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MINIMAX THEORY WITH TRANSVERSAL POINTS

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Abstract: This paper introduces and studies the notion of transversal points for a mapping with the domain in a partially ordered set. It generalizes the known results on saddle points and connects them with the results on fixed points and transversality.

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AMS (MOS) Subject Classification. 49, 47, 90.

1. INTRODUCTION

The problem of fixed point for a given mapping f of a partially ordered set P to itself is very easy to formulate: the question is if some $\zeta \in P$ verifies $f(\zeta) = \zeta$. It is interesting that many problems are reducible to the existence of fixed points of certain mappings. The question remains whether each statement could be equivalently expressed in the fixed point language as well. The answer is affirmative, the answers were given in [55].

Let X and Y be Hausdorff topological spaces and $S, T : X \to Y$ two set–valued transformations from X to Y. The *coincidence problem* for (S, T) is concerned with conditions which guarantee that the pair (S, T) has one or more coincidence points, that is points $(x_0, y_0) \in X \times X$ such that $S x_0 \cap T y_0$ is nonempty. Geometrical problems of this type in an approximate context turn out to be intimately related to some basic problems arising in convex analysis. This important fact was discovered by John von Neumann in 1937, who established a coincidence statement in R^n and made a direct use of it in the proof of his well–known Minimax Principle. cf [41]. In this sense, in paper [42] von Neumann investigated the concept of a saddle point for a mapping $f : A \times B \to \mathrm{R}$, where A and B are nonempty sets. A point $(x_0, y_0) \in A \times B$ is called a

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saddle point of $f : A \times B \to \mathrm{R}$ if

$$f(x_0, y) \le f(x_0, y_0) \le f(x, y_0) \qquad \text{for all } (x, y) \in A \times B.$$

This condition is equivalent with the following equality

$$\max_{x \in A} \min_{y \in B} f(x, y) = \min_{y \in B} \max_{x \in A} f(x, y),$$

i.e., with the following double equality

$$\max_{y \in B} f(x_0, y) = f(x_0, y_0) = \min_{x \in A} f(x, y_0).$$

Since then, geometrical problems of a similar kind (as well as their analytic counterparts) have attracted broad attention and remarkable progress has been made both in generalizing the original results as well as in finding new applications in a variety of mathematical areas, see [6] and [18].

In connection with the preceding, in this paper we consider a concept of transversal points, for the mapping f of a nonempty set X into a partially ordered set P. A map f of a nonempty set X into a partially ordered set P has a transversal point $\zeta \in P$ if there is a decreasing function $g : P^2 \to P$ such that the following equality holds

$$\max_{x, y \in X} \min \{ f(x), f(y), g(f(x), f(y)) \} =$$
$$\min_{x, y \in X} \max \{ f(x), f(y), g(f(x), f(y)) \} \ := \zeta. \tag{1}$$

Also, in this paper, we consider some other points of this type. Applications in nonlinear functional analysis, specially, in minmax theory and convex analysis are considered.

2. FUNDAMENTALS OF NEW MINIMAX THEORY

Let $(P, \le)$ be a partially ordered set by the ordering relation $\le$. The function $g : P^k \to P$ (k is a fixed positive integer) is *decreasing* on the ordered set P if $a_i, b_i \in P$ and $a_i \le b_i$ $(i = 1, \dots, k)$ implies $g(b_1, \dots, b_k) \le g(a_1, \dots, a_k)$.

Let L be a lattice and g a mapping from L^2 into L. For any $g : L^2 \to L$ it is natural to consider the following property of *local comparability*, which means, if $\zeta \in L$ is comparable with $g(\zeta, \zeta) \in L$ then ζ is comparable with every $t \in L$.

We begin with the following essential statements.

LEMMA 1. (Sup–Inf Inequalities). Let $(L, \le)$ be a lattice and let $g : L^2 \to L$ be a decreasing mapping. If L has property of local comparability, then for arbitrary functions $p : X \to L$ and $q : Y \to L$ (X and Y are arbitrary nonempty sets) the following relations are valid:

$$\zeta \le g(\zeta, \zeta) \qquad \text{implies} \quad \zeta \le \sup \{ p(x), q(y), g(p(x), q(y)) \}, \tag{2}$$

and

$$g(\zeta,\zeta)\le\zeta \qquad \text{implies} \quad \inf\{\, p(x), q(y), g(p(x), q(y))\,\}\le\zeta, \tag{3}$$

for all $x\in X$ and for all $y\in Y$. Hence, in particular, $\zeta = g(\zeta,\zeta)$ implies

$$\inf\{\, p(x), q(y), g(p(x), q(y))\,\}\le\zeta\le\sup\{\, p(x), q(y), g(p(x), q(y))\,\}, \tag{4}$$

for all $x\in X$ and for all $y\in Y$

From assertions (2), (3) and (4) we obtain the following interesting conclusions (which, incidentally are their equivalent formulations for $X = Y$):

$$\zeta\le g(\zeta,\zeta) \qquad \text{implies} \quad \zeta\le\inf_{x,y\in X}\sup\{\, p(x), q(y), g(p(x), q(y))\,\}, \tag{5}$$

and

$$g(\zeta,\zeta)\le\zeta \qquad \text{implies} \quad \sup_{x,y\in X}\inf\{\, p(x), q(y), g(p(x), q(y))\,\}\le\zeta, \tag{6}$$

and $g(\zeta,\zeta) = \zeta$ implies the following inequalities

$$\sup_{x,y\in X}\inf\{\, p(x), q(y), g(p(x), q(y))\,\}\le\zeta\le$$
$$\inf_{x,y\in X}\sup\{\, p(x), q(y), g(p(x), q(y))\,\}. \tag{7}$$

On the other hand, we note, that it is easy to construct a decreasing mapping on a complete lattice which is not a totally ordered set but the property of local comparability is fulfilled, see Figure 1.

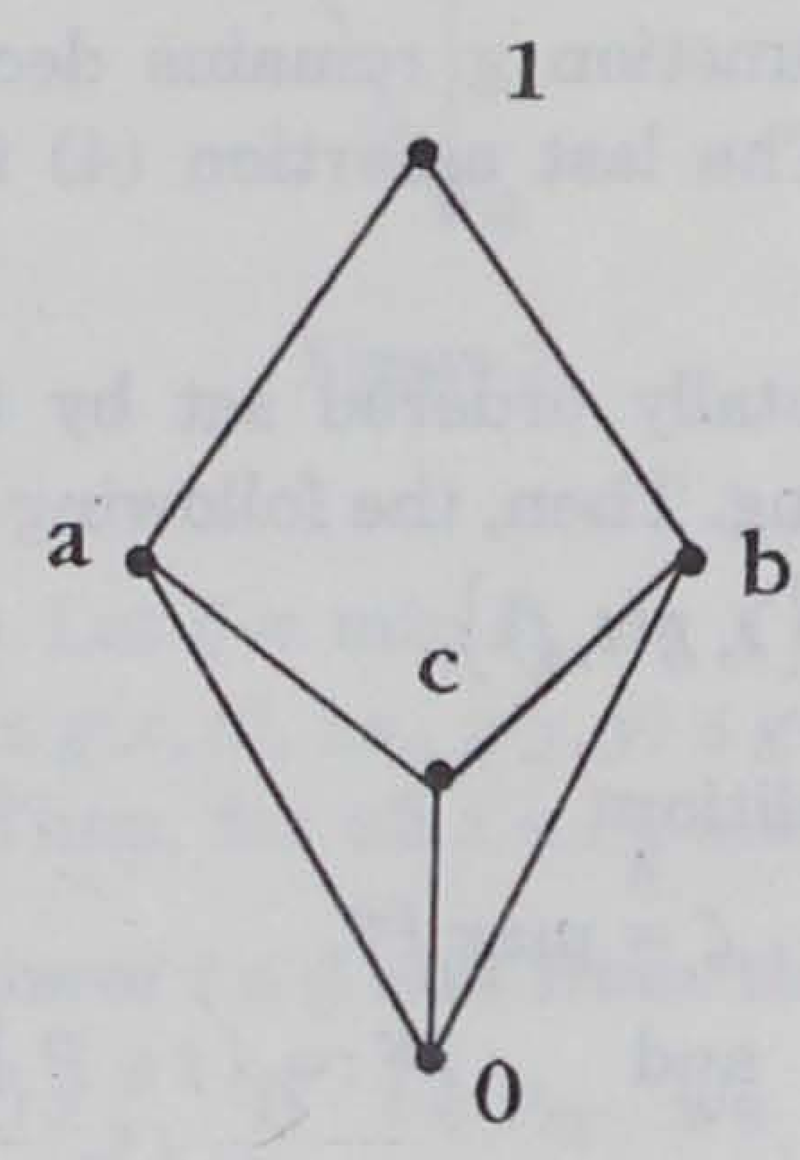

Figure 1.

Example 1. Let L be the lattice on Figure 1 and let $g : L\to L$ be defined by $g(0) = 1$, $g(a) = b$, $g(b) = a$, $g(c) = 0$, $g(1) = 0$. Evidently, g is decreasing and the property of local comparability is fulfilled, but the set L is not totally ordered.

REMARK. The above statements (Lemma 1) still hold when $g : L^k \to L$ (k is a fixed positive integer) is a decreasing function. The proof is quite similar; the assertions corresponding to (2) and (3) look as follows

$$\zeta \le g(\zeta, \dots, \zeta) \quad \text{implies} \quad \zeta \le \sup\{\lambda_1, \dots, \lambda_k, g(\lambda_1, \dots, \lambda_k)\}, \text{and} \tag{8}$$

$$g(\zeta, \dots, \zeta) \le \zeta \quad \text{implies} \quad \inf\{\lambda_1, \dots, \lambda_k, g(\lambda_1, \dots, \lambda_k)\} \le \zeta \tag{9}$$

for arbitrary functions $\lambda_1, \dots, \lambda_k : X \to L$, where X is an arbitrary nonempty set. Also, in particular, $\zeta = g(\zeta, \dots, \zeta)$ implies

$$\inf\{\lambda_1, \dots, \lambda_k, g(\lambda_1, \dots, \lambda_k)\} \le \zeta \le \sup\{\lambda_1, \dots, \lambda_k, g(\lambda_1, \dots, \lambda_k)\}, \tag{10}$$

for arbitrary functions $\lambda_i : X \to L$ ($i = 1, \dots, k$), where X is an arbitrary nonempty set. To simplify the notation we will give the proof only for the case $k = 2$.

PROOF OF LEMMA 1. Implication (2). Let $\zeta \le g(\zeta, \zeta)$ and $\lambda = \sup\{p(x), q(y)\}$, where the elements $x \in X$ and $y \in Y$ are arbitrarily chosen. If $\zeta \le \lambda$, then

$$\zeta \le \sup\{p(x), q(y), g(p(x), q(y))\}, \text{ for all } x \in X \text{ and } y \in Y, \tag{11}$$

obviously holds. If $\lambda \le \zeta$, then $\zeta \le g(\zeta, \zeta) \le g(p(x), q(y))$ and (11) holds too. We see that the comparability of elements λ and ζ is possible as a consequence of the property of local comparability.

One gets the implication (3) by applying the above results to the case where the relation $\le$ is replaced by the relation $\ge$; in fact, after this change, every supremum becomes an infimum and the function g remains decreasing with respect to each argument. Thus, we have (3). The last assertion (4) is evident. Thus, the proof is complete.

