Yugoslav Journal of Operations Research
26 (2016), Number 3, 381–392
DOI: 10.2298/YJOR140820021H

A NEW APPROACH TO THE LOGISTIC MODELING POPULATION HAVING HARVESTING FACTOR

Chao-Pao HO
Department of Applied Mathematics, Tunghai University Taichung, Taiwan
cpho@thu.edu.tw

Che-Hao LIN
Department of Applied Mathematics, Tunghai University, Taichung, Taiwan
linch@thu.edu.tw

Received: August 2014 / Accepted: May 2015

Abstract: The present paper deals with the logistic equation having harvesting factor by a new approach. This approach is an attempt to sketch the solutions to the model Rahmani Doust and Saraj [1].

Keywords: Equilibrium Points, Logistic Growth Model, Harvesting Model.

MSC: 34D23, 37N25, 78A70, 92D25, 92D40.

1. INTRODUCTION

M.H. Rahmani Doust and M. Staraj [1] solved, and obtained the solution $x(t)$ of the logistic equation having harvesting factor

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - h \tag{1.1}$$

They analyzed the solution of the system (1.1). However, the solution $x(t)$ of (1.1)is an implicit function, and the graph of the solution $x(t)$ with respect to t is not sketched.

We are trying to sketch the graph of the solution $x(t)$ with respect to t by a new approach. That is, we will sketch the solution $x(t)$ with respect to t directly. In general, before sketching the graph of the solution $x(t)$ with respect to t, we need to solve and obtain the solution $x(t)$ of (1.1). Then, we can use the data of $\dot{x}(t)$, $\ddot{x}(t)$ and $\lim_{t\to\infty} x(t)$ to sketch the graph of $x(t)$. Actually, now we know that

$$\dot{x}(t) = rx\left(1 - \frac{x}{K}\right) - h$$

and

$$\ddot{x}(t) = -\frac{2r}{K}\left(x - \frac{K}{2}\right)\dot{x}. \tag{1.2}$$

So, we can use the information (1.1) and (1.2) to sketch the graph of the solution $x(t)$ with respect to t even though we don't know the solution $x(t)$ of (1.1).

This paper is organized as follows. In section 2, we use the new approach to analyze the logistic population. In section 3, we use the new approach to analyze the Logistic equation having constant harvesting. In section 4, we use the new approach to analyze the logistic equation having variable (Holling type I) harvesting. Finally, we use the new approach to analyze the logistic equation having variable (Holling type II) harvesting.

2. THE LOGISTIC POPULATION

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \quad x(0) = x_0. \tag{2.1}$$

The equilibrium points of system (2.1) are $x = 0$ and $x = K$.

Lemma 2.1. If x is the solution of system (2.1), then
(a) x is increasing and concave upward, if $0 < x(0) < \frac{K}{2}$.
(b) x is increasing and concave downward, if $\frac{K}{2} < x(0) < K$.
(c) x is decreasing and concave upward, if $x(0) > K$.

Proof. From (2.1), we see that $\dot{x} > 0$ if $0 < x(0) < K$ and $\dot{x} < 0$ if $x(0) > K$. By direct calculation of the second order, derivatives may be found as follows:

$$\begin{aligned} \ddot{x} &= r\dot{x}\left(1 - \frac{x}{K}\right) + rx\left(-\frac{1}{K}\right)\dot{x} \\ &= r\dot{x}\left(1 - \frac{2x}{K}\right). \end{aligned} \tag{2.2}$$

We know from (2.1) and (2.2) that $\ddot{x} > 0$ if $0 < x < \frac{K}{2}$, $\ddot{x} < 0$ if $\frac{K}{2} < x < K$ and $\ddot{x} > 0$ if $x > K$. The proof is completed. □

Therefore, by Fundamental Existence-Uniqueness Theorem [2] and Lemma 2.1, we have the following theorem and the trajectory of system (2.1), shown in Figure 2.1.

Theorem 2.2. The equilibrium point $x = K$ of Logistic system (2.1) is asymptotically stable and the equilibrium point $x = 0$ is unstable.

