Yugoslav Journal of Operations Research
3 (1993), Number 1, 3-10

SEARCH GAME ON THE UNION OF N IDENTICAL GRAPHS JOINED AT ONE OR TWO POINTS

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Abstract. We give the conditions under which the value of strategy h on a graph which consists of n identical graphs joined at one or two points, where the hider uses the same strategy h/n on every single graph, is n times greater or equal then the value of strategy h on one graph. This gives the possibility that using the value of the game on one graph, we can obtain the value on the union of more identical graphs.

Key words: search game, graph, zero sum.

AMS(MOS) Subject Classification. 90D05.

1. INTRODUCTION

Let Q be a connected graph which consists of a finite number of arcs and edges in a three dimensional Euclidean space. The edges intersect only at vertices of the graph. Consider the next search game of two players: searcher and hider and the edge O is the starting point of the searcher. The hider chooses any point H on one of the arcs and hides at this point – that is a pure strategy of the hider. A pure strategy of the searcher is a continuous trajectory S with velocity not exceeding 1, i.e. $S:[0,\infty)\to Q$, $d(S(t_2)-S(t_1)) \leq t_2 - t_1$, $0 \leq t_1 < t_2$. The distance $d(z_1,z_2)$ between any points z_1, z_2 is defined as the minimum length among all paths in Q which connect z_1 and z_2. Since the hider is immobile, it is obvious that the searcher will always use the maximum velocity 1. The payoff $c(S,H)$ to the hider is the time spent until the hider is captured. Denote by TS (TH) the set of all pure searcher (hider) strategies. A mixed searcher strategy s is a random choice among his pure

Communicated by V.Kovačević-Vujčić

strategies, while a hider mixed strategy h is probability measure in the graph Q, that is to say a mixed strategy s (h) of the searcher (hider) is a regular Borel probability measure on the set of all pure strategies. If the players use mixed strategies, then the capture time is a random variable, so that each player cannot guarantee a fixed cost but only an expected cost $c(s,h)$. Denote the set of all mixed searcher (hider) strategies by Ts (Th), The maximal expected cost $v(s)$ of using a search strategy s: $v(s) = \sup_{h \in Th} c(s,h) = \sup_{H \in TH} c(s,H)$ will be called "the value of strategy s", and the minimal expected cost $v(h)$ of using a hider strategy h: $v(h) = \inf_{s \in Ts} c(s,h) = \inf_{S \in TS} c(S,h)$ will be called "the value of strategy h. If there exists a real number v which satisfies $v = \inf_{s \in Ts} v(s) = \sup_{h \in Th} v(h)$, then we say that the game has a value v. Gal has proved [4] that any search game on a graph described above has a value and an optimal search strategy.

2. SEARCH GAME ON THE UNION OF N IDENTICAL GRAPHS JOINED AT ONE POINT

Let nQ be a graph which consists of n identical graphs joined only at the starting point O. Let C be the value of strategy h on a single graph Q, i.e. $C = \min_{S \in TS(1)} c(S,h)$, where the minimum takes place over all pure search strategies on one graph $TS(1)$. This minimum is attained because the set $TS(1)$ is compact [4] and $c(S,h)$ is lower–semicontinuous function of S. The hider hides on n graphs using "the same" distribution function $h(x)/n$ on each graph. From now on, for computational convenience, we will take the probability measure on n graphs as n. At the end of calculation, we have to divide the result by n to obtain the right value. In this case the hider uses the probability distribution function $h(x)$ on each graph.

