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A LINEAR ALGORITHM FOR CONSTRUCTION OF OPTIMAL DIGITAL CONVEX 2K–GONS

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Abstract: This paper gives a linear algorithm (w.r.t. the number of vertices) for a construction of optimal digital convex $2k$–gons, that is, those digital convex polygons, which have the smallest possible diameter with a given even number of edges. The construction for k even is based on the efficient construction of Farey sequence, while the construction for k odd uses, in addition, two families of auxiliary 6–gons.

Keywords: Digital geometry, optimization, time complexity.

AMS Subject Classification. 68C25, 10A30.

1. INTRODUCTION

Recently there have been a lot of papers which deal with some optimization problems on digital shapes. One of such problems will be studied here. It is related to convexity, one of the basic computational geometry properties ([11]).

A *digital convex polygon* (shortly: *d.c. polygon*) is a polygon, all the vertices of which are points on the integer grid and all the interior angles of which are strictly less than π radians. The *diameter* of a digital convex polygon is the minimal edge size of the enscribed digital square with the edges parallel to the coordinate axes.

This paper gives a linear algorithm for the construction of optimal d.c. $2k$–gons ($k \in N$), in the sense that these polygons have the smallest possible diameter with respect to the given even number of vertices.

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A similar problem : What is the minimal possible *area* of a d.c. polygon with the given even number of vertices - has been studied in [13]. A method for the construction of such a d.c. polygon has been given in the same paper.

Some of the other optimization problems related to convexity on the integer grid have been considered in a number of recent papers (see, for example, [4], [5], [6], [12]).

Motivation for considering such questions comes from several sources, in particular from integer programming and computer graphics.

The relationship between the number of edges and the diameter of optimal d.c. polygons was studied in papers [14], [2], [3]. In particular, an effective lower bound for the diameter of a d.c. polygon has been introduced in the paper [3]. That lower bound is the key concept for the construction of optimal d.c. polygons and it is called "greedy lower bound", since it is derived by a variant of *greedy* approach, known from theory of matroids and greedoids ([15], [9]).

Time complexity of the algorithm proposed here is linear w.r.t. the number of vertices of the $2k$-gon, which is being constructed. Linearity is reached by using an efficient construction ([1]) of the Farey sequence as an auxiliary tool, as well as the observations from Section 3.

The input for the algorithm is an even number $2k$ of vertices. If the number k is even, then the algorithm for construction of optimal d.c. $2k$-gons is a generalization of the algorithm proposed in [2] for the construction of those optimal d.c. polygons, which are members of a special sequence $P(t)$, $t = 1, 2, \ldots$. Two families of auxiliary 6-gons ([3]) are used in the algorithm proposed here in order to cover the cases when the number k is odd. The algorithm also incorporates an efficient determination of the parameter of the Farey sequence.

2. PRELIMINARIES

The *diameter* of a d.c. polygon Q is equal to

$$\max\left\{\max\{|x_i - x_j|, |y_i - y_j|\}, \text{ where } \left((x_i, y_i), (x_j, y_j)\right) \text{ is a pair of vertices of } Q\right\}.$$

Note that the diameter is taken in the sense of the maximum distance.

Let y_{min} and x_{min} respectively denote the minimal y-coordinate and the maximal x-coordinate of the considered d.c. polygon Q. Generally, the *SE-arc* (*south-east* arc) of Q is the sequence of consecutive edges (V_i, V_{i+1}), $1 \le i \le k-1$, where:

- V_i denotes a vertex (x_i, y_i) of Q
- $x_1 < \ldots < x_k = x_{max}$; $y_{min} = y_1 < \ldots < y_k$;

In particular, if the polygon Q has a *lower horizontal edge* (V_0, V_1) ($V_0 = (x_0, y_1)$, $V_1 = (x_1 y_1)$, $x_0 < x_1$), then this edge is additionally considered to be the first edge of the SE–arc. The NE–arc, the NW–arc and the SW–arc of a d.c. polygon are defined in the analogous way [1].

