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SEARCH GAME ON AN ODD NUMBER OF ARCS WITH IMMOBILE HIDER

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Abstract. The search game for an immobile hider on an odd number of arcs of equal lengths, connecting two points is solved. The result obtained here solves the problem which has been open since 1980.

Key words: search game, graph, zero sum.

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1. INTRODUCTION

Let $Q(k)$ be a set of k nonintersecting arcs $b(1),b(2),...,b(k)$ of the unit length which join two points O and A, and k is an odd number greater than 1. The hider chooses any point H on one of the arcs and hides at this point – that is a pure strategy of the hider. A pure strategy of the searcher is a continuous trajectory S with velocity not exceeding 1 (A is the starting point). Since the hider is immobile it is obvious that the searcher will always use his maximal velocity 1. The payoff $c(S,H)$ to the hider is the time spent until the hider is captured. A mixed searcher strategy s is a random choice among his pure strategies, while a hider mixed strategy h is a probability measure in the graph Q, that is to say a mixed strategy s (h) of the searcher (hider) is a regular Borel probability measure on the set of all pure strategies. If the players use mixed strategies, then the capture time is a random variable and the payoff becomes an expected cost $c(s,h)$. The maximal expected cost $v(s)$ of using a search strategy s : $v(s) = \sup_h c(s,h) = \sup_H c(s,H)$ will be called "the value of strategy s",

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and the minimal expected cost $v(h)$ of using a hider strategy h: $v(h) = \inf_s c(s,h) = \inf_S c(S,h)$ will be called "the value of strategy h". If there exists a real number v which satisfies $v = \inf_s v(s) = \sup_h v(h)$, then we say that the game has a value v. Gal has proved [4] that any search game on a graph described above has a value and an optimal search strategy.

Gal showed in his book [4] that the value of this game satisfies: $k/2 < v < k/2 + 1/2$, where the hider can guarantee the lowest value by using the completely randomized strategy, while the searcher can guarantee the upper value by going along a random permutation of the arcs. He considered the search game on three arcs and a set of trajectories $S(i, j, \alpha)$, where i and j are two distinct integers in the set $\{1, 2, 3\}$ and $0 \le \alpha \le 1$. The trajectory $s(i, j, \alpha)$ starts from A, moves along arc $b(i)$ to O, moves along arc O(j) to the point A(α) which has a distance of α from O, moves back to O, moves to A along $b(m)$, where $m \in \{1, 2, 3\} \setminus \{i, j\}$, and then moves from A to A(α) along $b(j)$. The distance α is a random variable. He also showed that the search strategy $s^*(3)$ from the set of strategies $S(i, j, \alpha)$, where the searcher chooses equiprobably $i \in \{1, 2, 3\}, j \in \{1, 2, 3\} \setminus \{ i \}$ and α is a random variable with the following distribution function F^*: $F^*(\alpha) = 0$ for $\alpha < 0$, $F^*(\alpha) = 1/2 + 1/4 \exp(\alpha)$ for $0 \le \alpha \le \ln 2$ and $F^*(\alpha) = 1$ for $\alpha > \ln 2$, is optimal among search strategies of the similar type. The searcher can guarantee the value $(4 + \ln 2) / 3$ by using the strategy $s^*(3)$. On the other hand, the hider can guarantee the same value by using the strategy $h^*(3)$, namely, he makes equiprobably choice of an arc $b(i)$, $i \in \{1, 2, 3\}$, and hides at the point of $b(i)$ which has a distance β from O, where β is a random variable which has the probability density $2 \exp(-\beta)$ for $0 < \beta < \ln 2$ and zero otherwise.

Bostock [3] considered the three arcs game as a limiting case of a discrete game on three arcs and showed that $s^*(3)$ is optimal among a wider class of search strategies.

As for the case when k is even (the Eulerian case), Gal showed in his book [4] that the value of this game is $k/2$. The searcher can guarantee this value using the strategy $s^*(k)$, where the searcher chooses equiprobably any permutation between the arcs (for example: $b(i_1)$, $b(i_2)$, ... ,$b(i_k)$) and goes along them from one end point to another (the searcher starts from the point A, moves along the arc $b(i_1)$ to the point O, moves along the arc $b(i_2)$ to A and so on). The hider can guarantee this value using completely random strategy.

