A NOTE ON R-EQUITABLE K-COLORINGS OF TREES

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Abstract: A graph $G = (V, E)$ is $r$-equitably $k$-colorable if there exists a partition of $V$ into $k$ independent sets $V_1, V_2, \ldots, V_k$ such that $|V_i| - |V_j| \leq r$ for all $i, j \in \{1, 2, \ldots, k\}$. In this note, we show that if two trees $T_1$ and $T_2$ of order at least two are $r$-equitably $k$-colorable for $r \geq 1$ and $k \geq 3$, then all trees obtained by adding an arbitrary edge between $T_1$ and $T_2$ are also $r$-equitably $k$-colorable.

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1 INTRODUCTION

All graphs in this paper are finite, simple and loopless. Let $G = (V, E)$ be a graph. We denote by $|G|$ its order, i.e., the number of vertices in $G$. For a vertex $v \in V$, let $N(v)$ denote the set of vertices in $G$ that are adjacent to $v$. $N(v)$ is called the neighborhood of $v$ and its elements are neighbors of $v$. The degree of vertex $v$, denoted by $\text{deg}(v)$, is the number of neighbors of $v$, i.e., $\text{deg}(v) = |N(v)|$. $\Delta(G)$ denotes the maximum degree of $G$, i.e., $\Delta(G) = \max\{\text{deg}(v) | v \in V\}$. For a set $V' \subseteq V$, we denote by $G - V'$ the graph obtained from $G$ by deleting all vertices in $V'$ as well as all edges incident to at least one vertex of $V'$.

An independent set in a graph $G = (V, E)$ is a set $S \subseteq V$ of pairwise nonadjacent vertices. The maximum size of an independent set in a graph $G = (V, E)$ is called the independence number of $G$ and denoted by $\alpha(G)$.

A $k$-coloring $c$ of a graph $G = (V, E)$ is a partition of $V$ into $k$ independent sets which we will denote by $V_1(c), V_2(c), \ldots, V_k(c)$ and refer to as color classes.
The cardinality of a largest color class with respect to a coloring \( c \) will be denoted by \( \text{Max}_c \). A graph \( G \) is \( r \)-equitably \( k \)-colorable, with \( r \geq 1 \) and \( k \geq 2 \), if there exists a \( k \)-coloring \( c \) of \( G \) such that \( | |V_i(c)| - |V_j(c)| | \leq r \) for all \( i, j \in \{1, 2, \cdots, k\} \). Such a coloring is called an \( r \)-equitable \( k \)-coloring of \( G \). A graph which is 1-equitably \( k \)-colorable is simply said to be \( \text{equitably} \ k \)-colorable.

The notion of equitable colorability was introduced in [8] and has been studied since then by many authors (see [2, 3, 4, 5, 6, 7, 9]). In [3], the authors gave a complete characterization of trees which are equitably \( k \)-colorable. This result was then generalized to forests in [2]. More precisely, for a forest \( F = (V, E) \), let \( \alpha^*(F) = \min\{\alpha(F - N[v])| v \in V \text{ and } \deg(v) = \Delta(F)\} \)

**Theorem 1.1** ([2]) Suppose \( F = (V, E) \) is a forest and \( k \geq 3 \) is an integer. Then \( F \) is \( \text{equitably} \ k \)-colorable if and only if \( k \geq \lceil \frac{|F|+1}{\alpha^*(F)+2} \rceil \).

This result can easily be generalized to \( r \)-equitable \( k \)-colorings.

**Theorem 1.2** ([1]) Suppose \( F = (V, E) \) is a forest and \( r \geq 1, k \geq 3 \) are two integers. Then \( F \) is \( r \)-equitably \( k \)-colorable if and only if \( k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil \).

**Proof:** Suppose \( F \) is \( r \)-equitably \( k \)-colorable for \( r \geq 1 \) and \( k \geq 3 \). Let \( v \) be a vertex in \( F \) such that \( \deg(v) = \Delta(F) \) and \( \alpha(F - N[v]) = \alpha^*(F) \). Clearly, for such a coloring, there are at most \( \alpha^*(F) + 1 \) vertices in the color class that contains \( v \).