Given an edge $e = [\,(x_1, y_1), (x_2, y_2)\,]$ of a d.c. polygon, the *edge slope* of e denotes the fraction:

$$\frac{|x_1 - x_2|}{|y_1 - y_2|} \quad \text{if } e \in \text{NE - or SW - arc}; \qquad \frac{|y_1 - y_2|}{|x_1 - x_2|} \quad \text{if } e \in \text{SE - or } NW \text{ - arc},$$

while *bd–length* of the edge e denotes the sum $|x_1 - x_2| + |y_1 - y_2|$ (length in the sense of the block distance metrics).

A *digital square* $DS(p, q)$, where p and q are relatively prime natural numbers, is a d.c. 4–gon with the property that each arc has exactly one edge with the edge–slope q/p.

If the corresponding arcs of the two d.c. polygons Q_1 and Q_2 have not common edge slopes, then there exists a uniquely determined third d.c. polygon Q_3, called the *sum* (*Minkowski sum*) of Q_1 and Q_2 (for more details see [10] or [7]). Each arc of the polygon Q_3 includes all the edges of the corresponding arcs of Q_1 and Q_2, sorted so that the convexity condition is preserved. If Q_3 is the sum of Q_1 and Q_2, then Q_2 is the difference of Q_3 and Q_1. The diameter of Q_3 is equal to the sum of the diameters of Q_1 and Q_2.

Farey sequence of order t, (shortly $F(t)$, [8]), is a strictly increasing sequence of fractions, which includes all the fractions of the form b/a, where the integers a and b are relatively prime and $1 \le b < a \le t$. It is convenient for our purposes to add the fraction $0/1$ to be the first member of $F(t)$, for each t. Thus $F(5)$ looks as follows:

$$0/1,\ 1/5,\ 1/4,\ 1/3,\ 2/5,\ 1/2,\ 3/5,\ 2/3,\ 3/4,\ 4/5.$$

3. A CONSTRUCTION OF OPTIMAL DIGITAL CONVEX 2K–GONS

The diameter of a d.c. polygon Q cannot be smaller than one fourth of the perimeter of a minimal rectangle $MR(Q)$ with the edges parallel to the coordinate axes, in which the polygon Q can be included. On the other hand, this perimeter is equal to the sum S of all $2k$ summands of the form $p + q$, where q/p is the edge slope of an edge e of Q (Figure 1).

In order to minimize the diameter of Q, the summands $p + q$ of the sum S should be as small as possible. Such a choice of summands is naturally performed by the following "greedy" algorithm: choose as many summands equal to 1 as possible, then proceed with summands equal to 2 and so on. Note that each one of the edge slopes q/p may be used at most four times (once in each one of the four arcs), due to convexity of polygon Q. It is also obvious that the numbers p and q should be relatively prime with all the edge–slopes q/p of an optimal d.c. polygon.

[1]If the polygon Q has a right vertical edge, then it is considered to be the first edge of the NE–arc, and so on.