LEMMA 2. ([55]). Let P be a totally ordered set by the order relation $\le$, and let $g : L^2 \to L$ be a decreasing mapping. Then, the following conditions are equivalent:

$$\min\{t, g(t,t)\} \le \zeta \le \max\{t, g(t,t)\}, \tag{12}$$

for all $t \in P$ and the following condition

$$\zeta = \min P_g \qquad \text{or} \qquad \zeta = \max P^g, \tag{13}$$

where $P_g := \{t \in P \mid g(t,t) \le t\}$ and $P^g := \{t \in P \mid t \le g(t,t)\}$.

From this assertion as a direct consequence it follows that:

- The number of points $\zeta \in P$ with characteristic (12) can be 0, 1 or 2.
- Each of these cases can be realized.
- If P is an everywhere dense set of points, the number of points with characteristic (12) is 0 or 1.
- If the set P has the characteristic of density (:= that is for every Dedekind' s cross section the lower class has the maximum or the upper class has a minimum) the number of points is 1 or 2.

- If $\zeta \in P$ is the fixed point of the mapping $g : P^2 \to P$, then ζ is the point with characteristic (12), and then (12) holds if and only if
$$\max_{x\in P} \min\{ x, g(x, x) \} = \min_{x\in P} \max\{ x, g(x, x) \} = \zeta.$$

REMARK. In Lemma 2 the assumption that $(P, \leq)$ is totally ordered cannot be replaced by the weaker assumption that $(P, \leq)$ is a lattice. More precisely, the implication $(12) \Rightarrow (13)$ holds true in the case of any poset, while the implication $(13) \Rightarrow (12)$ is in general false even for lattices. Indeed, from (12) it follows that each element $t \in P$ is comparable with $g(t, t)$ so that $\zeta \in P_g$ or $\zeta \in P^g$. in the first case $t \in P_g$, i.e., $g(t, t) \leq t$; so we have $\zeta \leq \max\{ t, g(t, t) \} = t$, and hence $\zeta = \min P_g$. A symmetric proof shows that $\zeta \in P^g$ implies $\zeta = \max P^g$. On the other hand, the structure on Figure 2 is obviously a lattice and the function $g : P \to P$ defined by $g(a) = c$, $g(b) = g(d) = b$, $g(c) = a$, where $P = \{ a, b, c, d \}$, is decreasing. In this case we have also $P_g = \{ b, c \}$, $P^g = \{ a, b \}$ and thus $b = \min P_g = \max P^g$, i.e., (13) holds. However (12) is false since d is not comparable with $b = g(d)$.

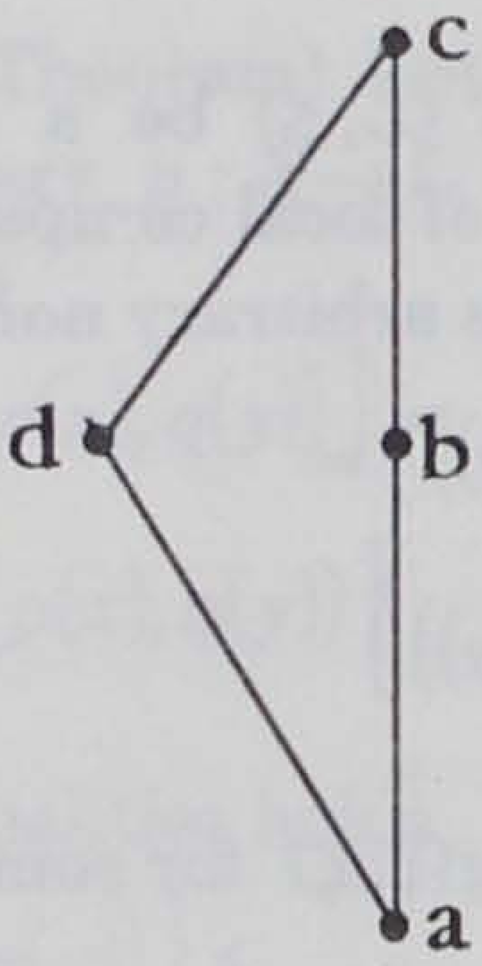

Figure 2.

PROOF OF LEMMA 2. $(13) \Rightarrow (12)$. Let $\zeta = \min P_g$ or $\zeta = \max P^g$. Now, let $x \in P_g$, $y \in P^g$, and $y < x$. Then $g(y, y) \leq y < x \leq g(x, x)$, i.e., $g(y, y) \leq g(x, x)$ is in contradiction with the decreasing of the function g. Then, for all $x \in P_g$ and $y \in P^g$ it follows that $x \leq y$. Let $\zeta = \max P^g$, then if $t \in P^g$ we have $t \leq \zeta$ and from that $\min\{ t, g(t, t) \} \leq x$,, and then $\max\{ t, g(t, t) \} = g(x, x) \geq g(\zeta, \zeta) \geq \zeta$. If $t \in Pg$, we have $\zeta < t$, and from that $\zeta \leq \max\{ t, g(t, t) \}$.

For $\zeta < t$, we have $g(t, t) \leq g(\zeta, \zeta) \leq \zeta$ and from that $\min\{ t, g(t, t) \} \leq \zeta$. The case $\zeta = \min P_g$ is symmetrical with previous one.

$(12) \Rightarrow (13)$. Let the point $\zeta \in P$ have characteristic (12). Then $x \in P^g$ implies $x \leq g(x, x)$, that is, $x = \min\{ x, g(x, x) \} \leq \zeta$, and $x \in P_g$ implies $g(x, x) \leq x$, that is,

$x = \max\{x, g(x,x)\} \geq \zeta$. Then for all $x \in P^g$ is $\zeta \leq x$, and for all $x \in P_g$ is $\zeta \leq x$. Accordingly, as for all $x \in P^g$ and $y \in P_g$ $x \leq y$ holds, we have the following: if $\zeta \in P^g$, then $\zeta = \max P^g$; if $\zeta \in P_g$ then $\zeta = \min P_g$. Owing to that, if the point $b > \zeta$ satisfies the condition (12), then we must have $\zeta = \max P^g$, $b = \min P_g$, and there cannot be any third point with characteristic (12). Thus, the proof is complete.

SOME COMMENTS. That two different points with characteristic (12) may exist proves the following example: $P = \{a, b\}$, $a < b$, $g(a) = b$, $g(b) = a$. In that case both points a and b have characteristic (12). But, if $(P, \leq)$ is an everywhere dense set ($x < y \Rightarrow (\exists z \in P)\ x < z < y$ for all $x, y \in P$), then there can be at most one point of characteristic (12).

Let us now give an example which shows that the points with characteristic (12) may not be fixed points. Let the mapping g be defined by $g(x) = 1$ $(0 \leq x \leq \frac{1}{2})$ and $g(x) = 0$ $(\frac{1}{2} < x \leq 1)$. Then on the set $P = [0,1]$ the point $\zeta = \frac{1}{2}$ has characteristic (12), but it is not the fixed point of the mapping $g : [0,1] \to [0,1]$.

With the help of the preceding statements we now obtain the fundamental fact of this section.

THEOREM 1. (Sup–Inf Theorem). Let $(L, \leq)$ be a lattice and let $g : L^2 \to L$ be a decreasing mapping. If L has property of local comparability, then for some arbitrary functions $p : X \to L$ and $q : X \to L$ (X is arbitrary nonempty set) the equality

$$\sup_{x,y \in X} \inf\{p(x), q(y), g(p(x), q(y))\} = \inf_{x,y \in X} \sup\{p(x), q(y), g(p(x), q(y))\} \tag{14}$$

holds if and only if $p(x_0) = q(y_0) := \zeta = g(\zeta, \zeta)$ for some $x_0, y_0 \in X$.

PROOF. This follows at once from (7) of Lemma 1 and the trivial fact that the strict inequality cannot hold in (7).

An immediate consequence (special case) of the preceding statement is the following principle.

THEOREM 2. (Minimax Principle). Let P be a totally ordered set by the order relation $\leq$, and let $g : P^2 \to P$ be a decreasing mapping. Then for some arbitrary function $p : X \to P$ and $q : X \to P$ (X is arbitrary nonempty set) the equality

$$\max_{x,y \in X} \min\{p(x), q(y), g(p(x), q(y))\} = \min_{x,y \in X} \max\{p(x), q(y), g(p(x), q(y))\} \tag{15}$$

holds if and only if $p(x_0) = q(y_0) := \zeta = g(\zeta, \zeta)$ for some $x_0, y_0 \in X$.

The statement above still holds when $g : P^k \to P$ (k is a fixed positive integer). The proof is quite similar. Therefore, let $(P, \leq)$ be a totally ordered set by the order relation $\leq$, and $g : P^k \to P$ ($k \in \mathrm{N}$) be a decreasing mapping. Then, the equality

$$\max_{\lambda_1,\dots,\lambda_k\in P}\min\{\lambda_1,\dots,\lambda_k,g(\lambda_1,\dots,\lambda_k)\}=$$
$$\min_{\lambda_1,\dots,\lambda_k\in P}\max\{\lambda_1,\dots,\lambda_k,g(\lambda_1,\dots,\lambda_k)\} \tag{16}$$

holds if and only if $\lambda_1(x_1) = \dots = \lambda_k(x_k) := \zeta = g(\zeta, \dots, \zeta)$ for some $x_1, \dots, x_k \in X$, where $\lambda_i\colon X\to P$ $(i = 1, \dots, k)$ are arbitrary functions and X is a nonempty set.

We remark that when $X = P$, $p(x) = x$ and $q(y) = y$ Theorem 2 reduces to the following result.

Corollary 1. ([55]). Let P be a totally ordered set by the order relation $\leq$, and let $g : P^2\to P$ be a decreasing mapping. Then the equality

$$\max_{x,y\in P}\min\{x,y,g(x,y)\}=\min_{x,y\in P}\max\{x,y,g(x,y)\},$$

holds, if and only if there is $\zeta\in P$ such that $g(\zeta,\zeta)=\zeta$.

In connection with the preceding, we note that we can give an extension of the preceding Theorem1, as a direct consequence of the preceding facts, in the following sense.

Theorem 3. (General Sup–Inf Theorem). Let $(L, \leq)$ be a lattice and let $g : L^2\to L$ be a mapping. Then for some arbitrary $p : X\to L$ and $q : X\to L$ (X is arbitrary nonempty set) the following equality holds

$$\sup_{x,y\in P}\inf\{p(x),q(y),g(p(x),q(y))\}=$$
$$\inf_{x,y\in P}\sup\{p(x),q(y),g(p(x),q(y))\} \tag{17}$$

if and only if the following inequalities holds

$$\inf\{p(x),q(y),g(p(x),q(y))\}\leq$$
$$p(x_0)=q(y_0)\quad :=\zeta=g(\zeta,\zeta)\leq$$
$$\sup\{p(x),q(y),g(p(x),q(y))\} \tag{18}$$

for some $x_0, y_0\in X$ and for all $x,y\in X$.