2. OPTIMAL STRATEGIES

We will prove that $s^*(3)$ and $h^*(3)$ are the optimal strategies without any restriction and that the optimal strategies for a game on an odd number of arcs are almost the same as for the game on three arcs. Namely, the hider optimal strategy is $h^*(k)$, where he equiprobably $(1/k)$ chooses an arc and hides at the point of this arc, which has a distance β from O, where β is a random variable with the probability density $2\exp(-\beta)$ for $0 < \beta < \ln 2$ and zero otherwise. As for the searcher, he can divide this game on two games: on the game on an even number of arcs $(k - 3)$ and the game on three arcs. First, he uses optimal strategy for a game on an even number of arcs and then strategy $s^*(3)$. In other words, the searcher chooses any permutation from the set $\{b(1), b(2), \ldots, b(k)\}$, for example $b(i_1)$, $b(i_2)$, $b(i_3)$, ... ,$b(i_{k-1})$, $b(i_k)$. The searcher starts from the point A, moves along the arc $b(i_1)$ to O, moves along the arc $b(i_2)$ to A, then moves along the arc $b(i_3)$ to O and so on until the arc $b(i_{k-1})$ (the last but one). Then he goes along the arc $b(i_{k-1})$ to the point A(α) which has a distance α from O, returns to O and goes along the arc $b(i_k)$ to A and finally moves along the arc $b(i_{k-1})$ from A to A(α). The distance α is a random variable which has $F^*(\alpha)$ as the distribution function. Let us denote this strategy by $s^*(k)$. The value of this game is:

$$v(k) = (k-3)^2 / 2k + 3\,[k - 3 + (4 + \ln 2) / 3] / k = (k^2 - 1 + 2\ln 2) / 2k \quad (1)$$

THEOREM 1. The strategies $s^*(k)$ and $h^*(k)$ are optimal in the game on $Q(k)$ without any restriction and the value is $(k^2 - 1 + 2\ln 2) / 2k$.

PROOF. In order to prove that we ought to show:

$$c(s^*(k), H) \le v(k), \qquad \forall H \in TH(Q(k)) \quad (2)$$

$$c(S, h^*(k)) \ge v(k), \qquad \forall S \in TS(Q(k)) \quad (3)$$

where $TS(Q(k))$ ($TH(Q(k))$) is the set of all pure searcher (hider) strategies on $Q(k)$.

Let the hider use his pure strategy H, namely he chooses one arc (for example, the arc $b(2)$) and hides at the point which has a distance x from O and let the searcher use his mixed strategy $s^*(k)$. This arc $(b(2))$ can occupy the first, or the second, or the third,..., or the last place in any permutation from the set $\{b(1), b(2), \ldots, b(k)\}$. The number of pure searcher strategies belonging to $s^*(k)$, which choose this arc on ith place is $(k - 1)!$ and taking in account that the probability mass of every pure searcher strategy entering $s^*(k)$ is k !, we have:

$$c\left(s^*(k),H\right)=(1-x)(k-1)!+(1+x)(k-1)!/k!+$$
$$(3-x)(k-1)!/k!+(3+x)(k-1)!/k!+$$
$$(5-x)(k-1)!/k!+(5+x)(k-1)!/k!+\ldots+(k-2-x)(k-1)!/k!+$$
$$\left(\int_{[0,x)}(k-2+2y+2-x)dF^*(y)+\int_{[x,\ln 2]}(k-2+x)dF^*(y)+\right.$$
$$\left.\int_{[0,\ln 2]}(k-2+2y+x)dF^*(y)\right)(k-1)!/k!$$
$$=\left(\left(k^2-3\right)/2+\int_{[0,x)}(2y+2-x)dF^*(y)+\int_{[x,\ln 2]}xdF^*(y)+\right.$$
$$\left.2\int_{[0,\ln 2]}ydF^*(y)\right)/k$$

$$\partial c/\partial x=\left(1-2F^*(x)+2\left(F^*(x)\right)'\right)/k=\begin{cases}0 & \text{for } 0<x<\ln 2\\ -1/k & \text{for } \ln 2<x\end{cases}$$

Thus, it is sufficient to calculate $c(s^*(k),H)$ for $x = \varepsilon$, where ε is small and calculation readily shows that:

$$c\left(s^*(k),H\right)\le c\left(s^*(k),\varepsilon\right)=$$
$$\left\{\left(k^2-3\right)/2+2(3/4)+2\int_{[0,\ln 2]}ydF^*(y)\right\}/k=$$
$$\left(k^2-1+2\ln 2\right)/2k \qquad (2')$$