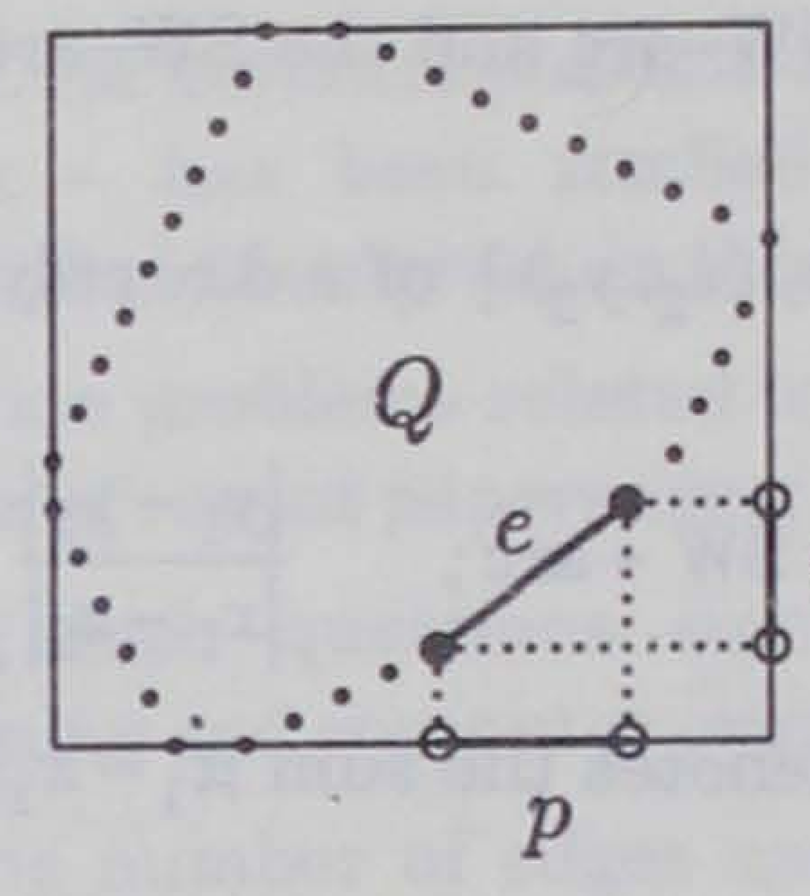

Figure 1. Orthogonal projection of edges of Q exactly cover the perimeter of the rectangle $MR(Q)$

When constructing optimal d.c. $2k$–gons, we can distinguish seven cases.

Case 1. k is even.

An optimal d.c. $2k$–gon can be constructed so that it has *four equal arcs*, i.e., so that all its arcs have the same number of edges with the same corresponding edge slopes. The edge slopes q/p within an arc are chosen so that the numbers p and q are relatively prime and that the sums $p + q$ are as small as possible.

An outstanding subcase for k even is the family of optimal d.c. $2k$–gons denoted by $P(t)$, for $t = 1, 2, ...$ The edge slopes of each arc of $P(t)$ are all different fractions q/p with relatively prime q and p, which satisfy that $2 \le p + q \le t$. In addition, the edge slope of the first edge in each arc of $P(t)$ is equal to $0/1$.

If $n(t)$ and $m(t)$ respectively denote the number of edges and the diameter of the polygon $P(t)$, then it is easy to show that

$$n(t) = 4 \cdot \sum_{s=1}^{t} \phi(s), \quad m(t) = \sum_{k=1}^{t} k \cdot \phi(k),$$

where $\phi(s)$ denotes the number of integers between 0 and s which are relatively prime with s (the Euler function from number theory; e.g. $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = 2$, $\phi(5) = 4$).

Note that $P(t)$ is the unique d.c. $n(t)$–gon, which has the diameter less or equal to $m(t)$. Namely, the only way to construct another d.c. $n(t)$–gon is to replace an edge with edge slope not greater than t by an edge with edge slope greater than t. Such a replacement necessarily increases the perimeter of $P(t)$ by at least 1, which implies that the diameter is also augmented by at least $\lceil 1/4 \rceil = 1$.

In the remaining part of this section, let t denote the natural number such that $n(t-1) < 2k < n(t)$.

If k is even, than an optimal d.c. $2k$–gon *with four equal arcs* is in fact equal to the sum of the polygon $P(t-1)$ and some arbitrarily chosen $(2k - n(t-1))/4$ digital squares of the form $DS(p, q)$, where $p + q = t$.

Case 2. k is odd, $t = 2u + 1$ for some $u \in N$.

An optimal d.c. $2k$–gon P can be obtained from an optimal d.c. $(2k - 2)$–gon Q with four equal arcs, which is constructed as in Case 1., with the additional requirement that the edge slope $u/(u+1)$ is not used within Q (equivalently, the digital square $DS(u+1, u)$ is not used as a summand of Q).

Namely, the minimal perimeter of a rectangle including some d.c. $2k$-gon is equal to (perimeter of $Q + 2 \cdot t$). This implies that the diameter of P cannot be smaller than the sum of diameter of Q and the summand

$$\left\lceil \frac{2 \cdot t}{4} \right\rceil = \left\lceil u + \frac{1}{2} \right\rceil = u + 1.$$

It follows that the addition of two edge slopes $u/(u+1)$, which are inserted into two opposite arcs of Q, is an optimal choice. Namely, such an addition augments the diameter of Q by exactly $u+1$, which is the minimal possible increase.

Case 3. k is odd, $2k = n(t-1) + 2$, $t = 2u$ for some $u \in N \setminus \{1\}$.