It follows that all other color classes contain at most \( \alpha^*(F) + r + 1 \) vertices. Thus \( |F| \leq \alpha^*(F) + 1 + (k - 1)(\alpha^*(F) + r + 1) = k(\alpha^*(F) + r + 1) - r \), and we therefore have \( k \geq \lceil \frac{|F|+1}{\alpha^*(F)+r+1} \rceil \).

Conversely, let \( k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil \). Consider the forest \( F' = (V', E') \) obtained from \( F \) by adding \( r-1 \) new isolated vertices. Then \( |F'| = |F|+r-1 \) and \( \alpha^*(F') = \alpha^*(F) + r-1 \). Thus \( k \geq \lceil \frac{|F'|+r}{\alpha^*(F')+r+1} \rceil = \lceil \frac{|F'|+1}{\alpha^*(F')+2} \rceil \). By Theorem 1.1, \( F' \) is \( \text{equitably} \ k \)-colorable. Restricting the color classes to \( V \) gives an \( r \)-equitable \( k \)-coloring of \( F \).

In this note, we are interested in a different sufficient condition for a tree to be \( r \)-equitably \( k \)-colorable. More precisely, given a tree \( T = (V, E) \) and an edge \( e \in E \) such that its removal from \( T \) creates two trees \( T_1 \) and \( T_2 \) of order at least two, we show that if both \( T_1 \) and \( T_2 \) are \( r \)-equitably \( k \)-colorable, for \( r \geq 1 \) and \( k \geq 3 \), then \( T \) is also \( r \)-equitably \( k \)-colorable. We also explain why \( |T_1|, |T_2| \geq 2 \) and \( k \geq 3 \) are necessary conditions.

## 2 A SUFFICIENT CONDITION

Consider a tree \( T \) and two integers \( r \geq 1 \) and \( k \geq 3 \). Let \( c \) be an arbitrary \( r \)-equitably \( k \)-coloring of the vertex set of \( T \) such that \( |V_i(c)| \geq |V_j(c)| \geq \cdots \geq |V_k(c)| \). Then there may be vertices in \( T \) which are forced to be colored with color \( k \). Indeed, if for instance \( T \) is a star on \( (k-1)r + k \) vertices, then the vertex \( v \) of degree \( \geq 1 \) necessarily belongs to \( V_k(c) \) and actually \( V_k(c) = \{v\} \). Furthermore, we have \( |V_i(c)| = r + 1 \) for \( i \in \{1, 2, \cdots, k-1\} \). It turns out that this is no longer true for colors \( 1, 2, \cdots, k-1 \), as shown in the following property.

**Lemma 2.1** Consider an \( r \)-equitably \( k \)-colorable tree \( T \) of order at least two, where \( r \geq 1 \) and \( k \geq 3 \). Also, let \( \ell \) be any element in \( \{1, 2, \cdots, k-1\} \). Then, for any vertex \( u \) in \( T \), there exists an \( r \)-equitable \( k \)-coloring \( c \) of \( T \) with \( |V_i(c)| \geq |V_j(c)| \) for all \( 1 \leq i < j \leq k \) such that \( u \notin V_i(c) \).
Suppose the lemma is false. We then clearly have $|T| \geq 3$. Let $c$ be an $r$-equitable $k$-coloring of $T$ with $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$. Among all such colorings we choose one such that, for each $t = 1, 2, \ldots, k$, there is no $r$-equitable $k$-coloring $c'$ of $T$ with $|V_t(c)| = |V_t(c')|$ for $i = 1, 2, \ldots, t - 1$ and $\max_{t=1}^k(|V_t(c')|) < |V_t(c)|$. In other words, $\text{Max}_c = |V_1(c)|$ is minimum among all $r$-equitable $k$-colorings of $T$, $|V_2(c)|$ is minimum among all $r$-equitable $k$-colorings $c'$ of $T$ with $\text{Max}_c = \text{Max}_{c'}$, and so on.