In order to prove $c(S,h^*(k)) \ge v(k)\ \forall S$, we will use the fact (which we will prove later) that the probability density function $h^*(k+1) = 2\exp(-x) / (k+1)$, $0 < x < \ln 2$ and 0 otherwise along each arc is optimal mixed strategy for the hider in the game on $k+1$ arcs (k is an odd number) both if the searcher starts from O or A. Let us take any pure search strategy $S \in TS(Q(k))$ in the game on k arcs. We associate with the strategy S one strategy $S' \in TS(Q(k+1))$, where the searcher starts from O and moves along one arc (the first, for example) to A and then moves further according to the strategy S. In such a way the searcher, arrived at A, plays the game on k arcs. Since $h^*(k+1)$ is optimal strategy for the hider on $Q(k+1)$, we have $c(S',h^*(k+1)) \ge (k+1) / 2$ and:

$$(k+1)/2 \le c\left(S', h^*(k+1)\right) =$$

$$\int_{[0,\ln 2]} x h^*(k+1)dx + k\left\{1 + c\left(S, h^*(k)\right)\right\} \Big/ (k+1) =$$

$$\left\{1 - \ln 2 + k\left(1 + c\left(S, h^*(k)\right)\right)\right\} \Big/ (k+1)$$

therefore

$$c(S', h^*(k)) \ge (k^2 - 1 + 2\ln 2) / 2k \qquad \forall S \in TS(Q(k)) \qquad (3')$$

3. OPTIMALITY OF h*

It remains to prove the former fact, namely that $h^*(k+1)$ is optimal strategy for the hider in the game on $Q(k+1)$ (k is an odd number). The graph $Q(k+1)$ is the union of $(k+1)$ / 2 graphs $Q(2)$ joined at two points O and A. Therefore, to prove optimality of $h^*(k+1)$ we will use Theorem 2 from [5], which we cite in a modified form:

THEOREM 2 [5]. The value of strategy $h^*(k+1)$ on the graph $Q(k+1)$ is $(k+1)$ / 2 times greater or equal than the value C of strategy $h^*(2)$ on $Q(2)$ under condition that: 1. the searcher can start from O or A and he can jump from one end point to another, but only after returning to the initial point; 2. $\inf_{P_i} T_i/P_i \ge 2C$, where infimum takes place over all possible combinations of the parts P_i of the trajectories, which begin at O (A) and end at O (A) or which begin at O (A) and end at A (O).

LEMMA 1. The strategy $h^*(2)$ guarantees the value ≥ 1 under conditions that the searcher starts from the point O (A).

$$1 = \min\left(\min_{S \in TS_O(Q(2))} c\left(S, h^*(2)\right), \min_{S \in TS_A(Q(2))} c\left(S, h^*(2)\right) \right) \qquad (4)$$

where $TS_O(Q(2))$ ($TS_A(Q(2))$) is the set of all pure searcher trajectories on $Q(2)$ where the searcher starts from O (A).

PROOF. At first, we will prove optimality of $h^*(2)$ in the game on two arcs under condition that the searcher starts from the point O. To simplify our notation we will denote $h^*(2)$ by h^*.

We will divide the set of all pure strategies $TS_O(Q(2))$, where the searcher starts from the point O, (the case where the searcher starts from the point A is similar) in the following way: $TS_O(Q(2)) = TS_O{}^0(Q(2)) \cup TS_O{}^1(Q(2)) \cup ... \cup TS_O{}^n(Q(2)) \cup ...$. Denote by A(Θ) a point in $Q(2)$ which has a distance Θ from O. The strategy S^n belongs to the set $TS_O(Q(2))$ iff the searcher, who starts from the point O, makes exactly n loops, for example, the searcher starts from O, moves along the first arc to the point $A(\alpha_1)$ ($0 \le \alpha_1 \le \ln 2$), comes back to O (the first loop), then moves along the other arc to the point $A(\beta_1)$, comes back to O (the second loop), moves along the first arc to the point $A(\alpha_2)$ $(\alpha_1 \le \alpha_2 \le \ln 2)$, comes back to O (the third loop), moves along the second arc to the point $A(\beta_2)$ $(\beta_1 \le \beta_2 \le \ln 2)$, comes back to O and so on the searcher moves up to the point $A(\alpha_{(n+1)/2})$ (if n is odd number and $\alpha_1 \le \alpha_2 \le ... \alpha_{(n+1)/2} \le \ln 2$) or $A(\beta_{n/2})$ (if n is even number and $\beta_1 \le \beta_2 \le ... \beta_{n/2} \le \ln 2$), comes back to O and then moves along the other arc to A and continues moving along the first arc to the point $A(\alpha_{(n+1)/2})$ or $A(\beta_{n/2})$. We have neglected unreasonable strategies where the searcher move twice along the same arc successively, for example, he moves to the point $A(\alpha_1)$, comes back to O and moves along the same arc to the point $A(\alpha_2)$. We can easily see that every reasonable strategy must satisfy the conditions: $\alpha_1 < \alpha_2 < ... \alpha_n < ... < \ln 2$ and $\beta_1 < \beta_2 < ... \beta_n < ... < \ln 2$. We will prove $c(S,h^*) \ge 1$ $\forall S \in TS_O{}^n(Q(2))$, $\forall n$ by induction.