The greedy argument gives that the diameter of a d.c. $(n(t-1)+2)$-gon cannot be smaller than

$$\left\lceil \frac{1}{4} \cdot (4 \cdot m(t-1) + 2 \cdot t) \right\rceil = \lceil m(t-1) + u \rceil.$$

We claim that this lower bound cannot be reached. Otherwise all the possible edges with bd-length not greater than $t-1$ must be used, together with two edges of bd-length $2u$. If $q + p = 2u$, then $\max\{p, q\} \geq u + 1$ (Figure 1); the edge slope u/u cannot be used for $u > 1$, since the edge slope $1/1$ has been already used. This implies that the addition of *two* edges with bd-length t to the polygon $P(t-1)$ augments its diameter $m(t-1)$ at least by $u + 1$ [2].

On the other hand, the insertion of two edges with edge slope $u/(u+1)$ into two opposite arcs of $P(t-1)$ produces a d.c. $2k$-gon with the diameter $m(t-1) + u + 1$.

Case 4. k is odd, $2k = n(t) - 2$, $t = 2u$ for some $u \in N \setminus \{1\}$.

This case is analogous to Case 3; we shall mention here only the differences between the two cases:

The lower bound for diameter is equal to

$$\left\lceil \frac{1}{4} \cdot (4 \cdot m(t) - 2 \cdot t) \right\rceil = \lceil m(t) - u \rceil.$$

The polygon $P(t-1)$ should be replaced by an optimal d.c. $(n(t) - 4)$-gon with four equal arcs.

Two families of auxiliary d.c. 6-gons $A_1(w)$ and $A_2(w)$, $w = 2, 3, \ldots$, are used for Cases 5 through 7: (Figure 2).

Case 5. k is odd, $2k \in [n(t-1) + 6, n(t) - 6]$, $t = 4w + 2$ for some $w \in N$.

An optimal d.c. $2k$-gon can be represented as the sum of an optimal d.c. $(2k-6)$-gon Q, constructed as in Case 1. and the d.c. 6-gon $A_1(w)$, where Q has not

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[2] On the contrary, observe that the addition of *four* such edges (with the same edge slope) can always increase the diameter by exactly $2u$.

common edge slopes with $A_1(w)$. Namely, the diameter of a d.c. $2k$–gon cannot be smaller than

$$\left\lceil \frac{1}{4}\cdot\left(4\cdot m(t-1)+\left(2k-6-n(t-1)\right)\cdot t\right)+6\cdot t\right\rceil = \text{diameter of } Q + (6w+3)\ .$$

The second summand is equal to the diameter of $A_1(w)$.

Note that the $(2k–6)$–gon Q cannot have more than $n(t) - 12$ edges, since it cannot use the edges of those *three* digital squares $DS(p, q)$, such that the edge slopes q/p are used within $A_1(w)$. This construction therefore cannot be applied for $2k = n(t) - 2$.