On the other hand, condition (18) is equivalent with the following equality

$$\max_{x,y\in X}\inf\{p(x),q(y),g(p(x),q(y))\}=$$
$$\min_{x,y\in X}\sup\{p(x),q(y),g(p(x),q(y))\}$$

Also, in connection with the preceding equality (16), if $g : P^k\to P$ (k is a fixed positive integer) is not decreasing mapping, we can extend equality (16). In this sense, if $g : P^k\to P$ (k is a fixed positive integer) is some arbitrary mapping then equality (16) holds if and only if the following inequalities hold

$$\min\{\lambda_1,\dots,\lambda_k, g(\lambda_1,\dots,\lambda_k)\} \le$$
$$\lambda_1(x_1) = \dots = \lambda_k(x_k) \quad := \zeta = g(\zeta,\dots,\zeta) \le$$
$$\max\{\lambda_1,\dots,\lambda_k, g(\lambda_1,\dots,\lambda_k)\},$$

for some $x_1, \dots, x_k \in X$, where $\lambda_i: X \to P$ $(i = 1, \dots, k)$ are arbitrary functions and X is a nonempty set.

On the other hand, the next result follows from the preceding statements.

Corollary 2. Let L be a lattice with the order relation $\le$. Then for some arbitrary mappings $p : X \to L$ and $q : X \to L$ (X is arbitrary nonempty set) the following equality holds

$$\sup_{x,y\in X} \inf\{p(x), q(y)\} = \inf_{x,y\in X} \sup\{p(x), q(y)\}$$

if and only if the following inequalities hold

$$\inf\{p(x), q(y)\} \le p(x_0) = q(y_0) \le \sup\{p(x), q(y)\}$$

for some $x_0, y_0 \in X$ and for all $x, y \in X$.

We note, in the preceding statements (as in Corollary 2) we can define the preceding functions $p, q : X \to L$ on different sets, in sense that $p : X \to L$ and $q : X \to L$ (X and Y are arbitrary nonempty sets). Then the preceding statements hold too. In this sense, for some arbitrary functions $f_i : X_i \to L$ $(i = 1, \dots, k)$ the following equality holds

$$\sup_{x_1\in X_1,\dots,x_k\in X_k} \inf\{f_1(x_1),\dots,f_k(x_k)\} = \inf_{x_1\in X_1,\dots,x_k\in X_k} \sup\{f_1(x_1),\dots,f_k(x_k)\}$$

if and only if the following inequalities hold

$$\inf\{f_1(x_1),\dots,f_k(x_k)\} \le f_1(t_1) = \dots = f_k(t_k) \le \sup\{f_1(x_1),\dots,f_k(x_k)\}$$

for some $t_i \in X_i$ $(i = 1, \dots, k)$ and for all $x_i \in X_i$ $(i = 1, \dots, k)$.

In this part of the section, we show that existence of separation in the preceding sense, is essential for applications of the preceding statements.

Theorem 4. (Statement of Separation). Let L be a lattice with the order relation $\le$. Then for some arbitrary mapping $p : X \to L$ and $q : Y \to L$ (X and Y are two arbitrary nonempty sets) the following equality holds

$$\sup_{x\in X} p(x) = \inf_{y\in Y} q(y) \tag{19}$$

if and only if there exists a function $g : L^2 \to L$ such that the following inequalities holds

$$p(x) \le g(p(x), q(y)) \le q(y) \tag{20}$$

for all $x \in X$ and $y \in Y$ and there is $\zeta \in L$ such that $\zeta = p(x_0) = q(y_0)$ for some $x_0 \in X$ and $y_0 \in Y$.

PROOF. Necessity. Let the inequalities (20) hold and let, from the conditions, there exist points $x_0 \in X$ and $y_0 \in Y$ such that $\zeta = p(x_0) = q(y_0)$. Thus, we obtain the following inequalities and equality

$$\inf\{\, p(x), q(y), g(p(x), q(y)) \,\} \le \zeta = g(\zeta, \zeta) \le \sup\{\, p(x), q(y), g(p(x), q(y)) \,\}$$

for some $x_0 \in X$ and $y_0 \in Y$, and for all $x \in X$ and $y \in Y$. This means, from Theorem 3, that the equality (17) holds, which gives the equality (19) of this statement.

Sufficiency. Assume that equality (19) holds. Thus, there is $\zeta \in L$ such that $p(x) \le \zeta \le q(y)$ for all $x \in X$ and $y \in Y$, where $p(x_0) = q(y_0) = \zeta$ for some $x_o \in X$ and $y_o \in Y$. If the function $g : L^2 \to L$ is defined by $g(s, t) = \zeta$, then, directly, we obtain inequalities (20). The proof is complete.

At the end of this section, based on the preceding statements, as an immediate consequence we have the following statement.

THEOREM 5. Let P be a set totally ordered by the order relation $\le$, and let $g : P^2 \to P$ be a decreasing mapping. Then the following equality holds

$$\max_{\zeta \le x} \min_{y \le \zeta} g(x, y) = \min_{y \le \zeta} \max_{\zeta \le x} g(x, y),$$

if and only if there is $\zeta \in P$ such that $g(\zeta, \zeta) = \zeta$.

3. SUP–INF INEQUALITIES

We give now some immediate applications of the preceding statements to Sup – Inf inequalities; other applications will be given in the next sections. As an immediate consequence of Lemma 1 we obtain the following inequalities.

LEMMA 3. Let P be a totally ordered set by the order relation $\le$ and let $g : P^2 \to P$ be a decreasing mapping. If for some arbitrary mapping $f : P^2 \to P$ is $f(\zeta, \zeta) \le \zeta$ and $f(\zeta, \zeta) \le g(\zeta, \zeta)$, then

$$f(\zeta, \zeta) \le \max\{\, p(x), q(y), g(p(x), q(y)) \,\}, \tag{21}$$

for all $x, y \in X$ where $p, q : X \to P$ (X is an arbitrary nonempty set).

Quantifying the preceding assertion (21) we obtain the following conclusion that $f(\zeta, \zeta) \le \zeta$ and $f(\zeta, \zeta) \le g(\zeta, \zeta)$ implies

$$f(\zeta, \zeta) \le \min_{x, y \in X} \max\{\, p(x), q(y), g(p(x), q(y)) \,\}.$$

PROOF. Let $\lambda = \max\{\, p(x), q(y) \,\}$ where the elements $x \in X$ and $y \in Y$ are arbitrarily chosen. If $f(\zeta, \zeta) \le \lambda$, then (21) obviously holds. If $\lambda \le f(\zeta, \zeta)$, then $f(\zeta, \zeta) \le g(\zeta, \zeta) \le g(p(x), q(y))$ and (21) holds too.

In connection with this, we now obtain the fundamental fact of this section.

Theorem 6. (Sup–Inf Inequality). Let $(L, \leq)$ be a lattice with zero and unit and let $A, B : X \times Y \to L$ (X and Y are arbitrary nonempty sets). Then for some arbitrary mappings $a, c : X \to L$ and $b, d : Y \to L$ the following inequality holds

$$\inf_{x \in X, y \in Y} \sup\{a(x), b(y), A(x,y)\} \leq \sup_{x \in X, y \in Y} \inf\{c(x), d(y), B(x,y)\}, \tag{22}$$

if and only if the following inequality holds

$$\sup\{a(x), b(y), A(x,y)\} \leq \inf\{c(x), d(y), B(x,y)\}, \tag{23}$$

for all $x \in X$ and $y \in Y$.

Proof. Since inequality (23) holds for all $x \in X$ and $y \in Y$, directly, quantifying this inequality we obtain the preceding inequality (22). On the other hand, if (22) holds, we assume that inequality (23) does not hold. Then there is some $x_o \in X$ and $y_o \in Y$ such that

$$\alpha := \inf\{c(x_0), d(y_0), B(x_0, y_0)\} < \sup\{a(x_0), b(y_0), A(x_0, y_0)\} := \beta.$$

Thus, we obtain the following consequences,

$$\inf\{c(x), d(y), B(x,y)\} \leq \alpha < \beta \leq \sup\{a(x), b(y), A(x,y)\}$$

for all $x \in X$ and for all $y \in Y$. Quantifying the inequalities we obtain the following inequality

$$\sup_{x \in X, y \in Y} \inf\{c(x), d(y), B(x,y)\} < \inf_{x \in X, y \in Y} \sup\{a(x), b(y), A(x,y)\}$$

which is a contradiction with (22). Thus, this statement is proved.

As an immediate consequence of the preceding statement we obtain the following statement.

Theorem 7. Let $(L, \leq)$ be a lattice with zero and unit, and let $A, B : X \times Y \to L$ (X and Y are arbitrary nonempty sets). Then for some arbitrary mappings $a, c : X \to L$ and $b, d : Y \to L$ the following inequality holds

$$\inf_{x \in X, y \in Y} \sup\{a(x), b(y), A(x,y)\} \leq \sup_{x \in X, y \in Y} \sup\{c(x), d(y), B(x,y)\},$$

if and only if the following inequality holds

$$\sup\{a(x), b(y), A(x,y)\} \leq \sup\{c(x), d(y), B(x,y)\}, \tag{24}$$

for all $x \in X$ and $y \in Y$.

At the end of this section, we give a separation statement for separation of the preceding inequalities.

Theorem 8. (Separation of Inequalities). Let L be a lattice with the order relation $\leq$, with zero and unit, and let the functions $c : X \to L$ and $b : Y \to L$ (X and Y are two arbitrary nonempty sets) satisfy the inequality $b(y) \leq c(x)$ for all $x \in X$ and $y \in Y$. Then the following inequality holds

$$\inf_{y\in Y} b(y) \le \sup_{x\in X} c(x) \tag{25}$$

if and only if there exist functions $A, B : X \times Y \to L$, $a : X \to L$ and $d : Y \to L$ such that the following inequalities hold

$$a(x) \le A(x, y) \le b(y) \le c(x) \le B(x, y) \le d(y), \tag{26}$$

for all $x \in X$ and for all $y \in Y$.

PROOF. Let inequality (25) holds, and let $\alpha := \inf_{y\in Y} b(y)$ and $\beta := \sup_{x\in X} c(x)$. If we define functions $A(x,y) = a(x) = \alpha$ and $B(x,y) = d(y) = \beta$, we obtain, directly, that inequalities (26) hold. On the other hand, if the inequalities (26) hold, from Theorem 6 and inequality (23), we directly obtain inequality (22), i.e., inequality (25) of this statement.

4. VON NEUMANN'S MINIMAX THEORY

John von Neumann's Minimax Theorem [42] can be stated as follows: if X and Y are finite dimensional simplices and f is a bilinear function on $X \times Y$, then f has a saddle point, i.e.,

$$\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y). \tag{27}$$

There have been several generalizations of this theorem. Ville [61] and Wald [66] extended in various ways von Neumann's result to cases where X and Y were allowed to be subsets of certain infinite dimensional linear space. The functions they considered, however, were still linear. Shiffman [53] seems to be the first to consider concave–convex functions in a minimax theorem. Kneser [26], Fan [18] and Berge [8] (using induction and the method of separating two disjoint convex sets in Euclidean space by a hyperplane) got minimax theorems for concave–convex functions that are appropriately semi–continuous in one of the two variables. Although these theorems include the previous results as special cases, they can also be shown to be rather direct consequence of von Neumann's theorem. Kakutany [29] and Nikaido [44], on the other hand, using Brouwer's fixed point theorem, proved the existence of a saddle point for functions satisfying the weaker algebraic condition of being quasi concave–convex, but the stronger topological condition of being continuous in each variable.