Let the searcher use some pure search strategy S on n graphs. We can represent this trajectory in the following way $S = (\mathcal{P}_1^{\,1}, \mathcal{P}_1^{\,2}, \mathcal{P}_1^{\,3}, \dots, \mathcal{P}_1^{\,n}, \mathcal{P}_2^{\,1}, \mathcal{P}_2^{\,2}, \mathcal{P}_2^{\,3}, \dots, \mathcal{P}_2^{\,n}, \dots, \mathcal{P}_m^{\,1}, \mathcal{P}_m^{\,2}, \mathcal{P}_m^{\,3}, \dots, \mathcal{P}_m^{\,n})$, where the $\mathcal{P}_i^{\,j}$ represents the given i–th part of the trajectory on the j–th graph. Each part $\mathcal{P}_i^{\,j}$ (except the last) begins and ends at the starting point O, because the searcher has to pass through this point when he changes the graph. Denote by $T_i^{\,j}$ the length of the accomplished path which corresponds to the part $\mathcal{P}_i^{\,j}$ and by $P_i^{\,j}$ the total probability disposed on this part of the trajectory which has not been discovered before, namely $P_i^j = \int_{\wp_i^j} dh(x)$ where the integral takes place over undiscovered points. For example, suppose that the hider went along one arc the first time from the point O to the point A(α), which has a distance of α from O along this arc, he came back (denote this individual trajectory by $\wp_\alpha$) and after some time he goes along the same arc the second time from the point O to the point A(β), β > α and comes back, (denote this individual trajectory by $\wp_\beta$, $\wp_\alpha$ and $\wp_\beta$ overlap on the segment between O and A(α) along this arc), then we have $T_\alpha = 2\alpha$, $P_\alpha = \int_{[0,\alpha]} dh(x)$, $T_\beta = 2\beta$ and $P_\beta = \int_{(\alpha,\beta]} dh(x)$. Denote further by $C_{\wp_1^j \wp_2^j \wp_3^j \dots \wp_m^j}$ the mathematical expectation of the time necessary to find an object on the j–th graph if the searcher uses the trajectory := $(\wp_1^j, \wp_2^j, \wp_3^j, \dots, \wp_m^j)$ and the hider strategy h, i.e. $C_{\wp_1^j \wp_2^j \wp_3^j \dots \wp_m^j} = c(\wp_1^j, \wp_2^j, \wp_3^j, \dots, \wp_m^j, h)$.

THEOREM 1. Let us have n identical graphs joined only by the starting point O of the searcher and the hider uses the same probability distribution function $h(x)/n$ on every graph. If the value of strategy h on a single graph is C under condition that $\inf_{\wp_i} T_i/P_i \geq 2C$, where we took infimum over all possible combinations of the parts $\wp_i$ of the trajectories, which begin and end at the point O, then the value of strategy h on nQ is greater or equal to nC.

PROOF. Let us choose any strategy of the searcher :$S = (\wp_1^1, \wp_1^2, \wp_1^3, \dots, \wp_1^n, \wp_2^1, \wp_2^2, \wp_2^3, \dots, \wp_2^n, \dots, \wp_m^1, \wp_m^2, \wp_m^3, \dots, \wp_m^n)$, where the upper index designs the number of the graph. The searcher starts from the point O, moves along the part $\wp_1^1$ of the trajectory S on the first graph, then switches to the second graph and moves along the part $\wp_1^2$, switches to the third graph and moves along the part $\wp_1^3$and so on. It is possible that any individual trajectory $\wp_i^j$ might be empty, with the consequence that $T_i^j = P_i^j = 0$. We have the following equalities:

$$\sum_{i=1}^{m} P_i^j = 1, \qquad j=1,\dots,n \tag{1}$$

$$
\begin{aligned}
c(S,h) = {} & \sum_{i=1}^{m} \int_{\rho_i^1} \left(\sum_{k=1}^{i-1} T_k^{\,1} + \sum_{k=1}^{i-1} T_k^{\,2} + \sum_{k=1}^{i-1} T_k^{\,3} + \ldots + \sum_{k=1}^{i-1} T_k^{\,n} + t \right) dh(t) + \\
& \sum_{i=1}^{m} \int_{\rho_i^2} \left(\sum_{k=1}^{i} T_k^{\,1} + \sum_{k=1}^{i-1} T_k^{\,2} + \sum_{k=1}^{i-1} T_k^{\,3} + \ldots + \sum_{k=1}^{i-1} T_k^{\,n} + t \right) dh(t) + \\
& \sum_{i=1}^{m} \int_{\rho_i^3} \left(\sum_{k=1}^{i} T_k^{\,1} + \sum_{k=1}^{i} T_k^{\,2} + \sum_{k=1}^{i-1} T_k^{\,3} + \ldots + \sum_{k=1}^{i-1} T_k^{\,n} + t \right) dh(t) + \ldots + \\
& \sum_{i=1}^{m} \int_{\rho_i^n} \left(\sum_{k=1}^{i} T_k^{\,1} + \sum_{k=1}^{i} T_k^{\,2} + \sum_{k=1}^{i} T_k^{\,3} + \ldots + \sum_{k=1}^{i-1} T_k^{\,n} + t \right) dh(t) = \\
& C_{\rho_1^1 \rho_2^1 \ldots \rho_m^1} + C_{\rho_1^2 \rho_2^2 \ldots \rho_m^2} + C_{\rho_1^3 \rho_2^3 \ldots \rho_m^3} + \ldots + C_{\rho_1^n \rho_2^n \ldots \rho_m^n} + \\
& \sum_{k=1}^{m} T_k^{\,1} \left(n - 1 - \sum_{i=1}^{k-1} P_i^2 - \sum_{i=1}^{k-1} P_i^3 - \sum_{i=1}^{k-1} P_i^4 - \ldots - \sum_{i=1}^{k-1} P_i^n \right) + \\
& \sum_{k=1}^{m} T_k^{\,2} \left(n - 1 - \sum_{i=1}^{k} P_i^1 - \sum_{i=1}^{k-1} P_i^3 - \sum_{i=1}^{k-1} P_i^4 - \ldots - \sum_{i=1}^{k-1} P_i^n \right) + \ldots + \\
& \sum_{k=1}^{m} T_k^{\,n} \left(n - 1 - \sum_{i=1}^{k} P_i^1 - \sum_{i=1}^{k} P_i^2 - \sum_{i=1}^{k} P_i^3 - \ldots - \sum_{i=1}^{k} P_i^{n-1} \right) \geq
\end{aligned}
$$