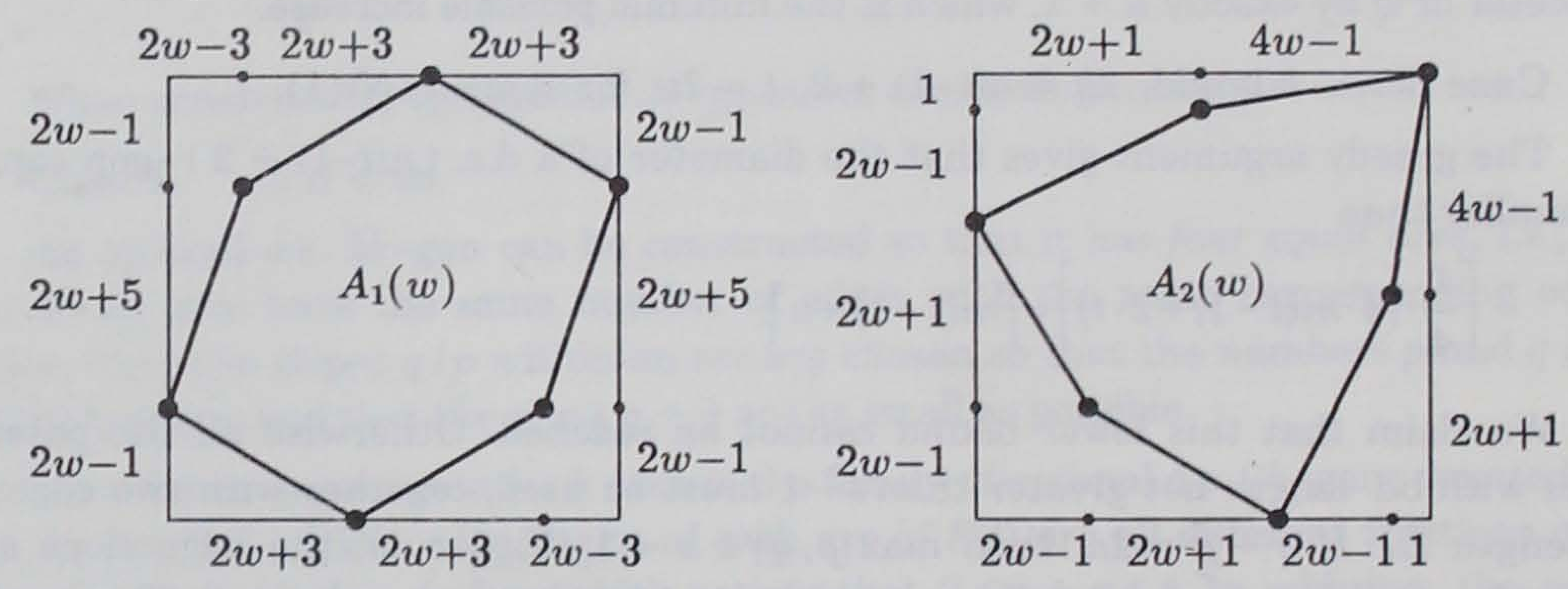

Figure 2. The families of auxiliary 6–gons

Case 6. k is odd, $2k\in[n(t-1)+6, n(t)-10]$, $t = 4w$ for some $w\in N$.

This case can be solved analogously to the Case 5., by replacing the 6–gon $A_1(w)$ with the 6–gon $A_2(w)$, the diameter of which is equal to $6w$. Note, however, that the 6–gon $A_2(w)$ uses *four* different edge–slopes, which implies that the $(2k - 6)$–gon Q cannot have more than $n(t) - 16$ edges. This requires a separate treatment of the last case:

Case 7. k is odd, $2k = n(t) - 6$, $t = 4w$ for some $w\in N$.

An optimal d.c. $(n(t) - 6)$–gon P can be represented as the difference of the polygon $P(t)$ and the 6–gon $A_2(w)$. The diameter of P is equal to $m(t) - 6w$. The optimality of this diameter follows from the optimality of $P(t)$ and from the fact that $MR(P(t))$ and $MR(A_2(w))$ are squares, which implies that $MR(P)$ is a square again.

4. ALGORITHM

The algorithm given in this section performs the construction proposed in Section 3.

Input: a natural number k.

Output: a digital convex $2k$–gon P with the minimal possible diameter.

The algorithm has the following three stages:

Stage 1. Evaluation of the natural number t such that the number $2k$ belongs to the half–open interval ($n(t-1), n(t)$]

Stage 2. Recognition of the case from Section 3. (among **Case 1.**, ..., **Case 7.**)

Stage 3. Generation of each one of the four arcs of the polygon P by one pass through the Farey sequence $F(t)$

The number t in Stage 1. can be determined [3] by summing up the summands of the form $4 \cdot \phi(s)$, for $s = 1, 2, ...$, until the sum (equal to $n(t)$) becomes greater than or equal to the number $2k$.

Recognition of the case in Stage 2. is easily completed by using the numbers $2k$, t, $n(t-1)$ and $n(t)$. The value of $n(t-1)$ is determined as $n(t) - 4 \cdot \phi(t)$.

