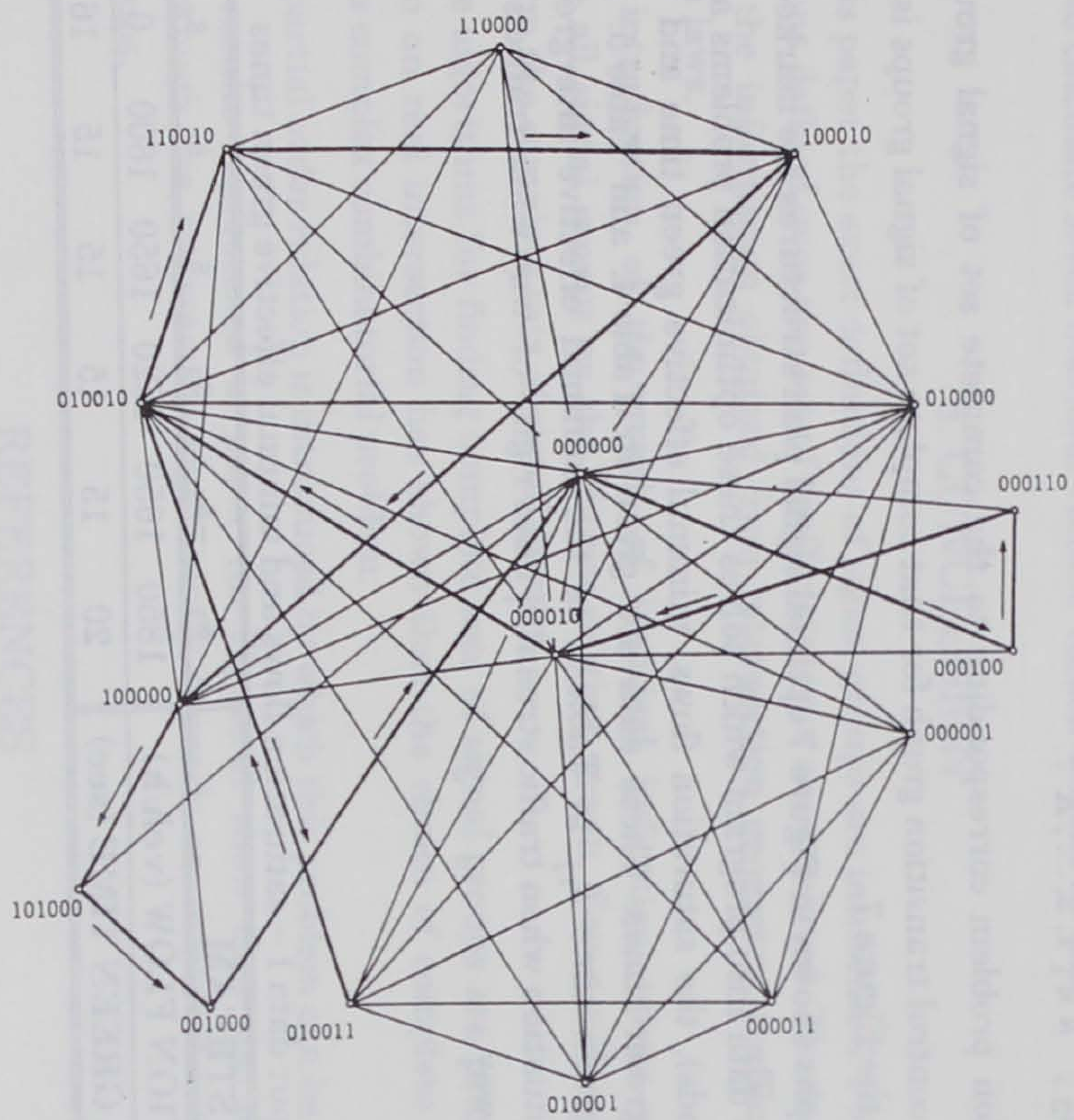
This mapping defines the correspondence between each control vector u^k and the set of all control vectors u^{k+1} which could be the next in the signal plan. That implies that the problem of signal plan optimization is the problem of finding the optimal closed path of the control transition graph.