Thus, there seems to be two essential types of arguments: one uses some form of separation of disjoint convex sets by a hyperplane and yields the theorem of Kneser–Fan, and the other uses the fixed point theorem and yields Nikaido's results.

In [52], Sion unified the two streams of thought by proving a minimax theorems for a function that is quasi–concave–convex and appropriately semi–continuous in each variable. The method of proof differs radically from any used previously. The key tool used is a theorem due to Knaster, Kuratowski, Mazurkiewicz based on Sperner's lemma.

In [24] and [14], Sion' s minimax theorem is extended for non-compact sets, and for certain two person zero - sum games on constrained sets a sequential unconstrained solution method is given.

Granas and Fon-Che Liu [23] discuss some new general minimax results which are of von Neumann' s type too.

In connection with the preceding, evidently, our statements of separation give methods for proof of all theorems of von Neumann' s type. Thus, this new minimax convex theory extends the theory of von Neumann' s type. In this section we give proofs for the preceding facts.

Let f be a real-valued function defined on the product set $X \times Y$ of two arbitrary sets, X, Y (not necessarily topologized). The function f is said to be convex on X, if for any two elements $x_1, x_2 \in X$ and two numbers $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, with $\zeta_1 + \zeta_2 = 1$, there exists an element $x_0 \in X$ such that $f(x_0, y) \leq \zeta_1 f(x_1, y) + \zeta_2 f(x_2, y)$ for all $y \in Y$. Similarly, f is said to be concave on Y, if for any two elements $y_1, y_2 \in Y$ and two numbers $\eta_1 \geq 0$, $\eta_2 \geq 0$ with $\eta_1 + \eta_2 = 1$, there exists an $y_0 \in Y$ such that $f(x, y_0) \geq \eta_1 f(x, y_1) + \eta_2 f(x, y_2)$ for all $x \in X$. Recall that a real valued function $f : X \to \mathrm{R}$ on a topological space is lower (respectively upper) semicontinuous if $\{x \in X : f(x) > r\}$ (respectively $\{x \in X : f(x) < r\}$) is open for each $r \in \mathrm{R}$; if X is a convex set in a linear space, then f is quasi-concave (respectively quasi-convex) if $\{x \in X : f(x) > r\}$ (respectively $\{x \in X : f(x) < r\}$) is convex for each $r \in \mathrm{R}$.

The following result of Sion [52] is the best representative of von Neumann' s theory.

Corollary 3. ([52]). Let $X \subset E$ and $Y \subset F$ be two nonempty compact convex sets in linear topological spaces E and F, and $f : X \times Y \to \mathrm{R}$ be a real-valued function satisfying:

(a) $y \mapsto f(x, y)$ is lower semicontinuous and quasi-convex on Y for each $x \in X$;

(b) $x \mapsto f(x, y)$ is upper semicontinuous and quasi-concave on X for each $y \in Y$.

Then the following equality holds

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y). \tag{28}$$

We shall use the notation as in [54]. Let $(P, \leq)$ be a partially ordered set. For $x, y \in P$ and $x < y$, the set $]x, y[$ is defined by $]x, y[\ := \{t \mid t \in P \text{ and } x < t < y\}$. Call a poset P conditionally complete when every nonempty subset of P with upper bound has its supremum in P. The proof of this fundamental statement in von Neumann' s minimax theory we begin with the following essential lemma.

Lemma 4. (Coincidence Lemma). Let $(P, \leq)$ or $]x, y[$ be a conditionally complete partially ordered set and let $f, g : P \to P$ be two mappings such that

$$a \leq b \ \text{ implies } \quad [f(a), g(b)] \subset [a, b], \ \text{for all } a, b \in \ ]x, y[. \tag{29}$$

If for $a \neq b$ the preceding inclusion $\subset$ is strict, then there exists a point $\zeta \in P$ with the property $f(\zeta) = g(\zeta) = \zeta$.

PROOF. From condition (29) we have $a \le f(a) \le g(b) \le b$. Applying again condition (29) we get the following relations (inequalities):

$$a \le f(a) \le f^2(a) \le \dots \le g^2(b) \le g(b) \le b. \tag{30}$$

Hence the set $P^f := \{a, f(a), f^2(a), \dots\}$ is nonempty and bounded from above, and $s := \sup P^f$ exists, by conditional completeness of P (or $]x, y[$). Also, $f(P^f) \subset P^f$. Since $x \le s$ for all $x \in P^f$, we have $x \le f(x) \le f(s)$ for all $x \in P^f$. Hence $f(s)$ is a majorant for the set $f(P^f)$. Also, from the preceding inequalities, we have $s \le \sup f(P^f) \le f(s)$. Thus $s \in P^f \subset]x, y[$. Applying condition (29) it follows that $f(s) \le s$. We conclude $s = f(s)$. Further, let $P_g := \{b, g(b), g^2(b), \dots\}$. Hence the set P_g is nonempty and bounded from above and $I := \inf P_g$ exists, by conditional completeness of P (or $]x, y[$). Also $g(P_g) \subset P_g$, and $I \le g(x) \le x$ for all $x \in P_g$. Analogous to the proof of the preceding fact, dually we have further that $g(I) = I$ with property $s \le I$. If $s < I$ (strictly), then from condition (29) we obtain $[f(s), g(I)] = [s, I]$ which is in contradiction with strict inclusion $[f(s), g(I)] \subset [s, I]$. We conclude $s = I$, i.e., there exists point $\zeta \in P$ with property $f(\zeta) = g(\zeta) = \zeta$. This completes the proof.

PROOF OF COROLLARY. (Application of Theorem 4 and Lemma 4). Because of upper and lower semicontinuity

$$p(x) := \min_{y \in Y} f(x, y) \text{ and } q(y) := \max_{x \in X} f(x, y) \text{ exist for each } x \in X \text{ and } y \in Y.$$

On the other hand, since $p(x) \le f(x,y) \le q(y)$ for all $x \in X$ and for all $y \in Y$, we obtain the following inequalities

$$p(x) \le \max_{x \in X} p(x) \le \min_{y \in Y} q(y) \le q(y), \text{ for all } x \in X \text{ and } y \in Y.$$

Suppose that $p(x) \ne q(y)$ for all $x \in X$ and $y \in Y$. Then we can find a strictly increasing sequence of real numbers $a_\alpha = p(x_\alpha)$ for $\alpha < w$, and a strictly decreasing sequence of real numbers $b_\alpha = q(y_\alpha)$ for $\alpha < w$, where w is any (finite or transfinite) ordinal.

Since X and Y are two nonempty compact (convex) sets, we obtain, also, that $p(X)$ and $q(Y)$ are two nonempty compact sets. Thus $p(X) \cup q(Y) \subset \mathrm{R}$ is a conditionally complete set. We define two mappings F and G from $p(X) \cup q(Y)$ into itself by

$$a_\beta = F(a_\alpha),\ a_\alpha = G(a_\alpha),\ b_\beta = G(b_\alpha) \text{ and } b_\alpha = F(b_\alpha) \text{ for } \alpha < \beta < w,$$

where w is any (finite or transfinite) ordinal. It is easy to see that F and G satisfy all the required hypotheses in Lemma 4. Applying Lemma 4, in this case, we obtain that then there exists a point $\zeta \in p(X) \cup q(Y)$ with property $F(\zeta) = G(\zeta) = \zeta$. From the construction of functions F and G it follows that point ζ must be element of $p(X) \cap q(Y)$, i.e., $\zeta \in p(X) \cap q(Y)$. On the other hand, if we define a function $g : \mathrm{R}^2 \to \mathrm{R}$ by

$$g(s, t) = \Theta \in \left[\max_{y \in Y} p(x), \min_{x \in X} q(y) \right] \tag{31}$$

it is easy to see that all the required hypotheses of Theorem 4 are satisfied and we obtain that equality (19) holds, which is equivalent, in this case, with the equality (28). The proof is complete.

In connection with the preceding, from the proof of Corollary 3 and from Theorem 4, we have directly two following statements:

THEOREM 9. Let $X \subset E$ and $Y \subset F$ be two nonempty compact sets in the topological spaces E and F; and let $f : X \times Y \to \mathbb{R}$ satisfy:

(a) For each fixed $x \in X$, $f(x, y)$ is lower semicontinuous on Y, and f is a mapping of Y onto R.

(b) For each fixed $y \in Y$, $f(x, y)$ is upper semicontinuous on X, and f is a mapping of X onto R.

Then the equality (28) holds.

THEOREM 10. Let $X \subset E$, $Y \subset F$ and $Z \subset G$ be three nonempty compact sets in the topological spaces E, F and G; and let $f : X \times Y \times Z \to \mathbb{R}$ satisfy

(a) For each fixed $x \in X$ and $y \in Y$, $f(x, y ,z)$ is lower semicontinuous on Z, and f is a mapping of Z onto R.

(b) For each fixed $z \in Z$, $f(x, y ,z)$ is upper semicontinuous on X and Y and f is a mapping of X, Y onto R.

Then the following equality holds

$$\max_{x\in X, y\in Y} \min_{z\in Z} f(x, y, z) = \min_{z\in Z} \max_{x\in X, y\in Y} f(x, y, z).$$

The proof is analogous to the proof of the preceding statements.

5. MINIMAX INEQUALITIES

In this part we establish some minimax inequalities in topological spaces, as consequences of former results of separation for inequalities.

As an immediate consequence of Theorems 6 and 8 we obtain the following result.

COROLLARY 4. ([22]). Let $k, h : X \times Y \to \mathbb{R}$ (X and Y are two nonempty compact convex subsets in linear topological spaces A and B, respectively) be two real-valued functions such that

$$k(x, y) \le h(x, y) \text{ for all } x \in X \text{ and } y \in Y.$$

If $x \mapsto k(x, y)$ is upper semicontinuous and quasi-concave on X for each $y \in Y$ and if $y \mapsto h(x, y)$ is lower semicontinuous and quasi-convex on Y for each $x \in X$, then

$$\inf_{y\in Y} \sup_{x\in X} k(x, y) \le \sup_{x\in X} \inf_{y\in Y} h(x, y) \tag{32}$$

PROOF. (Application of Theorem 8). Because of upper and lower semicontinuity

$$p_i(x) = \inf_{y\in Y} i(x, y) \text{ and } q_i(y) = \sup_{x\in X} i(x, y) \text{ for } i = k, h.$$

exists for each $x \in X$ and $y \in Y$. Since $p_k(x) \le k(x,y) \le q_k(y) \le p_h(x) \le h(x,y) \le q_h(y)$ for all $x \in X$ and $y \in Y$, it is easy to see that all the required hypotheses of Theorem 8 are satisfied. Applying Theorem 8 we have the preceding inequality (32) of Granas and Fon-Che Liu.

Also, as an immediate consequence of Theorems 7 and 8 we obtain the following results of Granas, Fon-Che Liu and Ky Fan.

COROLLARY 5. ([22]). Let X be a nonempty compact convex set in linear topological space E and Y be a subset of E containing X. Let $f, g : X \times Y \to \mathbb{R}$ be two real-valued functions satisfying the following inequality

$f(x,y) \le g(x,y)$ for all $x \in X$ and $y \in Y$,

where $x \mapsto g(x,y)$ is quasi-concave on X for each $y \in Y$ and $y \mapsto f(x,y)$ is lower semicontinuous on Y for each $x \in X$. Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X} g(x,x)$$

We remark that, in case $X = Y$, Corollary 5 gives a generalization of the Ky Fan Minimax Inequality due to Yen [64].