Stage 3., which is the most interesting, will be described in more detail:

4.1. GENERATION OF THE ARCS OF THE OPTIMAL D.C. 2K–GON

The construction of the optimal d.c. polygon P is separated into four independent constructions of its arcs (SE–, NE–, NW– and SW–arc in turn). Each arc is constructed by using only one pass through the Farey sequence $F(t)$. Each member b / a of $F(t)$ is mapped to $q / p = b / (a-b)$, a candidate for the edge slope of an edge within the arc. Note that the used mapping is a bijection which preserves the ordering and that the integers b and a are relatively prime if and only if the integers b and $a - b$ are. For example, the sequence $F(5)$ is bijected to the sequence

$$0/1,\ 1/4,\ 1/3,\ 1/2,\ 2/3,\ 1/1,\ 3/2,\ 2/1,\ 3/1,\ 4/1,$$

which includes all the edge slopes q/p of the edges of an arc of the polygon $P(5)$ in increasing order.

The generation of consecutive vertices of an arc becomes in this way computationally equivalent to the generation of consecutive members of the Farey sequence, but the latter generation (in increasing order) is possible in linear time ([1]). Thus the sorting of vertices within an arc is avoided.

Depending on the case and on the current arc, let $S = S(arc, case)$ denote an auxiliary set of specific edge slopes listed in the corresponding field of Table 1. Note that the edge slopes in the last two columns of the table are exactly those which are present in the corresponding arcs of $A_1(w)$ and $A_2(w)$ respectively.

The sequence $F(t)$ is primarily initialized and the following scheme is used for the general step of the construction of an arc of P:

- Construct the following member b / a of the sequence $F(t)$ from the previous member b^- / a^- by using the connections ([8]):

[3] in accordance with the formula for $n(t)$ given above

Table 1.

Case	2., 3. or 4.	5.		6. or 7.	
SE-arc	$\frac{u}{u+1}$	$\frac{2w-1}{2w+3}$	$\frac{2w+5}{2w-3}$	$\frac{2w+1}{2w-1}$	$\frac{4w-1}{1}$
NE-arc		$\frac{2w+3}{2w-1}$			
NW-arc	$\frac{u}{u+1}$	$\frac{2w-1}{2w+3}$	$\frac{2w+5}{2w-3}$	$\frac{1}{4w-1}$	$\frac{2w-1}{2w+1}$
SW-arc		$\frac{2w+3}{2w-1}$		$\frac{2w-1}{2w+1}$	$\frac{2w+1}{2w-1}$

$b = x_0 + r \cdot b^-$ and $a = y_0 + r \cdot a^-$, where

(x_0, y_0) is an integral solution of the equation: $a^- \cdot x - b^- \cdot y = 1$ and $r = \lfloor (n - y_0) / a^- \rfloor$. As suggested in [1], one solution (x_0, y_0) can be obtained as (b^{--}, a^{--}), where b^{--} / a^{--} is the predecessor of b^- / a^- in $F(t)$.

- Determine the corresponding edge slope q / p by using the equalities $q = b$ and $p = a - b$.
- If the current case is **Case 7.**, then **goto** the fourth part of the scheme. In all the other cases, let q^- / p^- denote the edge slope corresponding to the (previous) member b^- / a^- of $F(t)$. If there exists an edge slope q^* / p^* in $S(arc, case)$ such that $q^- / p^- < q^* / p^* < q / p$, then **register** the edge corresponding to q^* / p^*.
- If the edge slope q / p is **acceptable**, then **register** the corresponding edge.

We proceed with a more detailed description of the Boolean function **acceptable** and the procedure **register**:

In **Case 7.**, the value of **acceptable** is TRUE whenever $q / p \notin S(arc,$**Case 7.**$)$.

In **Case 3.**, the value of **acceptable** is TRUE whenever $q + p \le t - 1$.

In **Cases 1., 2., 4., 5., 6.**, let c denote a counter, which is initialized by 0, and the value of which is increased by 1 whenever $q + p = t$ and $q / p \notin S(arc, case)$.

The value of **acceptable** is TRUE IFF one of the following two conditions holds with the edge slope q / p:

1) $q + p \le t - 1$

2) $(q + p = t)$ and $q / p \notin S$ and $c \le (2k - n(t-1) - j) / 4$, where:

$j=0$ with **Case 1.**

$j=2$ with **Cases 2., 4.**

$j=6$ with **Cases 5., 6.**

(thus merely the lexicographically first $(2k - n(t-1) - j) / 4$ digital squares $DS(p, q)$ with $q + p = t$ are used for the construction of P).