The set of solutions of such formulated optimization problem makes possibility to analyze the influence of choice of different complete sets of signal groups to the value of the chosen performance index. This is the consequence of the fact that change from one complete set of signal groups to the another is equivalent to adding or deleting the constraints implying the control of two or more traffic streams by the same indications.

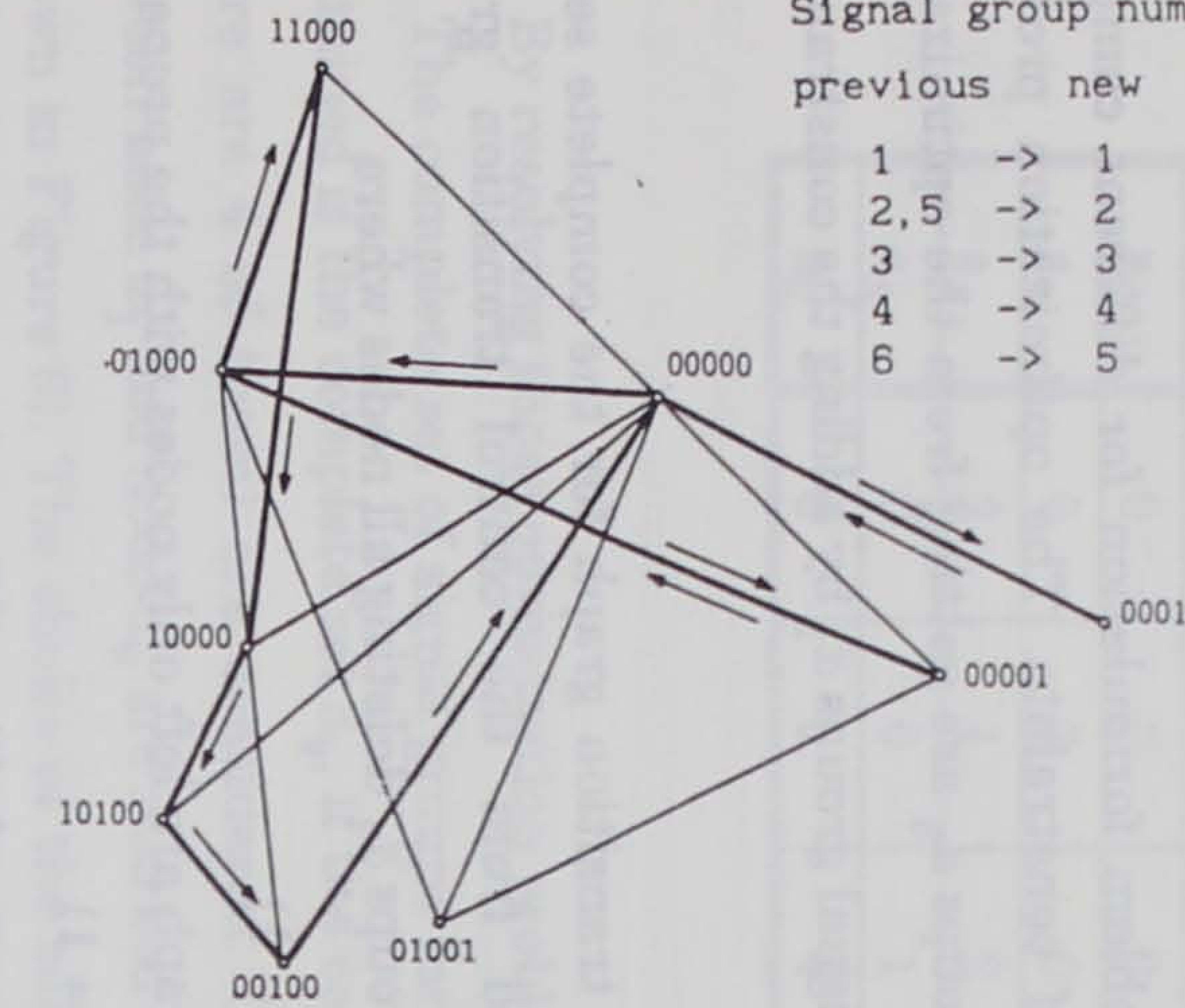
On the Figure 7 are presented control transition graphs for 3 complete sets of signal groups of the intersection at Figure 5. The graph (a) corresponds to the