COROLLARY 6. ([18]). Let X be a nonempty compact convex set in a linear topological space E and Y be a subset of E such that $X \subset Y$. Assume $f : X \times Y \to \mathbb{R}$ is a real-valued function such that $y \mapsto f(x,y)$ is lower semicontinuous on Y for each $x \in X$ and $x \mapsto f(x,y)$ is quasi-concave on X for each $y \in Y$. Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,x).$$

6. EXISTENCE OF TRANSVERSAL POINTS

In connection with the preceding, in this part we continue the study of the preceding minimax problems. In this section we consider a concept of transversal points for the mapping f of a nonempty set X into partially ordered set P. A map f of a nonempty set X into partially ordered set P has a transversal point $\zeta \in P$ if there is a decreasing function $g : P^2 \to P$ such that the following equality holds

$$\max_{x,y\in X} \min\left\{ f(x), f(y), g(f(x), f(y)) \right\} = \min_{x,y\in X} \max\left\{ f(x), f(y), g(f(x), f(y)) \right\} := \zeta. \tag{33}$$

On the other hand, in our paper [55] we investigated the concept of fixed apices for a mapping f of a set X into itself. A map f of a set X to itself has a fixed apex $u \in X$ if for $u \in X$ there is $v \in X$ such that $f(u) = v$ and $f(v) = u$. The points $u, v \in X$ are called fixed apices of f if $f(u) = v$ and $f(v) = u$. In this sense, a nonempty set X is apices set if each of its points is an apex of some mapping $T : X \to X$. If $T : S^n \to S^n$ is the map such that $Tx = -x$ for $x \in S^n$, then S^n is an apices set.

Otherwise, a function $f : X \to P$ has a SI-transversal point if the preceding equality (33) holds with sup and inf instead max and min, respectively. If the preceding equality (33) holds for points $x, -x \in X$ (X is a linear space) ζ is A–transversal point; more generally ζ is R–transversal point for $f: X \to P$ if the equality (33) holds for points $x, Tx \in X$. A function $f: X \to P$ (X is a linear space) has a pair of antipodal points $p, -p \in X$ if the following equality holds $f(p)=f(-p)$.

We note that from the second section, i.e., from Corollary 1, we obtain that the function $f(x)=\mathrm{id}_{\mathrm{R}}: \mathrm{R} \to \mathrm{R}$ has a transversal point $\zeta \in \mathrm{R}$ if and only if for some decreasing function $g: \mathrm{R}^2 \to \mathrm{R}$ we have $g(\zeta, \zeta) = \zeta$.

Let E be the normed space of all those sequences $x = (x_1, x_2, \ldots)$ of real numbers having at most finitely many $x_n \neq 0$, with the norm $\|x\| = \sum |x_i|$. The subset $\{x \in E \mid x_i = 0 \text{ for all } i>n\}$ is denoted by E^n or R^n; the unit n–ball is $V^n = \{x \in E^n : \|x\| \leq 1\}$. The unit n–sphere $S^n = \{x \in E^{n+1} : \|x\| = 1\}$; its upper hemisphere is $S_+{}^n = \{x \in S^n : x_{n+1} \geq 0\}$, and its lower hemisphere is $S_-{}^n = \{x \in S^n : x_{n+1} \leq 0\}$; clearly $S^n = S_+{}^n \cup S_-{}^n$. Observe that for any $k < n$, we have

$$S^k = \{x \in S^n \mid x_{k+2} = \ldots = x_{n+1} = 0\}$$

and that $S^{n-1} = S_+{}^n \cap S_-{}^n$. Recall that a map $f: S^n \to S^n$ is antipodal–preserving if $f(\alpha) = \alpha(f)$, for some $\alpha: S^n \to S^n$.

Results equivalent to the Lusternik Schnirelman and Borsuk statement use the notions of extendability and homotopy in their formulation. For the convenience of the reader, and to establish the terminology, we recall the relevant definitions. By space we understand a Hausdorff space; unless specifically stated otherwise, a map is a continuous transformation.

Let X, Y be two spaces and $A \subset X$. A map $f: A \to Y$ is called extendable over X if there is a map $F: X \to Y$ with $F|A = f$. Two maps $f, g: X \to Y$ are called homotopic if there is a map $H: X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each $x \in X$. The map H is called homotopy (or: continuous deformation) of f to g, and written $H: f \cong g$. For each t, the map $x \to H(x, t)$ is denoted by $H_t: X \to Y$; clearly the family $(H_t)_{0 \leq t \leq 1}$ determines H and vice versa. Thus, the relation of homotopy decomposes the set of all maps of X into Y into pairwise disjoint classes called homotopy classes and $f: X \to Y$ homotopic to a constant map is called nullhomotopic.

We now prove Borsuk' s antipodal theorem and also show that it is equivalent to various geometric results about the n–sphere.

THEOREM 11. Let S^n denote the n–sphere. Then the following statements are equivalent:

(a) (Lusternik–Schnirelman–Borsuk theorem). In any closed covering $\{M_1, \ldots, M_{n+1}\}$ of S^n by $(n+1)$–sets, at least one set M_i must contain a pair of antipodal points.

(b) (Borsuk antipodal theorem). An antipodal – preserving map $f: S^{n-1} \to S^{n-1}$ is not nullhomotopic.

(c) (Borsuk–Ulam type theorem). Every continuous map $f: S^n \to \mathrm{R}$ sends at least one pair of antipodal points to the same point.

(d) ([56]). Every continuous map $f: S^n \to \mathrm{R}$ has at least one A–transversal point.

In connection with the proof of this statement, first, we note that the following fact holds.

LEMMA 4. Let $f : S^n \to \mathrm{R}^n$ be a continuous mapping. Then f has an A-transversal point if and only if all projections $\mathrm{pr}^i(f) : S^n \to \mathrm{R}$ defined by $\mathrm{pr}^i(f) = \mathrm{pr}^i(\alpha_1, \dots, \alpha_n) = \alpha_i$ $(i = 1, \dots, n)$ have an A-transversal point.

PROOF. If f has an A-transversal point $\zeta \in \mathrm{R}^n$, then from Theorem 2, we obtain $\zeta := f(t) = f(-t)$ for some $t \in S^n$. Thus, $\mathrm{pr}^i(f(t)) = \mathrm{pr}^i(f(-t))$ for some $t \in S^n$ and $i = 1, \dots, n$. Therefore we choose for $g : \mathrm{R}^2 \to \mathrm{R}$ the function $g(s, t) = \eta := \mathrm{pr}^i(f(t)) = \mathrm{pr}^i(f(-t))$, such that $g(\eta, \eta) = \eta$. Thus, from Theorem 2 we obtain that all projections of f have an A-transversal point. On the other hand, if all projections of f have an A-transversal point $\eta \in \mathrm{R}$, from Theorem 2, we obtain $\eta := \mathrm{pr}^i(f(t)) = \mathrm{pr}^i(f(-t))$ for some $t \in S^n$ and $i = 1, \dots, n$. Again applying Theorem 2, analogous to the preceding case, we choose for $g : \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}$ the function $g(s, t) = \zeta := f(t) = f(-t)$ for some $t \in S^n$ and we obtain that f has an A-transversal point.

PROOF OF THEOREM 11. From Dugundji–Granas [17], (a) is equivalent to (b). Thus, we need only to show that (b) implies (c), (c) implies (d) and that (d) implies (a).

First, (b) $\Rightarrow$ (c). Assume $f : S^n \to \mathrm{R}$ is such that $f(x) \neq f(-x)$ for every $x \in S^n$. Define $F : S^n \to S^{n-1}$ by

$$F(x) = \frac{\|f(x) - f(-x)\|}{f(x) - f(-x)} \quad \text{for } x \in S^n.$$

Then $F|S^{n-1} : S^{n-1} \to S^{n-1}$ is antipodal-preserving and, since $F|S_+{}^n$ is an extension over V^n, $F|S^{n-1}$ would be null-homotopic, contradicting (b).

Second, (c) $\Rightarrow$ (d). Suppose that the map $f : S^n \to \mathrm{R}$ does not have A-transversal points, i.e., assume $f : S^n \to \mathrm{R}$ is such that

$$\max_{x \in S^n} \min \left\{ f(x), f(-x), g\big(f(x), f(-x)\big) \right\} \neq$$
$$\min_{x \in S^n} \max \left\{ f(x), f(-x), g\big(f(x), f(-x)\big) \right\}, \tag{34}$$

On the other hand, from (c) f has a pair of antipodal points such that $f(t) = f(-t)$ for some $t \in S^n$. Thus, from Theorem 2 (Minimax Principle) for $\zeta := f(t) = f(-t)$ and some decreasing map $g : \mathrm{R}^2 \to \mathrm{R}$ the following equality holds

$$\max_{x \in S^n} \min \left\{ f(x), f(-x), g\big(f(x), f(-x)\big) \right\} =$$
$$\min_{x \in S^n} \max \left\{ f(x), f(-x), g\big(f(x), f(-x)\big) \right\},$$

contradicting (34). Thus, $f : S^n \to \mathrm{R}$ has at least one A-transversal point, i.e., (d) holds.

Third, (d) $\Rightarrow$ (a). Assume there were some closed covering $M_1, \dots, M_{n+1}$ of S^n with no M_i containing a pair of antipodal points, i.e., $M_i \cap \alpha(M_i) = \varnothing$ for each i, where $\alpha(x) = -x$. Let $h_i : S^n \to [0,1]$ be a function with $h_i|M_i = 0$ and $h_i|M_{i+1} = 1$ for each $i = 1, \dots, n$. Define $h : S^n \to \mathrm{R}^n$ by $h(x) = (h_1(x), \dots, h_n(x))$. Then, according to (d) all projections of h have an A–transversal point. Therefore, from Lemma 4, we obtain that h has an A–transversal point. According to Theorem 2, there must be a $z \in S^n$ with $h(z) = h(\alpha(z))$, so that $h_i(z) = h_i(\alpha(z))$ for $i = 1, \dots, n$ and therefore

$z \in S^n - \bigcup_{i=1}^{n} M_i - \bigcup_{i=1}^{n} \alpha(M_i)$. Since both $\{M_i\}_{i=1}^{n+1}$ and $\{\alpha(M_i)\}_{i=1}^{n+1}$ cover S^n, the point $z \in S^n$ must belong to both M_{n+1} and $\alpha(M_{n+1})$, which is the desired contradiction.

This completes the proof.

In the connection with the former results of Lusternik, Schnirelman, Borsuk and Theorem 11, as an immediate consequence we obtain the following fact.

COROLLARY 7. Let S^n denotes the n–sphere. Then the following statements are equivalent:

(a) (Borsuk–Ulam Theorem). Every continuous map $f : S^n \to \mathrm{R}^n$ sends at least one pair of antipodal points to the same point.

(b) ([56]). Every continuous map $f : S^n \to \mathrm{R}$ has at least one A–transversal point.

On the other hand, analogous to the preceding statement, we obtain the following extension of the former results.

THEOREM 12. Let X be an apices set in sense of fixed point free map $T : X \to X$ and let Card $X \geq$ continuum; then the following statements are equivalent:

(a) In any closed covering $\{M_1, \dots, M_{n+1}\}$ of X by $(n+1)$–sets, at least one set M_i must contain a pair of points $x, Tx \in X$.