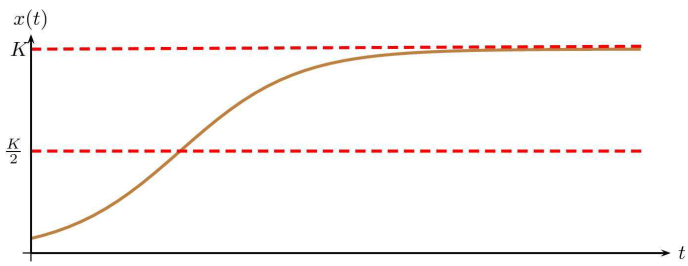

Figure 2.1: The trajectory of system (2.1).

3. HAVING CONSTANT HARVESTING

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - h, \quad x(0) = x_0. \tag{3.1}$$

Case I. $0 < h < \dfrac{rK}{4}$.

The equilibrium points of system (3.1) are

$$\bar{x} \equiv \frac{K - \sqrt{K^2 - \frac{4Kh}{r}}}{2} \quad \text{and} \quad \hat{x} \equiv \frac{K + \sqrt{K^2 - \frac{4Kh}{r}}}{2}.$$

Lemma 3.1. If $0 < h < \dfrac{rK}{4}$ and x is the solution of system (3.1), then
(a) x is decreasing and concave downward, if $0 < x(0) < \bar{x}$.
(b) x is increasing and concave upward, if $\bar{x} < x(0) < \frac{K}{2}$.
(c) x is increasing and concave downward, if $\frac{K}{2} < x(0) < \hat{x}$.
(d) x is decreasing and concave upward, if $x(0) > \hat{x}$.

Proof. Since $\bar{x}$ and $\hat{x}$ are the roots of $rx(1 - \frac{x}{K}) - h = 0$, we get

$$\dot{x} = -\frac{r}{K}(x - \bar{x})(x - \hat{x})$$

and

$$\ddot{x} = -\frac{r}{K}[2x - (\bar{x} + \hat{x})]\dot{x} = -\frac{2r}{K}\left(x - \frac{K}{2}\right)\dot{x}$$

It is easy to check that

	$0 < x < \bar{x}$	$\bar{x} < x < \hat{x}$	$x > \hat{x}$
$\dot{x}$	$-$	$+$	$-$
	decreasing	increasing	decreasing

and

	$0 < x < \bar{x}$	$\bar{x} < x < \frac{K}{2}$	$\frac{K}{2} < x < \hat{x}$	$x > \hat{x}$
$\ddot{x}$	$-$	$+$	$-$	$+$
Concavity	downward	upward	downward	upward

The proof is completed. □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 3.1, we have the following theorem and the trajectory of system (3.1), shown in Figure 3.1.

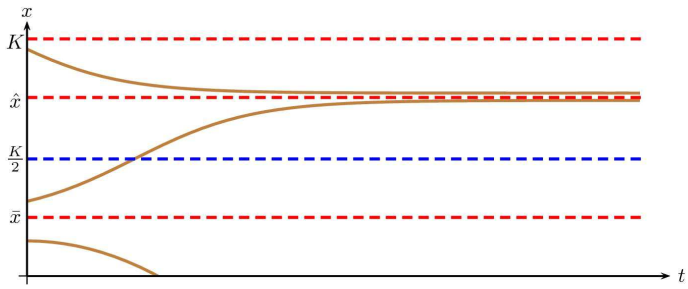

Figure 3.1: The trajectory of system (3.1) with $0 < h < \dfrac{rK}{4}$.

Theorem 3.2. Let $0 < h < \dfrac{rK}{4}$ and x be the solution of system (3.1).
(a) If $\hat{x} < x_0 < K$ or $x_0 \in (\bar{x}, \hat{x})$, then $\lim_{t\to\infty} x(t) = \hat{x}$.
(b) If $0 < x_0 < \bar{x}$, then there is $t_0 > 0$ such that $x(t_0) = 0$.

Case II. $h = \dfrac{rK}{4}$

The equilibrium point of system (3.1) is $x^* = \frac{K}{2}$.

Lemma 3.3. If $h = \dfrac{rK}{4}$ and x is the solution of system (3.1), then
(a) x is decreasing and concave downward, if $0 < x(0) < \frac{K}{2}$.
(b) x is decreasing and concave upward, if $x(0) > \frac{K}{2}$.