Let us start with the strategy S^0 from the set $TS_O{}^0(Q(2))$, where the searcher moves simply along one arc to A and then along the other arc to O.

$$c\left(S^0,h^*\right)=\int_0^{\ln 2} t\exp(-t)dt+\int_0^{\ln 2}(2-t)\exp(-t)dt \qquad (5)$$

Let us regard any strategy from the set $TS_O{}^1(Q(2))$, where the searcher starts from O, moves along one arc to the point $A(\alpha_1)$, comes back to O and moves along the other arc to A and along the first arc to the point $A(\alpha_1)$.

$$c\left(S^1,h^*\right)=\int_0^{\alpha_1} t\exp(-t)dt+\int_0^{\ln 2}(2\alpha_1+t)\exp(-t)dt+$$
$$\int_{\alpha_1}^{\ln 2}(2\alpha_1+2-t)\exp(-t)dt=1 \qquad (6)$$

In the same way, for any $S^2 \in TS_O{}^2(Q(2))$, where the searcher starts from O, moves along one arc to $A(\alpha_1)$, comes back to O and then moves along the first arc to A and continues moving along the second arc to $A(\beta_1)$, we have:

$$c\left(S^2,h^*\right)=\int_0^{\alpha_1} t\exp(-t)dt+\int_0^{\beta_1}(2\alpha_1+t)\exp(-t)dt+$$

$$\int_{\alpha_1}^{\ln 2}(2\alpha_1+2\beta_1+t)\exp(-t)dt+\int_{\beta_1}^{\ln 2}(2\alpha_1+2\beta_1+2-t)\exp(-t)dt=$$

$$2\exp(-\alpha_1)(\alpha_1+\beta_1)+1-2\beta_1$$

$\partial c / \partial \beta_1 = 2\,(\exp(-\alpha_1) - 1) \le 1$ and $c(S^2,h^*)$ attains min at $\beta_1 = \ln 2$

$c(S^2,h^*) \ge 2\exp(-\alpha_1)(\alpha_1 + \ln 2) + 1 - 2\ln 2 \ge$ (7)

(because this function is concave at α_1) ≥ 1

Let us suppose that $c(S,h^*) \ge 1, \forall S \in TS_O{}^0(Q(2)),\ TS_O{}^1(Q(2)), \ldots, TS_O{}^{2n-1}(Q(2))$, for example $S^{2n-1} := (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \ldots, \alpha_{n-1}, \beta_{n-1}, \alpha_n)$, we have:

$$c\left(S^{2n-1},h^*\right)=\int_0^{\alpha_1} t\exp(-t)dt+$$

$$\int_0^{\beta_1}(2\alpha_1+t)\exp(-t)dt+\int_{\alpha_1}^{\alpha_2}(2\alpha_1+2\beta_1+t)\exp(-t)dt+$$

$$\int_{\beta_1}^{\beta_2}(2\alpha_1+2\beta_1+2\alpha_2+t)\exp(-t)dt+\ldots+ \qquad (8)$$

$$\int_{\alpha_{n-1}}^{\alpha_n}(2\alpha_1+2\beta_1+2\alpha_2+\ldots+2\alpha_{n-1}+2\beta_{n-1}+t)\exp(-t)dt+$$

$$\int_{\beta_{n-1}}^{\ln 2}(2\alpha_1+2\beta_1+2\alpha_2+\ldots+2\alpha_{n-1}+2\beta_{n-1}+2\alpha_n+t)\exp(-t)dt+$$

$$\int_{\alpha_n}^{\ln 2}(2\alpha_1+2\beta_1+2\alpha_2+\ldots+2\alpha_{n-1}+2\beta_{n-1}+2\alpha_n+2-t)\exp(-t)dt=$$

$$=2\exp(-\alpha_1)(\alpha_1+\beta_1)+2\exp(-\beta_1)(\beta_1+\alpha_2)+$$

$$2\exp(-\alpha_2)(\alpha_2+\beta_2)+\ldots+2\exp(-\beta_{n-1})(\beta_{n-1}+\alpha_n)+$$