(b) Every continuous map $f : X \to \mathrm{R}$ has at least one pair of points $p, Tp \in X$ such that $f(p) = f(Tp)$.

(c) Every continuous map $f : X \to \mathrm{R}$ has at least one R–transversal point.

The proof is analogous to the proof of the preceding Theorem 11.

In connection with the transversal points, in this part we consider some other concepts of points for the mapping f of a nonempty set X into a partially ordered set P. A map $f : X \to P$ has a *furcate point* $\zeta \in P$ if for some function $T : X \to X$ the following equality holds

$$\max_{x,y \in X} \min \{ f(x), f(Ty) \} = \min_{x,y \in X} \max \{ f(x), f(Ty) \} := \zeta. \tag{35}$$

Otherwise, a function $f : X \to P$ has a SI–furcate point if the preceding equality (35) holds when instead max and min stand sup and inf, respectively. If the preceding equality (35) holds for points $x, -x \in X$ (X is a linear space) then ζ is A–furcate point; or generally ζ is R–furcate point for $f : X \to P$ if the equality (35) holds for points $x, Tx \in X$.

From the second section, i.e., from Theorem 3, we obtain that for the function $f : X \to L$ (X is an arbitrary nonempty set and $(L, \leq)$ is a lattice) the following inequalities hold $\inf \{ f(x), f(Ty) \} \leq f(x_0) = f(Ty_0) \leq \sup \{ f(x), f(Ty) \}$, for some $x_0, y_0 \in X$ and for all $x, y \in X$.

Thus, if $f: X \to L$ has an R–furcate-point then f has at least one pair of points $p, Tp \in X$ such that $f(p) = f(Tp)$. Reverse does not hold. Figure 3 shows the mapping f of complete lattice I into itself with $f(p) = f(Tp)$ for some $p \in I$, but without furcate points.

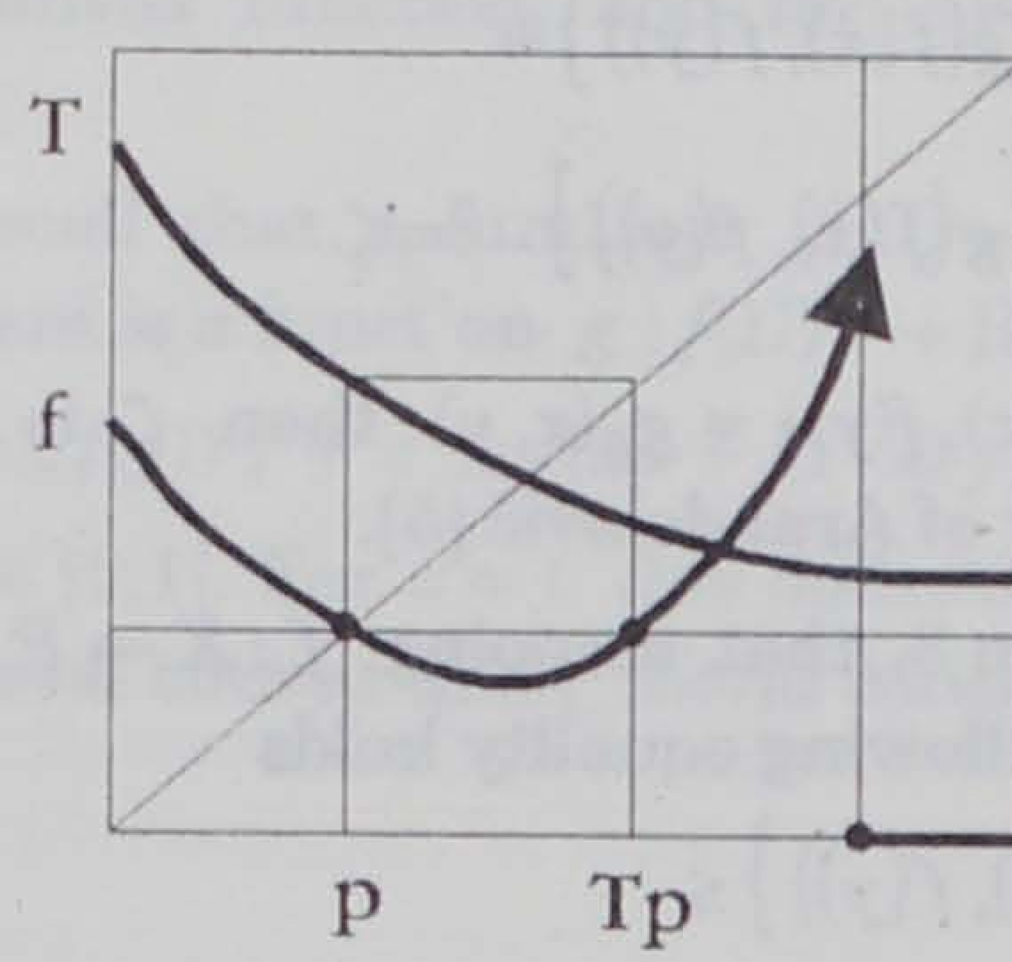

Figure 3.

For two mappings $f: X \to P$ and $g: Y \to P$ (X and Y are arbitrary nonempty sets and P is a partially ordered set) we have common (coincidence) furcate points. Namely, two mappings $f: X \to P$ and $g: Y \to P$ have a coincidence furcate point $\zeta \in P$, if the following equality holds

$$\max_{x \in X, y \in Y} \min\{ f(x), g(y) \} = \min_{x \in X, y \in Y} \max\{ f(x), g(y) \} := \zeta.$$

In general, the mappings $f_i : X_i \to P$ ($i = 1, \ldots, k$) (X_i are arbitrary nonempty sets and P is a poset) have a coincidence furcate point $\zeta \in P$ if the following equality holds

$$\max_{x_1 \in X_1, \ldots, x_k \in X_k} \min\{ f(x_1), \ldots, f(x_k) \} = \min_{x_1 \in X_1, \ldots, x_k \in X_k} \max\{ f(x_1), \ldots, f(x_k) \} := \zeta.$$

We notice that from Theorem 3, we obtain that the function $f: X \to P$ and $g: Y \to P$ have coincidence furcate point if and only if the following inequalities holds

$$\min\{ f(x), g(y) \} \le f(x_0) = g(y_0) \le \max\{ f(x_0), g(y_0) \},$$

for some $x_0 \in X$, $y_0 \in Y$ and for all $x \in X$, $y \in Y$. In connection with this, we notice that there are some continuous functions $f, g: I \to I$ (Figure 4) which map compact interval into itself, but f and g have not coincidence furcate points.

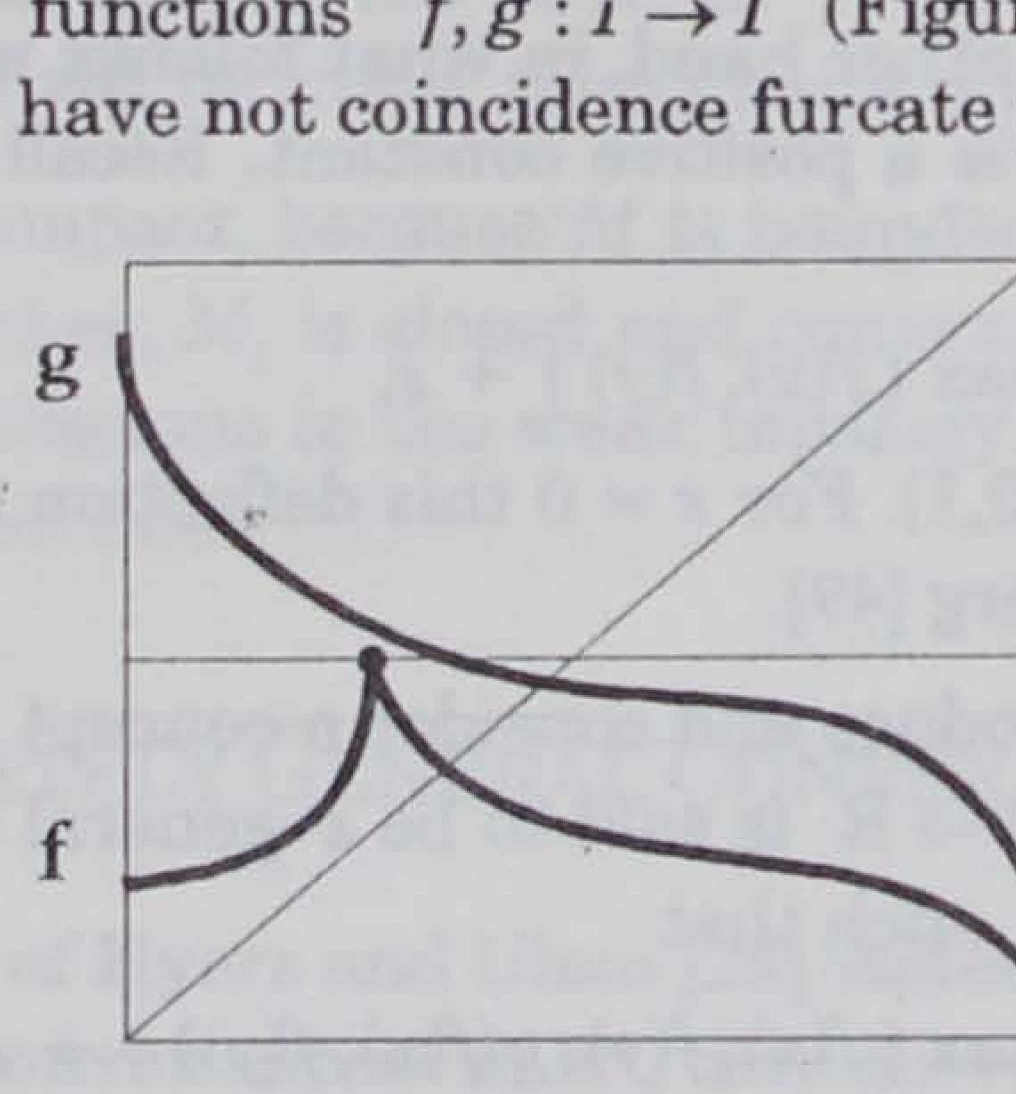

Figure 4.

It is also possible to introduce the concept of general transversal point in the following sense: the function $f: X \to P$ (X is a nonempty set and P a poset) has a quasi transversal point $\zeta \in P$, if for some function $g: P^2 \to P$ the following equality holds

$$\max_{x,y\in X} \min\left\{ f(x), f(y), g(f(x), f(y)) \right\} =$$

$$\min_{x,y\in X} \max\left\{ f(x), f(y), g(f(x), f(y)) \right\} \ := \zeta.$$

If in this equality $g(f(x), f(y)) = g_0(x, y)$ then ζ is TD-transversal point. In connection with this see paper of Aranđelović [5].

We notice, from Theorem 3, that mapping $f: X \to P$ has a general transversal point $\zeta \in P$ if and only if the following equality holds

$$\min\left\{ f(x), f(y), g(f(x), f(y)) \right\} \le$$

$$f(x_0) = f(y_0) \ := \zeta = g(\zeta, \zeta) \le$$

$$\max\left\{ f(x), f(y), g(f(x), f(y)) \right\},$$

for some $x_0, y_0 \in P$ and for all $x, y \in P$. Also, we can introduce SI-general transversal points, A-general and R-general transversal points.