Proof. By direct calculation

$$\dot{x} = -\frac{r}{K}\left(x - \frac{K}{2}\right)^2$$

and

$$\ddot{x} = -\frac{2r}{K}\left(x - \frac{K}{2}\right)\dot{x}$$

It implies x is decreasing for all $x \in (0, \frac{K}{2}) \cup (\frac{K}{2}, \infty)$ and

	$0 < x < \frac{K}{2}$	$x > \frac{K}{2}$
$\ddot{x}$	$-$	$+$
Concavity	downward	upward

The proof is completed. □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 3.3, we have the following theorem and the trajectory of system (3.1), shown in Figure 3.2.

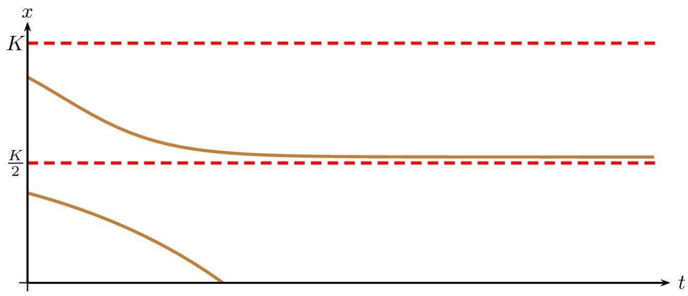

Figure 3.2: The trajectory of system (3.1) with $h = \dfrac{rK}{4}$.

Theorem 3.4. Let $h = \dfrac{rK}{4}$ and x be the solution of system (3.1).
(a) If $\frac{K}{2} < x_0 < K$, then $\lim_{t\to\infty} x(t) = \frac{K}{2}$.
(b) If $x_0 < \frac{K}{2}$, then there is $t_0 > 0$ such that $x(t_0) = 0$.

Case III. $h > \dfrac{rK}{4}$

There is no equilibrium point of system (3.1).

Lemma 3.5. If $h > \dfrac{rK}{4}$ and x is the solution of system (3.1), then
(a) x is decreasing and concave downward, if $0 < x < \frac{K}{2}$.
(b) x is decreasing and concave upward, if $x > \frac{K}{2}$.

Proof. By direct calculation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - h < 0$$

and

$$\ddot{x} = -\frac{2r}{K}\left(x - \frac{K}{2}\right)\dot{x}$$

It implies x is decreasing for all $x \in (0, \infty)$ and

	$0 < x < \frac{K}{2}$	$x > \frac{K}{2}$
$\ddot{x}$	$-$	$+$
Concavity	downward	upward

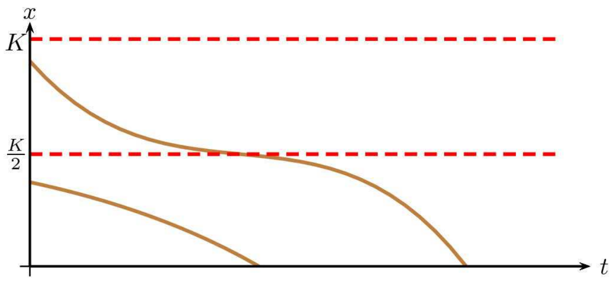

Figure 3.3: The trajectory of system (3.1) with $h > \dfrac{rK}{4}$.

This completes the proof. □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 3.5, we have the following theorem and the trajectory of system (3.1), shown in Figure 3.3.

Theorem 3.6. Let $h > \dfrac{rK}{4}$ and x be the solution of system (3.1). If $0 < x_0 < K$, then there is $t_0 > 0$ such that $x(t_0) = 0$.

4. HAVING VARIABLE (HOLLING TYPE I) HARVESTING

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - hx, \quad x(0) = x_0. \tag{4.1}$$

Case I. $r - 1 < h < r$

The equilibrium points of system (4.1) are 0 and

$$\tilde{x} \equiv \frac{K}{r}(r - h).$$

Lemma 4.1. If $r - 1 < h < r$ and x is the solution of system (4.1), then
(a) x is increasing and concave upward, if $0 < x(0) < \frac{\tilde{x}}{2}$.
(b) x is increasing and concave downward, if $\frac{\tilde{x}}{2} < x(0) < \tilde{x}$.
(c) x is decreasing and concave upward, if $x(0) > \tilde{x}$.