Signal group numbering

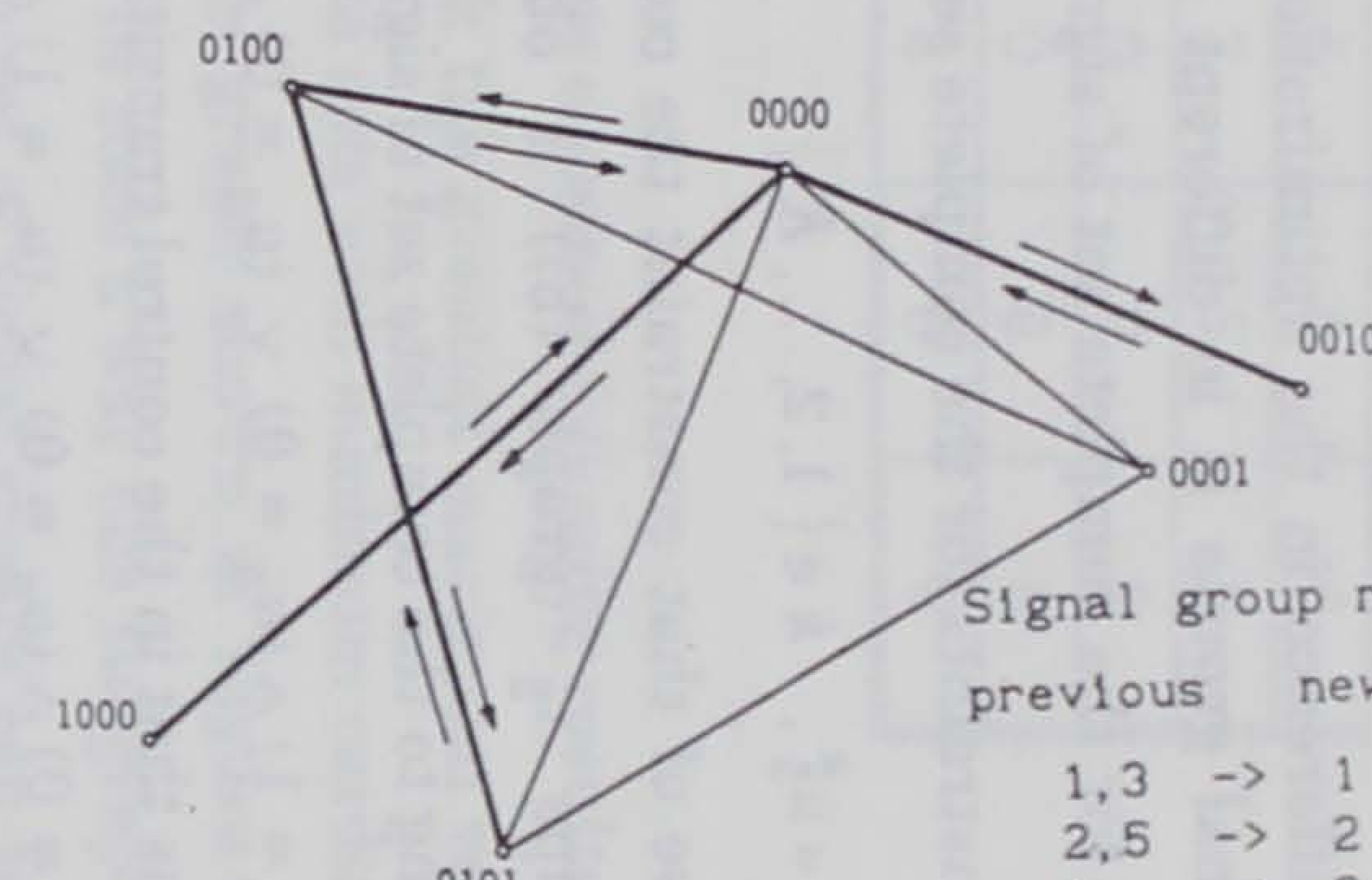
previous		new
1	->	1
2,5	->	2
3	->	3
4	->	4
6	->	5