7. GENERALIZED CONVEX FUNCTIONS

In this section we introduce and consider a new concept of convexity. First, we introduce a concept of ε-general convex functions and general convex functions.

In second part we prove a stability theorem of Hyers–Ulam type for general convex functions. First, we prove an extremal principle. Hyers and Ulam [25] have introduced the notion of approximately convex function. A function $f: D \to \mathrm{R}$ where R denotes the real line and D is a convex subset of R^n, is said to be ε-approximately convex provided it satisfies the inequality

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y) + \varepsilon,$$

for all $x, y \in D$, for all $\lambda \in [0,1]$, and some $\varepsilon > 0$. For $\varepsilon = 0$ the definition reduces to that of convex function. On the other hand, in what follows we assume that D is nonempty convex subset of R^n and ε is a positive constant. Recall that a function $f: D \to \mathrm{R}$ is said to be ε-quasiconvex if

$$f(\lambda x + (1-\lambda) y) \le \max\{ f(x), f(y) \} + \varepsilon,$$

for all $x, y \in D$, and all $\lambda \in [0,1]$. For $\varepsilon = 0$ this definition reduces to that of quasiconvex function, cf. Roberts–Varberg [49].

In this section we introduce and consider a concept of ε-general convex functions. Recall that a function $f: D \to \mathrm{R}$ is said to be ε-general convex if for some $\varepsilon > 0$ there is a function $g: f(D)^2 \to \mathrm{R}$ such that

$$f(\lambda x + (1-\lambda) y) \le \max\{ f(x), f(y), g(f(x), f(y)) \} + \varepsilon, \tag{36}$$

for all $x, y \in D$, and for all $\lambda \in [0,1]$. For $\varepsilon = 0$ this definition reduces to that of general convex function.

We notice that the set of all convex and quasiconvex functions can be a proper subset of the set of all general convex functions. Also, an ε–quasiconvex function is an ε–general convex function. If the preceding inequalities hold for all $x, y \in D$ and $\lambda = \frac{1}{2}$ then $f(x)$ is ε–general J–convex function, and for $\varepsilon = 0$, $f(x)$ is general J–convex function.

On the other hand, recall that a function $f : D \to \mathrm{R}$ is said to be ε–general concave if for some $\varepsilon > 0$ there is a function $g : f(D)^2 \to \mathrm{R}$ such that

$$\min \{ f(x), f(y), g(f(x), f(y)) \} + \varepsilon \le f(\lambda x + (1-\lambda) y) \tag{37}$$

for all $x, y \in D$ and for all $\lambda \in [0,1]$. For $\varepsilon = 0$ this definition reduces to that of general concave function. If f is general convex and general concave function then f is general inner function.

7.1. An Extremal Principle

Let X be a Banach space and let $f : M \to \mathrm{R} \cup \{+\infty\}$ be a map from a closed convex set M in X, and set

$$M_t := \{x \in M \mid \max \{ f(x), g(f(x), f(x)) \} \le t\}$$

where $g : f(M)^2 \to \mathrm{R}$ is a continuous function. The map f is called g–lower semicontinuous if the set M_t is closed for all $t \in \mathrm{R}$; and the map f is called g–quasi convex if the set M_t is convex for all $t \in \mathrm{R}$.

Let $f : M \to \mathrm{R}$ be continuous and convex on a closed, convex set M in the Banach space X. Then the map f is both g–lower semicontinuous and g–quasi convex. Indeed, the map f is g–lower semicontinuous, for if (x_n) is a sequence in M_t such that $x_n \to x$ as $n \to \infty$, then $x \in \mathrm{M}$. The map f is also g–quasi convex, for if $x, y \in M_t$, then for all $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda) y) \le \max \{ f(x), f(y), g(f(x), f(y)) \} \le t,$$

which means that $\lambda x + (1-\lambda) y \in M$, so that M_t is convex.

Lemma 5. (Extremal Principle). Let X be a reflexive Banach space, and let M be a nonempty, closed, bounded and convex set in X. If $f : M \to \mathrm{R} \cup \{+\infty\}$ is a g–lower semicontinuous and g–quasiconvex function, than f has a minimum on M.

Proof. The set M is weakly compact, because M is bounded, closed and convex set in reflexive Banach space X. Further, M_t is closed and convex, and hence weakly closed. Therefore f is g–lower semicontinuous in the weak topology on M. The conclusion now follows from the Weierstrass theorem.

7.2. Approximately General Convex Functions

The classical stability theorem of Hyers and Ulam [25] states that, if $f : D \to \mathrm{R}$ is an ε–convex function, where D is a convex subset of R^n, then there exists a convex function $\eta : D \to \mathrm{R}$ such that $\eta(x) \le f(x) \le \eta(x) + k_n \varepsilon$ for all $x \in D$, where $k_n = 1 + (n-1)(n+2) / 2(n+1)$, i.e., where the constant k_n depends only on the dimension of the domain. Also, Nikodem [45] has proved a stability theorem of Hyers–Ulam type for quasiconvex functions.

In this section we prove an analogous statement for general convex functions. Assume that $f : D \to \mathrm{R}$ is an ε–general convex function and consider the level sets

$$L_a := \{x \in D \mid \max\{ f(x), g(f(x), f(x)) \} \le a\},$$

for $a \in \mathrm{R}$ where $g : f(D)^2 \to \mathrm{R}$. It is clear that $\bigcup_{a \in R} L_a = D$, and for $a \le b$ we have $L_a \subseteq L_b$.

We are now in a position to formulate the main statement.

THEOREM 13. If a function $f : D \to \mathrm{R}$ is ε–general convex function, then there exists a general convex function $\eta : D \to \mathrm{R}$ such that

$$\eta(x) \le f(x) \le \eta(x) + \varepsilon k(n),$$

for all $x \in \mathrm{D}$, where $k(n) := [\log_2{}^n] + 1$ and D is a convex subset of R^n.

We note, if $x_1, \dots, x_{m+1} \in L_a$ for $a \in \mathrm{R}$; $m \in \mathrm{N}$, where $\alpha_1, \dots, \alpha_{m+1} \in [0,1]$ and $\alpha_1 + \dots + \alpha_{m+1} = 1$, then $\alpha_1 x_1 + \dots + \alpha_{m+1} x_{m+1} \in L_{a + \varepsilon k(n)}$ where $k(m) = [\log_2{}^n] + 1$.

PROOF. By induction we can show that $\alpha_1 x_1 + \dots + \alpha_{2^s} x_{2^s} \in L_{a + \varepsilon s}$ for all $s \in \mathrm{N}$ and assume that $x_1, \dots, x_m \in D$, $\alpha_1, \dots, \alpha_m \in [0,1]$ and $\alpha_1 + \dots + \alpha_m = 1$. Take the minimal $s \in \mathrm{N}$ such that $m + 1 \le 2^s$. One can easily check that $s = [\log_2 m] + 1 := k(m)$. In the case $m + 1 < 2^s$, let us put $\alpha_{m+2} = \dots = \alpha_{2^s} = 0$ and $x_{m+2} = \dots = x_{2^s} := x_1$. Then by the preceding fact, we obtain $\alpha_1 x_1 + \dots + \alpha_{m+1} x_{m+1} = \alpha_1 x_1 + \dots + \alpha_{2^s} x_{2^s} \in L_{a + \varepsilon k(m)}$ and the proof is complete.

PROOF OF THEOREM 12. By the Caratheodory theorem ([49]) $x = \alpha_1 x_1 + \dots + \alpha_{n+1} x_{n+1}$ for some $x_1, \dots, x_{n+1} \in L_a$, and $\alpha_1, \dots, \alpha_{n+1} \in [0,1]$ with $\alpha_1 + \dots + \alpha_{n+1} = 1$, where we take an $a \in \mathrm{R}$ such that x is included in convex hull of L_a i.e., $x \in \operatorname{conv} L_a$. From the preceding facts we get $x \in L_{a + \varepsilon k(n)}$, which means that $f(x) \le a + \varepsilon k(n)$ for every $a \in \mathrm{R}$. Since this inequality holds for every $a \in \mathrm{R}$ such that $x \in L_a$, we have also the following inequality

$$f(x) \le \inf\{a \in \mathrm{R} \mid x \in \operatorname{conv} L_a\} + \varepsilon k(n). \tag{38}$$

Let us define a function $\eta : D \to \mathrm{R}$ putting $\eta(x) := \inf\{a \in \mathrm{R} \mid x \in \operatorname{conv} L_a\}$, for $x < D$. By (38) we obtain $f(x) \le \eta(x) + \varepsilon k(n)$, for all $x \in D$. Since $\{a \in \mathrm{R} \mid x \in L_a\} \subset \{a \in \mathrm{R} \mid x \in \operatorname{conv} L_a\}$, we have $h(x) \le \inf\{a \in \mathrm{R} \mid x \in L_a\} = f(x)$, for all $x \in D$. Now we shall show that η is a general convex function. Assume that $\eta(x) \le \eta(y)$ for $x, y \in D$. Take an arbitrary $t > \max\{ \eta(y), g(\eta(y), \eta(y)) \}$. Then, by the definition of $\eta(y)$, there exists and $a < t$ such that $y \in \operatorname{conv} L_a$. Also, then $y \in \operatorname{conv} L_t$, because $L_a \subset L_t$. Analogously we show that $y \in \operatorname{conv} L_t$. Hence $\lambda x + (1-\lambda) y \in \operatorname{conv} L_t$ for every $\lambda \in [0,1]$. Since this relation holds for all $t > \max\{ f(y), g(f(y), f(y)) \}$, we obtain

$$\eta(\lambda x + (1-\lambda) y) = \inf\{t \in \mathrm{R} \mid \lambda x + (1-\lambda) y \in \operatorname{conv} L_t\} \le$$
$$\eta(y) \le \max\{ \eta(y), g(\eta(y), \eta(y)) \}.$$

This shows that η is a general convex function and the proof is complete.

8. MISCELLANEOUS RESULTS AND PROBLEMS

This part is given in the form of exercises at the end of the paper. It describes extensions and related developments of the theory and indicates further applications not treated in the text.

8.1. NEW MATHEMATICAL GAMES

Game theory is a mathematical search for the optimal balance of conflicting interests, such as between two partners. As such, it is applicable to a wide variety of situations: social games, economic competition between organizations, conflicts in nature, and so on. The optimal strategies for both partners turn out to be described by saddle points, whose existence we established in Section 4. This key observation goes back to John von Neumann [41].

In new mathematical games, the optimal strategies for both partners turn out to be described by transversal points, whose existence we established in Section 6.

Let $(P, \leq)$ be a totally ordered set and let $g : P^2 \to P$ be a decreasing mapping. We consider two players, A and B. Players A and B have available sets of strategies $X \subset P$ and $Y \subset P$, respectively. Each point $x \in X$ and $y \in Y$ represents a possible choice by A and B, respectively. If A chooses x, and B chooses y, than the function $(x, y) \to \max\{x, y, g(x,y)\}$ represents the gain by A and the function $(x, y) \to \min\{x, y, g(x,y)\}$ represents gain by B. The point $\zeta \in P$ is called an optimal strategy if the following equality holds

$$\zeta := \max_{x\in X, y\in Y} \min\{x, y, g(x,y)\} = \min_{x\in X, y\in Y} \max\{x, y, g(x,y)\}.$$

We note that the existence of optimal strategy is established in Corollary 1.