Proof. We get

$$\dot{x} = -\frac{rx}{K}(x - \tilde{x})$$

and

$$\ddot{x} = -\frac{2r}{K}\left(x - \frac{\tilde{x}}{2}\right)\dot{x}$$

It implies

	$0 < x < \tilde{x}$	$x > \tilde{x}$
$\dot{x}$	+	−
	increasing	decreasing

and

	$0 < x < \frac{\tilde{x}}{2}$	$\frac{\tilde{x}}{2} < x < \tilde{x}$	$x > \tilde{x}$
$\ddot{x}$	+	−	+
Concavity	upward	downward	upward

The proof is completed. □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 4.1, we have the following theorem and the trajectory of system (4.1), shown in Figure 4.1.

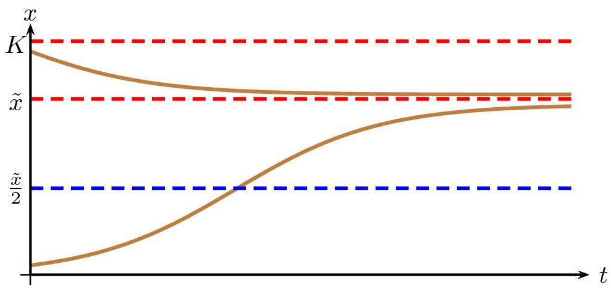

Figure 4.1: The trajectory of system (4.1) with $r - 1 < h < r$.

Theorem 4.2. Let $r - 1 < h < r$ and x be the solution of system (4.1). If $0 < x_0 < \tilde{x}$ or $\tilde{x} < x_0 < K$, then $\lim_{t\to\infty} x(t_0) = \tilde{x}$.

Case II. $h = r$

The equilibrium point of system (4.1) is $x^* = 0$.

Lemma 4.3. If x is the solution of system (4.1), then x is decreasing and concave upward for all $x > 0$.

Proof. We have

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - hx = -\frac{r}{K}x^2$$

and

$$\ddot{x} = -\frac{2r}{K}x\dot{x}$$

It implies for all $x > 0$, $\dot{x} < 0$ and $\ddot{x} > 0$. The proof is completed. □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 4.3, we have the following theorem and the trajectory of system (4.1), shown in Figure 4.2.

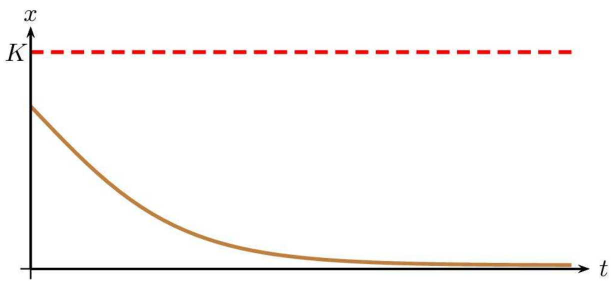

Figure 4.2: The trajectory of system (4.1) with $h = r$.

Theorem 4.4. Let $h = r$ and x be the solution of system (4.1). If $0 < x_0 < K$, then $\lim_{t\to\infty} x(t) = 0$.

Case III. $h > r$

The equilibrium point of system (4.1) is $x^* = 0$.

Lemma 4.5. If x is the solution of system (4.1), then x is decreasing and concave upward for all $x > 0$.

Proof. We have

$$\dot{x} = -\frac{rx}{K}\left[x + \frac{K}{r}(h-r)\right],$$

and

$$\ddot{x} = -\frac{2r}{K}\left[x + \frac{K}{2r}(h-r)\right]\dot{x}$$

It implies for all $x > 0$, $\dot{x} < 0$ and $\ddot{x} > 0$. The proof is completed. □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 4.5, we have the following theorem and the trajectory of system (4.1), shown in Figure 4.3.