(a) Complete set of signal groups d_1



(b) Complete set of signal groups d_5



(c) Complete set of signal groups d_9

Signal group numbering

previous		new
1,3	->	1
2,5	->	2
4	->	3
6	->	4

Figure 7: Control transition graph

complete set of signal groups d_1, when each traffic stream is one signal group. The graph (b) corresponds to the complete set of signal groups d_5. This graph is subgraph of the graph (a).

The difference in the optimization problem formulation for different complete sets of signal groups is in different set of constraints. The optimization problem constraints for the complete set of signal groups d_5 are getting from the optimization problem constraints for the complete set of signal groups d_1 by adding the constraint:

$$u_2^k = u_5^k, \quad k \in \{1, 2, \dots, K\}$$

Because of that constraint the control transition graph for the complete set of signal groups d_5 (graph (b)) is obtained from the control transition graph corresponding to the complete set of signal groups d_1 deleting all nodes where

$$(u_2 = 1) \wedge (u_5 = 0) \vee (u_2 = 0) \wedge (u_5 = 1)$$

It means that in the control transition graph are left only nodes with the property

$$(u_2 = 0) \wedge (u_5 = 0) \vee (u_2 = 1) \wedge (u_5 = 1)$$

When, now, this new set of constraints is extended by adding the constraint

$$u_1^k = u_3^k, \quad k \in \{1, 2, \dots, K\}$$

the optimization problem corresponding to the complete set of signal groups d_9 is obtained. The control transition graph for that complete set of signal groups is given as the case (c) on the Figure 7.

On all graphs shown in Figure 7 optimal signal plan structures are marked bold.

The input data into program which solves these optimization problems are: cycle time (90 seconds), the saturation flows, minimal effective green time and minimal effective intergreen times. These data are given in Table 4. and Table 5. Minimal effective intergreen time θ_{ij} in Table 5. is the minimal effective intergreen time necessary for situation when traffic stream s_i loses right-of-way while traffic stream s_j gains right-of-way.

Table 4. Input data I - saturation flows and minimal effective green times

TRAFFIC STREAM	s_1	s_2	s_3	s_4	s_5	s_6
SATURATION FLOW (veh/h)	1850	1650	1620	1650	1600	0
MINIMAL GREEN TIME (sec)	20	15	15	15	15	16

Table 5. Input data II – minimal effective intergreen times z_{ij}

	s_1	s_2	s_3	s_4	s_5	s_6
s_1	0	0	0	4	0	4
s_2	0	0	3	5	0	0
s_3	0	3	0	3	5	2
s_4	2	1	2	0	0	2
s_5	0	0	1	0	0	0
s_6	8	0	4	8	0	0

By resolving those optimization problems we obtained following results:

The complete set of signal groups for which the greatest maximum volume value is attained is the complete set d_9, if the complete sets with minimum cardinal number (there are 4 of them) are considered. On the graph shown in Figure 6 all paths between node d_1 and node d_9 are marked bold. In such a way one subgraph is formed (shown in Figure 6). The choice of the complete set of signal groups can be narrowed to the nodes of this subgraph only. In the example, since the relation R_p is applied, it is sufficient to calculate seven maximal volume values instead of ten.

8. CONCLUSION

In this paper the exact definitions of signal groups are introduced. For that reason the relations defined over the set of traffic streams are analyzed. These relations are based on the information obtained from intersection geometry, traffic process and regulation laws.

The introduction of the graph of identical indications gives possibility to determine all signal groups as the cliques of that graph. Some collections of those cliques form the complete sets of signal groups.

Three algorithms for finding complete sets of signal groups are presented. Their application on real intersection has shown that the choice of complete set of signal groups is a complex combinatorial problem.

The partial order relation is introduced by which this problem can be simplified.

The results represented in paper (in Figure 5) show that the maximal flow volume through the intersection significantly depends on choice of complete set of signal groups.

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