$$1-2\beta_1-2\alpha_2-2\beta_2-\ldots-2\beta_{n-1}-2\alpha_n \ge 1$$

Let us choose $S^{2n} \in TS_O{}^{2n}(Q(2))$, where $S^{2n} := (\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{n-1}, \beta_{n-1}, \alpha_n, \beta_n)$. We have:

$$c\left(S^{2n},h^*\right)=c\left(S^{2n-1},h^*\right)+\int_{\beta_{n-1}}^{\beta_n}(2\alpha_1+2\beta_1+2\alpha_2+\ldots+2\alpha_n+t)\exp(-t)dt+$$

$$\int_{\alpha_n}^{\ln 2}(2\alpha_1+2\beta_1+2\alpha_2+\ldots+2\alpha_n+2\beta_n+t)\exp(-t)dt+$$

$$\int_{\beta_n}^{\ln 2}(2\alpha_1+2\beta_1+2\alpha_2+\ldots+2\alpha_n+2\beta_n+2-t)\exp(-t)dt-$$

$$\int_{\beta_{n-1}}^{\ln 2}(2\alpha_1+2\beta_1+2\alpha_2+\ldots+2\alpha_n+t)\exp(-t)dt-$$

$$\int_{\alpha_n}^{\ln 2}(2\alpha_1+2\beta_1+2\alpha_2+\ldots+2\alpha_n+2-t)\exp(-t)dt=$$

$$c\left(S^{2n-1},h^*\right)+2\exp(-\alpha_n)(\alpha_n+\beta_n)-2\beta_n \ge 1$$

$c(S^{2n},h^*) \ge 1$ (9)

We have to prove also that if the result holds $\forall S \in TS_O^0(Q(2)), TS_O^1(Q(2)), \ldots, TS_O^{2n}(Q(2))$, then it holds for $S \in TS_O^{2n+1}(Q(2))$. Let us choose $S^{2n+1} \in TS_O^{2n+1}(Q(2))$, where $S^{2n+1} := (\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n, \alpha_{n+1})$. In the same way, we have:

$$c(S^{2n+1},h^*) = c(S^{2n},h^*) + 2\exp(-\beta_n)(\beta_n+\alpha_{n+1}) - 2\alpha_{n+1} \geq 1 \qquad (10)$$

Thus,

$$c(S,h^*) \geq 1 \qquad \forall S \in TS_O(Q(2)) \qquad (11)$$

We can prove in the same way, using mathematical induction, that the strategy h^* is optimal in the game on two arcs under the condition that the searcher starts from the point A. The best behavior for the searcher is to move along one arc to O and then along the other arc to A. All other strategies give mathematical expectation greater than 1. Using the similar notations, we have:

$$c(S,h^*) \geq c(S^0,h^*) = \int_0^{\ln 2}(1-t)\exp(-t)dt + \int_0^{\ln 2}(1+t)\exp(-t)dt = 1 \qquad (12)$$

The proof is similar to that for the case when the searcher starts from the point O.

Lemma 2. The strategy $h^*(2)$ guarantees the value ≥ 1 under conditions that the searcher can jump from one end point to another, but only after returning to the initial point.

$$1 = \min\left(\inf_{S\in\overline{TS}_O(Q(2))} c(S,h^*(2)), \inf_{S\in\overline{TS}_A(Q(2))} c(S,h^*(2)) \right) \qquad (13)$$

where $\overline{TS}_O(Q(2))$ ($\overline{TS}_A(Q(2))$) is a set of all pure searcher trajectories on Q(2), where the searcher starts from O (A) and it is allowed to him to jump from one end point O or A to another one, but only after returning to the initial point.

We will omit the proof of Lemma 2. because it is similar to the one of Lemma 1., but more possibilities have to be considered.

It is easy to see that the strategy h^* satisfies condition 2 of Theorem 2.

$$T_i/P_i \geq 2\alpha \Big/ \int_0^{\alpha}\exp(-x)dx = 2\alpha/(1-\exp(-\alpha)) \geq 2 \qquad (14)$$

$$T_j/P_j \geq 2\beta \Big/ \int_{1-\beta}^{\ln 2}\exp(-x)dx = 2\beta/(-1/2+\exp(-1+\beta)) \geq 2 \qquad (15)$$

$$T_k/P_k \geq 1/1/2 = 2 \qquad (16)$$

Thus, the value of strategy $h^*(k+1)$ (k is odd number) is greater or equal $(k+1)/2$ on the base of Theorem 2, which means that the strategy $h^*(k+1)$ is optimal in the game on $Q(k+1)$.

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