$$
nC + \quad (\text{because } \forall i = 1,..,m, j = 1,..,n, T_i^{\,j} \geq 2CP_i^{\,j}, \text{except } i = m, j = n)
$$

$$
\begin{aligned}
2C \Bigg\{ & \sum_{k=1}^{m} P_k^{\,1} \left(n - 1 - \sum_{i=1}^{k-1} P_i^2 - \sum_{i=1}^{k-1} P_i^3 - \sum_{i=1}^{k-1} P_i^4 - \ldots - \sum_{i=1}^{k-1} P_i^n \right) + \\
& \sum_{k=1}^{m} P_k^{\,2} \left(n - 1 - \sum_{i=1}^{k} P_i^1 - \sum_{i=1}^{k-1} P_i^3 - \sum_{i=1}^{k-1} P_i^4 - \ldots - \sum_{i=1}^{k-1} P_i^n \right) + \ldots + \\
& \sum_{k=1}^{m} P_k^{\,n} \left(n - 1 - \sum_{i=1}^{k} P_i^1 - \sum_{i=1}^{k} P_i^2 - \sum_{i=1}^{k} P_i^3 - \ldots - \sum_{i=1}^{k} P_i^{n-1} \right) \Bigg\}
\end{aligned}
$$

Let us denote the expression in the brace by $R(n)$. We will prove that $R(n) = n\,(n-1)\,/\,2$ by induction. It is easy to see that $R(2) = 1$. Let us suppose that $R(n-1) = (n-1)\,(n-2)\,/\,2$.

$$
\begin{aligned}
R(n) = {} & R(n-1) + \sum_{i=1}^{m} P_i^1 + \sum_{i=1}^{m} P_i^2 + \sum_{i=1}^{m} P_i^3 + \ldots + \sum_{i=1}^{m} P_i^{n-1} + \\
& \sum_{i=1}^{m} P_i^n \left(n - 1 - \sum_{k=1}^{m} P_k^{\,1} - \sum_{k=1}^{m} P_k^{\,2} - \sum_{k=1}^{m} P_k^{\,3} - \ldots - \sum_{k=1}^{m} P_k^{\,n-1} \right) =
\end{aligned}
$$

(because of (1))

$$
= (n-1)\,(n-2)\,/\,2 + n-1 = n\,(n-1)\,/\,2
$$

Now it follows easily: $c(S,h) \geq n^2\,C$. Since the total measure is n, dividing by n we will obtain the desired result.

3. SEARCH GAME ON THE UNION OF N IDENTICAL GRAPHS JOINED AT TWO POINTS

Let us suppose that we have n identical graphs joined only at two points O and O', where the point O is starting point for the searcher. Let:

$$C = \min\left(\inf_{S\in\overline{TS}_{O}(1)} c(S,h), \inf_{S\in\overline{TS}_{O'}(1)} c(S,h)\right), \tag{2}$$

$\overline{TS}_{O}(1)$ ($\overline{TS}_{O'}(1)$) is the set of all pure searcher trajectories on one graph, where the searcher starts from the point O (O') and it is allowed for him to jump from one end point O or O' to another end point. (3)

We have to take into consideration the strategies where the jumping from one end point to another is allowed, because the graphs are joined only by two points O and O' and the searcher can switch from one graph to another only at these points, which creates unusual strategies.

Suppose that

$$2C \le \min\left(\inf_{\wp_i} T_i/P_i, \inf_{\wp_j} T_j/P_j, \inf_{\wp_k} T_k/P_k\right), \tag{4}$$

where the index i corresponds to the parts of the trajectories where the searches starts from the point O and comes back to O, $\wp_j$ corresponds to the parts of the trajectories where the searcher starts from the O' and comes back to O' and $\wp_k$ corresponds to the parts of the trajectories where the searcher starts from the point O (O') and comes to O' (O).