Let $\ell \in \{1, 2, \ldots, k - 1\}$ be an integer for which the lemma does not hold. We define $x = 1$, $y = 2$, $z = 3$ if $\ell = 1$, and $x = \ell - 1$, $y = \ell$, $z = \ell + 1$ if $\ell > 1$. Since we assume that the lemma is false, it follows that $u \in V_t(c)$, which means that $u \in V_{2t}(c)$ if $\ell = 1$ and $u \in V_{2\ell}(c)$ if $\ell > 1$. Then $|V_{2t}(c)| > |V_{2\ell}(c)|$, otherwise we could assign color $x$ to all vertices in $V_{2t}(c)$ and color $y$ to all vertices in $V_{2\ell}(c)$ to obtain an $r$-equitable $k$-coloring $c'$ with $u \notin V_{2\ell}(c')$, a contradiction. Similarly, we must have $|V_{2\ell}(c)| > |V_{2t}(c)|$ when $\ell > 1$ since otherwise we could assign color $y$ to all vertices in $V_{2\ell}(c)$ and color $z$ to all vertices in $V_{2t}(c)$, and thus the lemma would hold.

We define $F$ as the subgraph of $T$ induced by $V_{2t}(c) \cup V_{2\ell}(c) \cup V_{2t}(c)$. If $F$ is disconnected, we add some edges to make $F$ become a tree $T'$ such that no two adjacent vertices have the same color with respect to $c$; otherwise we set $T' = F$. Let $V'$ denote the vertex set of $T'$. Moreover, for $q = y$ or $z$, we denote $\overline{q} = y + z - q$. This implies that $\overline{\overline{q}} = \overline{y} = q = \overline{q} + \overline{z} - \overline{q} = z$. We start by proving the following two claims.

Claim 1: There exists no $r$-equitable $3$-coloring $c'$ of $T'$ (using colors $x, y, z$) with $c'(u) = c(u)$, $|V_{2t}(c')| = |V_{2t}(c)| - 1$, $|V_{2\ell}(c')| = |V_{2\ell}(c)| + 1$ and $|V_{\overline{z}}(c')| = |V_{\overline{z}}(c)|$ for $q = y$ or $z$.

Indeed, if such a coloring $c'$ exists, then the assumption on $c$ implies $|V_{2t}(c')| = |V_{2t}(c)| > |V_{2\ell}(c')|$. Now we can obtain an $r$-equitable $k$-coloring $c^*$ of $T$ by letting $V_{2t}(c^*) = V_{2t}(c')$, $V_{2\ell}(c^*) = V_{2\ell}(c')$, and $V_{\overline{z}}(c^*) = V_{\overline{z}}(c')$ if $i \neq x, q$. We distinguish two cases:

- If $\ell = 1$, we have $|V_1(c^*)| > \max_{t=2}^k(|V_t(c^*)|)$ and $u \notin V_1(c^*)$.
- If $\ell > 1$, we have $q = y$ since otherwise $|V_2(c')| = |V_2(c)| + 1 = |V_2(c)|$ which contradicts $|V_{2t}(c)| > |V_{2\ell}(c)| > |V_{2t}(c)|$. Then $|V_1(c^*)| \geq \cdots \geq |V_{2t-1}(c^*)| > |V_{2t}(c')| \geq |V_{2t+1}(c')| \geq \cdots \geq |V_{k}(c^*)|$ and $u \notin V_{2t-1}(c')$. Thus, in both cases, $c^*$ is an $r$-equitable $k$-coloring of $T$ such that $|V_1(c^*)| \geq |V_j(c^*)|$ for all $1 \leq i < j \leq k$ and $u \notin V_{2t}(c^*)$, a contradiction.

Claim 2: No leaf of $T'$, except possibly $u$, is in $V_{2\ell}(c)$.

Indeed, assume $T'$ has a leaf $v \neq u$ in $V_{2\ell}(c)$ and let $w$ be its unique neighbor in $T'$. We can change the color of $v$ from $x$ to $c(w)$ to obtain an $r$-equitable $3$-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_{2t}(c')| = |V_{2t}(c)| - 1$, $|V_{\overline{z}}(c')| = |V_{\overline{z}}(c)| + 1$ and $|V_{\overline{z}}(c')| = |V_{\overline{z}}(c')|$, contradicting Claim 1.