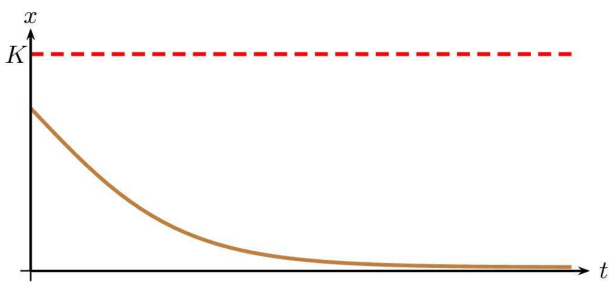

Figure 4.3: The trajectory of system (4.1) with $h = r$.

Theorem 4.6. Let $h > r$ and x be the solution of system (4.1). If $0 < x_0 < K$, then $\lim_{t\to\infty} x(t) = 0$.

5. HAVING VARIABLE (HOLLING-TYPE II) HARVESTING

In this type, we assume that the carrying capacity $K = 1$.

$$\dot{x} = rx(1-x) - \frac{hx}{1+x}, \quad x(0) = x_0. \tag{5.1}$$

Case I. $h < r$.

The equilibrium points of system (5.1) are 0 and

$$x^* \equiv \sqrt{1 - \frac{h}{r}}$$

Lemma 5.1. If x is the solution of system (5.1) and $h < r$, then there is a number x^{**} in $(0, x^*)$ such that
(a) x is increasing and concave upward for all $x(0) \in (0, x^{**})$.
(b) x is increasing and concave downward for all $x(0) \in (x^{**}, x^*)$.
(c) x is decreasing and concave upward for all $x(0) \in (x^*, \infty)$.

Proof. From (5.1),

$$\begin{aligned} \dot{x} &= \frac{x}{1+x}[r(1-x)(1+x) - h] \\ &= -\frac{rx}{1+x}\left[x^2 - \left(1 - \frac{h}{r}\right)\right] \\ &= -\frac{rx}{1+x}(x - x^*)(x + x^*) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \ddot{x} &= \frac{-r\dot{x}}{(1+x)^2}(x - x^*)(x + x^*) - \frac{2rx^2\dot{x}}{1+x} \\ &= -\frac{r\dot{x}}{(1+x)^2}(2x^3 + 3x^2 - x^{*2}) \end{aligned} \tag{5.3}$$

Let $G(x) = 2x^3 + 3x^2 - x^{*2}$, we have $G(0) = -x^{*2} < 0$, $G(x^*) = 2x^{*3} + 2x^{*2} > 0$ and $G'(x) = 6x^2 + 6x > 0$. By the Intermediate theorem and Mean Value Theorem, there is a unique number $x^{**} \in (0, x^*)$ such that $G(x^{**}) = 0$. Hence, there is a quadratic function $H(x)$ such that $H(x) > 0$ and $G(x) = (x - x^{**})H(x)$. Rewrite (5.3), then we have

$$\ddot{x} = -\frac{r\dot{x}}{(1+x)^2}(x - x^{**})H(x) \tag{5.4}$$

From (5.2) and (5.4), it is easy to see

	$0 < x < x^*$	$x > x^*$
$\dot{x}$	+	−
	increasing	decreasing

and

	$0 < x < x^{**}$	$x^{**} < x < x^{*}$	$x > x^{*}$
$\ddot{x}$	+	−	+
Concavity	upward	downward	upward

The proof is completed . □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 5.1, we have the following theorem and the trajectory of system (5.1), shown in Figure 5.1.

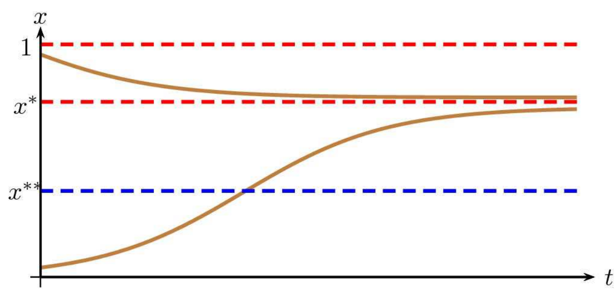

Figure 5.1: The trajectory of system (5.1) with $h < r$.

Theorem 5.2. Let $h < r$ and x be the solution of system (5.1). If $0 < x_0 < x^*$ or $x^* < x_0 < 1$, then $\lim_{t\to\infty} x(t_0) = x^*$.

Case II. $h = r$.

The equilibrium point of system (5.1) is $x^* = 0$.