THEOREM 2. Under upper conditions (2), (3) and (4), the value of strategy h on the union of n identical graphs joined only at two points, where the hider uses the same probability distribution function h/n on every single graph is n times greater or equal than the value of strategy h on a single graph under condition that the searcher can start from the point O and O' and it is allowed for him to jump from one end point to another.

The proof is identical with the proof of Theorem 1.

In the case when a single graph $Q(2)$ has two arcs of equal lengths which join two points O and O' and the hider uses the same strategy $h/2$ on every arc, condition (3) is transformed into (3'):

$\overline{TS}_O(1)$ ($\overline{TS}_{O'}(1)$) is the set of all pure trajectories on the one graph, where the searcher starts from the point O (O') and it is allowed for him to jump from one end point O or O' to another end point, but only after returning to the initial point. (3')

Initial point means end point O or O' ; at the beginning when the searcher starts from the point O (for example), the point O is the starting and the initial point, but after jumping to another end point O', this point O' becomes the initial point and so on. It is not possible that the searcher, who left the point O and arrived at the point O', sets off again from the point O. The searcher, who left the point O, arrived at the point A(α), which has a distance of α from O, returned to the point O can jump from O to another end point O'. Because of the symmetry, we can always choose the arcs in such a way, depending on the searcher trajectory, that is not allowed to leave the point O (O'), arrive at the point O' (O) and start again from O (O').

At first, we eliminate all searcher trajectories where he goes twice along the same arc from one end point to another using the next lemma. Let $Q(k)$ be a set of k nonintersecting arcs b(1), b(2),...,b(k), of equal lengths which join two points O and O'. Then, we have:

LEMMA 1. In the game on $Q(k)$, a pure searcher strategy S where he goes twice along some arc b(i) from one end point to another and then continues moving along some trajectory $\mathcal{T}_S$ is dominated by a pure searcher strategy S', where he follows the strategy S up to the second passing over arc b(i) and, instead to go over arc b(i), chooses the first unexplored arc, goes along it to the second end point and continues moving following $\mathcal{T}_S$.

PROOF. Let H be any pure hider strategy, namely the hider chooses one arc and hides at a point H of it. If the point H is on an arc different from the arc which the searcher has chosen in the strategy S' as the first unexplored arc, then $c(S', H) = c(S,H)$, otherwise $c(S', H) < c(S,H)$.

Let us come back to any "reasonable" strategy S of the searcher on $nQ(2)$. At the moment, we neglect "small" loops (different of OO'O and O'OO') which the searcher makes following S. The searcher arrives at the another end point O' (in general case), comes back to O and so on. The first time, when he makes loop OO'O or O'OO' moving along two arcs, we will say that these arcs form one graph and so on.

4. PRACTICAL USE

We will give one consequence of Theorem 1.

CONSEQUENCE 1. If the symmetrical mixed strategy $h(x)$ of the hider is optimal in the game on the graph $Q(2)$, which contains two arcs of unit length which join two points O and O' (O is the starting point), then the strategy $h(x)/n$ on every graph is optimal in the game on n such graphs joined by the point O.

PROOF. To prove it we will take any strategy S_α, where the searcher starts from O, goes along one arc (equiprobably chosen) to the point A(α) which has a distance of α from O, comes back to O and then goes along the other arc to O' and continues to A(α). The graph $Q(2)$ is Eulerian and the value on it is equal 1 [4].

$$\begin{aligned}
1 &\le c(S_\alpha, h) \quad \text{(because } h(t) \text{ is optimal)}\\
&= \int_{[0,\alpha]} t\,dh(t) + \int_{[0,1]} (2\alpha + t)\,dh(t) + \int_{[\alpha,1]} (2\alpha + 2 - t)\,dh(t)\\
&= \int_{[0,\alpha]} t\,dh(t) + \int_{[0,1]} (2\alpha + t)\,dh(t) +\\
&\quad \int_{[0,1]} (2\alpha + 2 - t)\,dh(t) - \int_{[0,\alpha]} (2\alpha + 2 - t)\,dh(t)\\
&= 2\int_{[0,\alpha]} t\,dh(t) + 2\alpha + 1 - 2\alpha \int_{[0,\alpha]} dh(t) - 2\int_{[0,\alpha]} dh(t) \le
\end{aligned}$$

(because $t \le \alpha$, for $0 \le t \le \alpha$)

$$2\alpha + 1 - 2\int_{[0,\alpha]} dh(t)$$

thus, $2\alpha \big/ \int_{[0,\alpha]} dh(t) \ge 2$, which means that $h(t)$ satisfies the conditions of Theorem 1.

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