Let $\text{vec}T$ be the oriented rooted tree obtained from $T'$ by orienting the edges from root $u$ to the leaves. Let us partition the vertices in $V_{2\ell}(c)$ into subsets $U_1, \ldots, U_p$ such that $U_q$ ($q = 1, 2, \ldots, p$) contains all vertices in $V_{2\ell}(c)$ having no successor in $V_{2\ell}(c)$, i.e., $|\bigcup_{j=1}^{p-1} U_j| = 1$. For a vertex $v \in U_1$, let $L(v)$ denote the set of leaves in $\text{vec}T$ having $v$ as predecessor.
In summary, we have Let an arbitrary edge between vertices in $V'_z(c)$, which means that $V_q(c) = V_z(c) = \emptyset$ since $|V_q(c)| > |V_z(c)| \geq |V_z(c)|$. Thus $T'$ has only one vertex, namely $u$, and since $u \in V_1(c)$ this implies that $T$ has only one vertex, a contradiction. Hence $v \neq u$.

Let $w$ be the predecessor of $v$ in $\text{vec}T$:

- If $c(w) = c(s_1)$, we change the color of $v$ to $c(w)$ to obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_z(c')| = |V_z(c)| - 1$, $|V_{c(w)}(c')| = |V_{c(w)}(c)| + 1$ and $|V_{c(w)}(c')|$, contradicting Claim 1;

- If $c(w) \neq c(s_1)$, we assign color $c(s_1)$ to $v$, color $c(s_{j+1})$ to $s_j$ ($j = 1, 2, \ldots, a-1$), and color $x$ to $s_a$; we obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $|V_z(c')| = |V_z(c)|(i = x, y, z)$, $c'(u) = c(u)$ and a leaf $s_a \in V_z(c')$. But this contradicts Claim 2.

We therefore conclude that $|L(v)| \geq 2$ for all $v \in U_1$. By denoting $W_1 = \bigcup_{v \in U_1} L(v)$, we get $|W_1| \geq 2|U_1|$. For each set $U_q$, with $q > 1$, we will now construct a set $W_q$ containing vertices in $V'_y(c) \cup V_z(c)$ that are successors of vertices in $U_q$ but not successors of vertices in $U_{q-1}$. So let $v$ be any vertex in $U_q$ ($q > 1$). If $v$ has at least 2 immediate successors in $\text{vec}T$, we add two of them to $W_q$. If $v$ has a unique immediate successor in $\text{vec}T$, then let $P = v \rightarrow s_1 \rightarrow \cdots \rightarrow s_a \rightarrow v'$ denote a path from $v$ to a vertex $v' \in U_{q-1}$. If $a > 1$, we add $s_1$ and $s_2$ to $W_q$. If $a = 1$ and $s_1$ has an immediate successor $w \notin V_z(c)$, then we add $s_1$ and $w$ to $W_q$. Assume now that $a = 1$ and all the immediate successors of $s_1$ are in $V_z(c)$. We will prove that such a case is not possible.

- If $v \neq u$, then $v$ has a predecessor $w$ in $\text{vec}T$. We must have $c(w) = c(s_1)$, otherwise we could assign color $c(s_1)$ to $v$ to obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_z(c')| = |V_z(c)| - 1$, $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)| + 1$ and $|V_{c(s_1)}(c')|$, contradicting Claim 1. But now we can assign color $c(s_1)$ to $v$ and assign color $c(s_1)$ to $s_1$ to obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_z(c')| = |V_z(c)| - 1$, $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)| + 1$ and $|V_{c(s_1)}(c')|$, contradicting Claim 1.

- If $v \equiv u$, then $\ell = 1$ since $u \in V_z(c)$. By assigning color $c(s_1)$ to $u$ and color $c(s_1)$ to $s_1$, we obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $|V_z(c')| = |V_z(c)| - 1$, $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$. It follows from the assumptions on $c$ that $|V_{c(s_1)}(c')| = |V_z(c)| > |V_{c(s_1)}(c)| = |V_{c(s_1)}(c')|$. Thus the lemma would hold, a contradiction.

In summary, we have $|W_q| \geq 2|U_q|$. Since all sets $W_q$ are disjoint, we have

$$|V_y(c)| + |V_z(c)| \geq \sum_{q=1}^p |W_q| \geq \sum_{q=1}^p 2|U_q| = 2|V_z(c)|.$$

Hence $|V_y(c)|$ or $|V_z(c)|$ is larger than or equal to $|V_z(c)|$, a contradiction.

Lemma 2.1 allows us to show our main result.