Lemma 5.3. If x is the solution of system (5.1) and $h = r$, then x is decreasing and concave upward for all $x > 0$.

Proof. From (5.1), we obtain for all $x > 0$

$$\begin{aligned}\dot{x} &= rx\left(1 - x - \frac{1}{1+x}\right)\\ &= -\frac{rx^3}{1+x}\\ &< 0\end{aligned}$$

and

$$\begin{aligned}\ddot{x} &= \frac{-3rx^2(1+x)\dot{x} + rx^3\dot{x}}{(1+x)^2}\\ &= -\frac{rx^2(2x+3)\dot{x}}{(1+x)^2}\\ &> 0.\end{aligned}$$

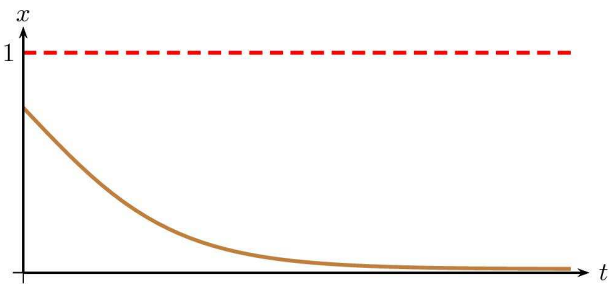

Figure 5.2: The trajectory of system (5.1) with $h = r$.

The proof is completed. □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 5.3, we have the following theorem and the trajectory of system (5.1), shown in Figure 5.2.

Theorem 5.4. Let $h = r$ and x be the solution of system (5.1). If $0 < x_0 < 1$, then $\lim_{t\to\infty} x(t) = 0$.

Case III. $h > r$

The equilibrium point of system (5.1) is $x^* = 0$.

Lemma 5.5. If x is the solution of system (5.1) and $h = r$, then x is decreasing and concave upward for all $x > 0$.

Proof. For all $x > 0$ from (5.1) that

$$\dot{x} = -\frac{rx}{1+x}\left(x^2 + \frac{h}{r} - 1\right)$$

and

$$\begin{aligned}\ddot{x} &= \frac{-r\dot{x}}{(1+x)^2}\left(x^2 + \frac{h}{r} - 1\right) - \frac{2rx^2\dot{x}}{1+x} \\ &= -\frac{r\dot{x}(2x^3 + 3x^2 + \frac{h}{r} - 1)}{(1+x)^2}\end{aligned}$$

Since $h > r$, we have $x^2 + \frac{h}{r} - 1 > 0$ and $2x^3 + 3x^2 + \frac{h}{r} - 1 > 0$ for all $x > 0$. It implies $\dot{x} < 0$ and $\ddot{x} > 0$ for all $x > 0$. The proof is complete. □

By Fundamental Existence-Uniqueness Theorem [2] and Lemma 5.5, we have the following theorem and the trajectory of system (5.1), shown in Figure 5.3.

Theorem 5.6. Let $h > r$ and x be the solution of system (5.1). If $0 < x_0 < 1$, then $\lim_{t\to\infty} x(t) = 0$.

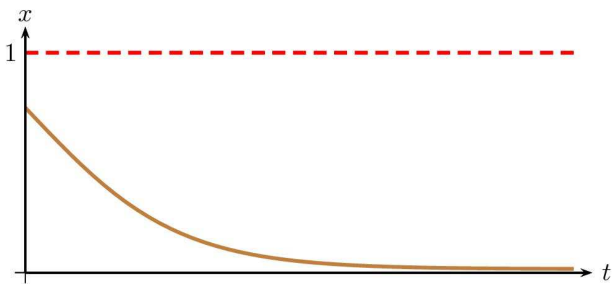

Figure 5.3: The trajectory of system (5.1) with $h > r$.

REFERENCES

[1] Rahmani Doust, M.H., and Saraj, M., "The logistic modeling population; having harvesting factor", *Yugoslav Journal of Operations Research*, 25(1)(2015) 107-115.

[2] Perko, L., *Differential Equations and Dynamical Systems*, second edition. Texts in Applied Mathematics 7, Springer-Verlag New York, 1996. xiv+519 pp. ISBN: 0-387-94778-7.