On the other hand, we can consider the following game with the decreasing function $g(x, y)$. In this sense, the point $\zeta \in P$ solves the game if the following equality holds

$$\zeta := \max_{\zeta \leq x} \min_{y \leq \zeta} g(x,y) = \min_{y\leq\zeta} \max_{\zeta\leq x} g(x,y).$$

The existence of optimal strategy we established in Theorem 5.

Finally, let $(P, \leq)$ be a totally ordered set, let $g : P^k \to P$ (k is a fixed positive integer) be a decreasing function and consider players $A_1, \dots, A_k$ with sets of strategies $X_1, \dots, X_k$ in P, respectively. Each point $\lambda_1 \in X_1, \dots, \lambda_k \in X_k$ represents a possible choice by $A_1, \dots, A_k$, respectively. The point $\zeta \in P$ is called an optimal strategy, in this case, if the following equality holds

$$\zeta := \max_{\lambda_1\in X_1,\dots,\lambda_k\in X_k} \min\{\lambda_1, \dots, \lambda_k, g(\lambda_1, \dots, \lambda_k)\} = \min_{\lambda_1\in X_1,\dots,\lambda_k\in X_k} \max\{\lambda_1, \dots, \lambda_k, g(\lambda_1, \dots, \lambda_k)\}.$$

We notice that the existence of optimal strategy we established in Theorem 2, case (16).

8.2. General Transversal Points

In connection with former facts on transversal points, we have the following extensions. A map $f: X \to P$ (X is an arbitrary nonempty set and P is a poset) has a general transversal point $\zeta \in P$ if there is a decreasing function $g: P^k \to P$ (k is a fixed positive integer) such that the following equality holds

$$\max_{x_1,\dots x_k \in P} \min\left\{ f(x_1),\dots,f(x_k), g\big(f(x_1),\dots,f(x_k)\big)\right\} =$$
$$\min_{x_1,\dots x_k \in P} \max\left\{ f(x_1),\dots,f(x_k), g\big(f(x_1),\dots,f(x_k)\big)\right\} \ := \zeta \tag{39}$$

From the second section, i.e., from Theorem 2, case (16), we obtain that the function $f: X \to P$ has a general transversal point if and only if $f(t_1) = \dots = f(t_k) := \zeta = g(\zeta, \dots, \zeta)$ for some $t_1, \dots, t_k \in P$.

On the other hand, a map $f: X \to P$ (X is an arbitrary nonempty set and P is a poset) has a quasi general transversal point $\zeta \in P$ if there is a function $g: P^k \to P$ (k is a fixed positive integer) such that the equality (39) holds. Also, from the former results, we obtain that the function $f: X \to P$ has a quasi general transversal point if and only if the following equality holds

$$\begin{aligned}&\min\left\{ f(x_1),\dots,f(x_k), g\big(f(x_1),\dots,f(x_k)\big)\right\} \le \\ &f(t_1) = \dots = f(t_k) \ := \zeta = g(\zeta,\dots,\zeta) \le \\ &\max\left\{ f(x_1),\dots,f(x_k), g\big(f(x_1),\dots,f(x_k)\big)\right\}\end{aligned} \tag{40}$$

for some $t_1, \dots, t_k \in P$ and for all $x_1, \dots, x_k \in P$.

If the function $g: P^k \to P$ (k is a positive integer) is a decreasing function with the property $f(t_1) = \dots = f(t_k) := \zeta = g(\zeta, \dots, \zeta)$ for some $t_1, \dots, t_k \in P$, then, from Theorem 2 (case (16)) and Lemma 1, we obtain that the preceding condition (40) holds.

8.3. Roots of Algebraic Equations

We note that, by the application of Lemma 1 (in fact of (4), i.e., (10)), one can simultaneously obtain the upper and lower bounds of the roots of the equation

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n \quad \left(a_1,\dots,a_n \ge 0, \ \sum_{i=1}^{n} a_i > 0 \right). \tag{41}$$

As an immediate consequence of Theorem 2, case (16), we obtain the following statement.

Theorem 14. A point $\zeta \in \mathrm{R}_+ := (0, +\infty)$ is the root of equation (41) if and only if the following equality holds

$$\xi = \max_{\lambda_2,\ldots,\lambda_n \in \mathrm{R}_+} \min\left\{ \lambda_2, \ldots, \lambda_n, a_1 + \frac{a_2}{\lambda_2} + \ldots + \frac{a_n}{\lambda_n^{n-1}} \right\}$$

$$\min_{\lambda_2,\ldots,\lambda_n \in \mathrm{R}_+} \max\left\{ \lambda_2, \ldots, \lambda_n, a_1 + \frac{a_2}{\lambda_2} + \ldots + \frac{a_n}{\lambda_n^{n-1}} \right\}$$

In connection with the preceding facts about transversal points, from Theorem 14, we obtain that the equation (41) has a root $\zeta \in \mathrm{R}_+$ if and only if the point ζ is a general transversal point of the function $f(x) = \mathrm{id}_{\mathrm{R}} : \mathrm{R} \to \mathrm{R}$.

PROOF. From Theorem 2, case (16), we may choose the function $g : P^n \to P$ $(n \in \mathrm{N})$ for $P := \mathrm{R}_+$, defined by

$$g(x_1, \ldots, x_n) = a_1 + \frac{a_2}{x_2} + \ldots + \frac{a_n}{x_n^{n-1}} \quad \text{for } x_1, \ldots, x_n \in \mathrm{R}_+.$$

Applying Theorem 2, case (16), we obtain directly the preceding equality for positive root of equation (41).

THEOREM 15. Let $I_1, \ldots, I_n$ be indices sets and $\Theta_{i_j} \geq 0$ be real numbers which satisfy the following condition

$$\sum_{i_j \in I_j} \Theta_{i_j} = j - t \qquad \text{for } j = 1, \ldots, n \text{ and } 0 < t < 1.$$

Then $\xi \in \mathrm{R}_+$ is the root of the algebraic equation $x^t = a_1 x^{t-1} + \ldots + a_n x^{t-n}$ ($(a_1, \ldots, a_n) \neq (0, \ldots, 0)$) if and only if the following equality holds

$$\max_{M_{i_j}} \min \left\{ M_{i_j}, \left(\sum_{j=1}^{n} \frac{a_j}{\prod_{i_j \in I_j} M_{i_j}^{\Theta_{i_j}}} \right)^{1/t} \right\} = \min_{M_{i_j}} \max \left\{ M_{i_j}, \left(\sum_{j=1}^{n} \frac{a_j}{\prod_{i_j \in I_j} M_{i_j}^{\Theta_{i_j}}} \right)^{1/t} \right\} := \xi.$$

PROOF. In order to prove this statement we may choose the function $g : \mathrm{R}^n_+ \to \mathrm{R}_+$ (n is a positive integer) defined by

$$g(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} \frac{a_j}{\prod_{i_j \in I_j} x_{i_j}^{\Theta_{i_j}}} \right)^{1/t}, \quad \text{for } x_1, \ldots, x_n \in \mathrm{R}_+$$

and then apply Theorem 2, case (16).

8.4. METHOD FOR INEQUALITIES

In this part we give a method for proving convex type inequalities. We consider our general method on the following inequality.

HOLDER'S INEQUALITY. Let $x, y \in \mathrm{R}_+ \cup \{0\}$ and $p > 1$ such that $1/p + 1/q = 1$. Then the following inequality holds

$$x^{1/p} y^{1/q} \le \frac{x}{p} + \frac{y}{q}, \text{ where equality holds if and only if } x = y. \tag{42}$$

METHOD OF PROOF. In Theorem 2 or Lemma 1, we may choose for function $g : P \to P$, $P = \mathrm{R}+$, the following function $g(t) = \left(x^{1/p} y^{1/q}\right)^2 \big/ t$ where $\zeta = x^{1/p} y^{1/q}$ is a fixed point of mapping g. Then, from Lemma 1 we have

$$\zeta = x^{1/p} y^{1/q} \le \max\{\lambda, g(\lambda)\} \quad \text{for arbitrary } \lambda \in \mathrm{R}_+. \tag{43}$$

Thus, for $\lambda = x/p + y/q$ we have two cases. First, if $\max\{\lambda, g(\lambda)\} = x/p + y/q$, then inequality (42) holds. If not, then from (43) we have

$$\zeta = x^{1/p} y^{1/q} \le g\left(\frac{x}{p} + \frac{y}{q}\right) = \left(x^{1/p} y^{1/q}\right)^2 \bigg/ \left(\frac{x}{p} + \frac{y}{q}\right).$$

But, this is not possible by Theorem 2, since g is a decreasing function, which tends to zero. Thus, the inequality (42) holds.

REMARK. The method of the preceding proof of Holder's inequality can be used for all convex type inequalities (Hadamard, Jansen, Abel, Holder, Canchy).

8.5. DOUBLE LIMIT CONDITION

In the following, let X and Y be two nonempty sets and let

$$a : X \times Y \to [-\infty, \alpha], \quad b : X \times Y \to [\beta, +\infty] \quad (\alpha, \beta \in \mathrm{R}),$$

be functions on the Cartesian product $X \times Y$ into extended reals [32]. If for all sequences (x_m) in X and (y_n) in Y

$$\limsup_{n\to\infty} \liminf_{m\to\infty} a(x_m, y_n) \le \liminf_{m\to\infty} \limsup_{n\to\infty} b(x_m, y_n)$$

holds, then (X, Y, a, b) is said to satisfy the "double limit condition" (DLC for short, see [33]).

Let P_X (P_Y) denote the set of all probability measures on $X(Y)$ with finite support. We extend a and b on $P_X \times P_Y$ according to

$$a(p, q) = \int_{X \times Y} a \, dp \otimes q \quad \text{and} \quad b(p, q) = \int_{X \times Y} b \, dp \otimes q.$$

As an immediate consequence of Theorem 6, we obtain the following statement.

COROLLARY 8. ([32]). (X, Y, a, b) satisfies the DLC if and only if the following inequality holds

$$\inf_{q \in P_T} \sup_{p \in P_S} a(p, q) \le \sup_{p \in P_S} \inf_{q \in P_T} b(p, q)$$

for all nonempty subsets $S \subset X$ and $T \subset Y$.

We note, that Kindler gets the following Ptak' s combinatorial lemma as a special case, see [32].

Corollary 9. ([48]). Let F be a nonempty system of subsets of an infinite set Y. Then there is an infinite $T \subset Y$ such that

$$\inf_{q \in P_T} \sup_{G \in F} q(G) > 0$$

if and only if there exists sequences (G_m) in F and (y_n) in Y such that y_n are distinct and $\{y_1, \ldots, y_m\} \in G_m$ for every $m \in \mathrm{N}$.

8.6. An Illustrative Example

From the proof of Corollary 3 it is easy to see that if f satisfies the condition (28) then f also satisfies the condition (33).

Now we shall give an example which shows that there exists such mapping which does not satisfy (28), but satisfies the condition (33). Let $f(x,y)$ be a mapping defined by $f(x,y) = x$ for all $x, y \in \mathrm{R}$, where R denotes the real line. The mapping f does not satisfy (28) on whole R, but it satisfies (33) for every decreasing mapping $g : \mathrm{R}^2 \to \mathrm{R}$ with property $g(\xi, \xi) = \xi$.

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