The coordinates (x_0, y_0) of the first vertex of the SE–arc of P are given in advance. Given a current vertex (x_i, y_i), an edge with the edge slope q/p is **register**–ed by producing the next vertex (x_{i+1}, y_{i+1}) in accordance with the connections

$$x_{i+1} = x_i + x_{dif} \quad \text{and} \quad y_{i+1} = y_i + y_{dif},$$

where the pair (x_{dif}, y_{dif}) is equal to $(+p, +q)$, $(-q, +p)$, $(-p, -q)$, $(+q, -p)$ within the SE–arc, NE–arc, NW–arc, SW–arc respectively.

We finish this section with some comments on small cases:

The smallest values of $2k$ to which **Cases 1., 2., ..., 7.** are applied are in order: 4, 10, 18, 22, 118, 78 and 82.

The value $2k = 6$ exceptionally satisfies all the conditions of **Cases 3.** and **4.**, except for $u > 1$. This implies that the minimal diameter is equal to $2 = m(1) + 1$; the whole perimeter is covered by orthogonal projections of edges (Figure 1), which is not satisfied with **Cases 3.** and **4.**.

The auxiliary 6–gons would not be well–defined for $w = 1$. Fortunately, they are not necessary with that value. Namely, the intervals $(n(3), n(4)) = (16, 24)$, (for $t = 4 \cdot 1$) and $(n(5), n(6)) = (40, 48)$ (for $t = 4 \cdot 1 + 2$) do not allow the use of auxiliary 6–gons; the values 18 and 42, respectively 22 and 46, are covered by **Cases 3.** and **4.**. On the other hand, if $w \geq 2$, then there always exists an edge with a smaller bd–length, which can be inserted between the two edges of auxiliary 6–gons belonging to the same arc; thus two edge slopes q^* / p^* in the third part of the general step of the construction never exist.

5. COMPLEXITY OF THE ALGORITHM

Theorem 1. *The algorithm given in Section 3. is linear with respect to the number of edges of the constructed optimal d.c. polygon.*

Proof. The following asymptotic estimation for the number $n(t)$ has been derived in [2]:

$$n(t) = \frac{12t^2}{\pi^2} + O(t \log t)$$

Since $n(t-1) < 2k \leq n(t)$, the number of edges of the constructed polygon P is of the same order of magnitude $(O(t^2))$. The presented construction of optimal d.c. polygon P is asymptotically optimal in the sense that the number of elementary steps of the construction is also of order $O(t^2)$. Such a conclusion can be derived by analyzing the stages of the algorithm:

Stage 1. The number of elementary steps for calculating $\phi(s)$ (using the factorization of s) is known to be bounded by $O(\sqrt{s})$. It follows that calculating $\phi(s)$ for $s = 1, 2, ..., t$, and consequently the calculating of t, $n(t)$ and $n(t-1)$ – requires $O(t\sqrt{t})$ elementary steps.

Stage 2. Distinguishing between the cases **0.**, **1.**, ..., **6.** can obviously be performed in constant time.

Stage 3. Given a member of the Farey sequence, the calculation of the next member is performed in a constant time [1]. On the other hand, the necessary calculations concerning q/p and related to each member of the Farey sequence can also be performed in a constant time; they may include merely the search of edges of a fixed auxiliary 6-gon.

The sequence $F(t)$ is passed four times during the generation of P. Thus the number of elementary steps used in Stage 3. is asymptotically equal to the 4-fold number of members of the sequence $F(t)$. The latter number has been estimated as $3t^2/\pi^2 + O(t \log t)$ ([8], Theorems 330. and 331.).

Since the complexity of Stages 1. and 2., is smaller than $O(t^2)$, it follows that the number of elementary steps of the whole algorithm is linear with respect to the number of edges of the constructed polygon.

6. AN EXAMPLE

Let be given $2k = 78$.

It can be derived in turn that $t = 8$, $n(t) = 88$, $n(t-1) = 72$. Case 6. is recognized and the auxiliary polygon $A_2(2)$ should be used with $c = 0$. Thus the optimal d.c 78-gon P can be represented as the sum of polygons $P(7)$ and $A_2(2)$ (Figure 3).