**Theorem 2.2** Let $T_1$ and $T_2$ be two trees or order at least two. If both $T_1$ and $T_2$ are $r$-equitably $k$-colorable for $r \geq 1$ and $k \geq 3$, then a tree $T$ obtained by adding an arbitrary edge between $T_1$ and $T_2$ is also $r$-equitably $k$-colorable.
Proof: Consider an $r$-equitable $k$-coloring $c$ of $T_1$ and an $r$-equitable $k$-coloring $c'$ of $T_2$ such that $|V_i(c)| ≥ |V_j(c')|$ and $|V_i(c')| ≥ |V_j(c')|$ for all $1 ≤ i < j ≤ k$. Let $u$ be a vertex in $T_1$ and $v$ a vertex in $T_2$, and let $T$ be the tree obtained by adding an edge which joins $u$ and $v$. According to Lemma 2.1, we may assume that $v ∉ V_1(c')$. Hence $v ∈ V_{k−ℓ+1}(c')$ for some $ℓ ∈ \{1, 2, \ldots, k−1\}$ and it follows from Lemma 2.1 that we may assume that $u ∉ V_ℓ(c)$. We can therefore construct a $k$-coloring $c^*$ of $T$ such that $V_i(c^*) = V_i(c) ∪ V_{k−i+1}(c')$, $i = 1, 2, \ldots, k$. For $i > j$, we have:

$$|V_i(c^*)| − |V_j(c^*)| = |V_i(c)| + |V_{k−i+1}(c')| − (|V_j(c)| + |V_{k−j+1}(c')|)$$

$$= (|V_i(c)| − |V_j(c)|) + (|V_{k−i+1}(c')| − |V_{k−j+1}(c')|).$$

Since $V_i(c)| ≥ |V_j(c)|$ and $|V_{k−j+1}(c')| ≤ |V_{k−i+1}(c')|$, we have:

- $|V_i(c^*)| − |V_j(c^*)| ≥ |V_i(c)| − |V_j(c)| ≥ −r$;
- $|V_j(c^*)| − |V_i(c^*)| ≤ |V_{k−i+1}(c')| − |V_{k−j+1}(c')| ≤ r$.

This proves that the considered $k$-coloring $c^*$ of $T$ is $r$-equitable.

Note that the condition $k ≥ 3$ in Theorem 2.2 is necessary. Indeed, if both $T_1$ and $T_2$ are isomorphic to a star on 3 vertices (with $u$ being the vertex of degree two in $T_1$ and $v$ a leaf in $T_2$) then clearly $T_1$ and $T_2$ are 1-equitably 2-colorable. But by adding an edge which joins $u$ and $v$, we obtain a tree $T$ which is not 1-equitably 2-colorable.

Note also that the condition in Theorem 2.2 on the number of vertices in each tree is necessary. Indeed, if $T_1$ is an $r$-equitably $k$-colorable tree for some $k ≥ 3$ and $r ≥ 1$, and if $T_2$ contains a single vertex $v$, then the tree $T'$ obtained by adding an edge which joins $v$ and a vertex $u$ of $T_1$ is possibly not $r$-equitably $k$-colorable. For example, if $u$ is the vertex of degree four in the star $T_1$ on five vertices, and if we add a neighbor $v$ (the single vertex in $T_2$) to $u$, we obtain a star $T'$ on six vertices. While $T_1$ and $T_2$ are clearly 1-equitably 3-colorable, $T'$ is not 1-equitably 3-colorable. It is however not difficult to prove that if $T$ is an $r$-equitably $k$-colorable tree for some $k ≥ 2$ and $r ≥ 1$, then the tree $T'$ obtained by adding a new vertex $v$ and making it adjacent to some vertex $u$ of $T$ is $(r+1)$-equitably $k$-colorable. Indeed, given an $r$-equitable $k$-coloring $c$ of $T$, we can extend it to a $k$-coloring $c'$ of $T'$ by assigning any color $j ≠ c(u)$ to $v$ with $j ∈ \{1, 2, \ldots, k\}$. If $|V_j(c)| ≥ |V_i(c)|$ for all $i ≠ j$, then $c'$ is $(r+1)$-equitable, otherwise $c'$ is $r$-equitable.

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