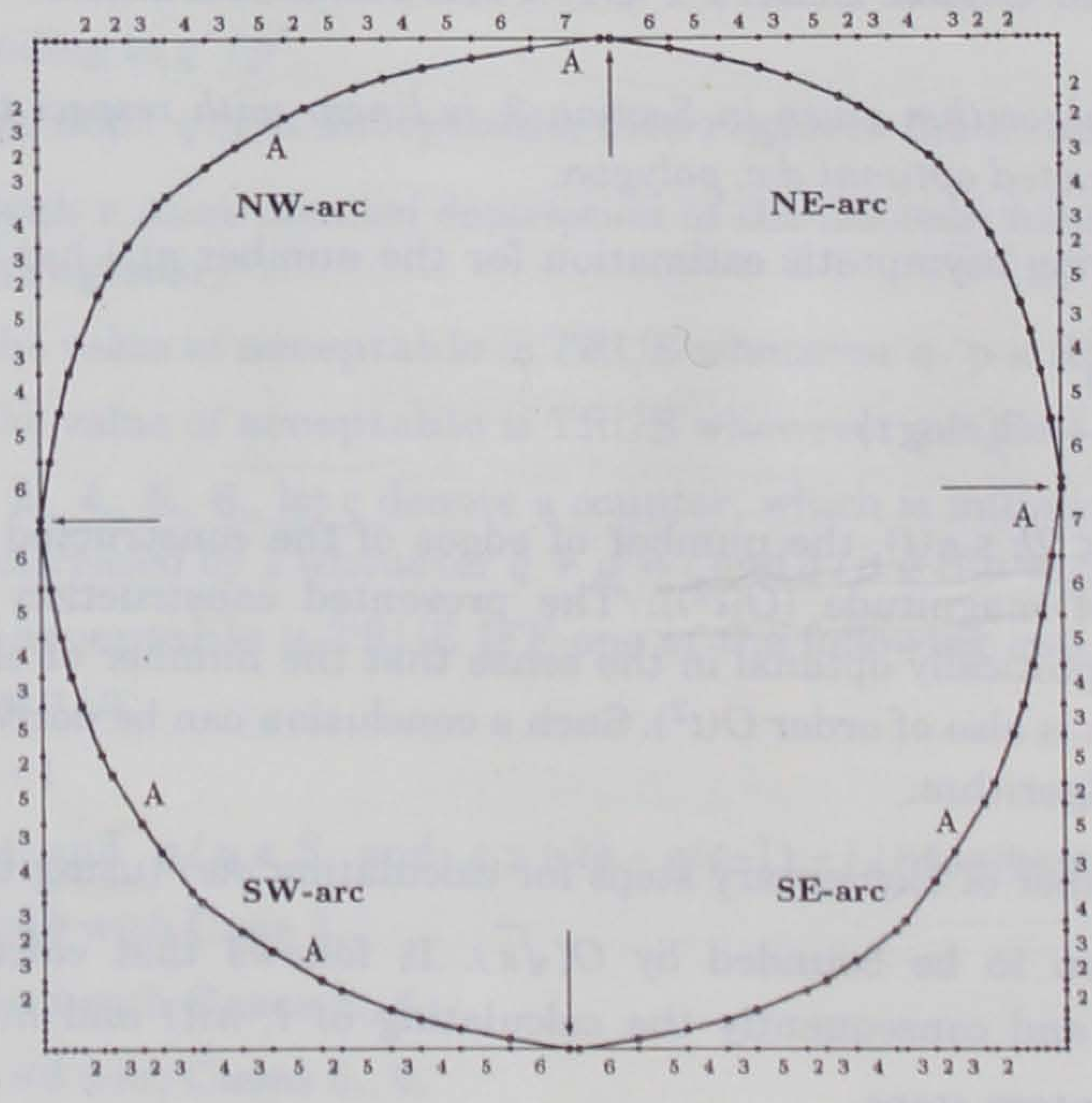

Figure 3. P(7) + $A_2(2)$, an optimal d.c. 78-gon

Those edges of P, the edge slopes of which belong to the auxiliary polygon $A_2(2)$, are marked by the letter "A". Those lengths of orthogonal projections of edges of P to the square $MR(P)$ (see also Figure 1), which are greater than 1, are written down. The common points of neighbouring arcs are pointed to by arrows.

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