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APPLICATIONS OF THE FINITE STATE AUTOMATA FOR COUNTING RESTRICTED PERMUTATIONS AND VARIATIONS

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Abstract: In this paper, we use the finite state automata to count the number of restricted permutations and the number of restricted variations. For each type of restricted permutations, we construct a finite state automaton able to recognize and enumerate them. We, also, discuss how it encompasses the other known methods for enumerating permutations with restricted position, and in one case, we establish connections with some other combinatorial structures, such as subsets and compositions.

Keywords: Finite state automata, restricted permutations, restricted variations, exact enumeration.

MSC: 11B85, 05A15, 05A05, 11B39.

1. INTRODUCTION

For the beginning, we give the concepts of the finite state automaton, restricted permutations and restricted variations.

A **finite state automaton** M consists of five parts:

1. a finite set (alphabet) T of inputs;
2. a finite set S of (internal) states;
3. a subset Y of S (whose elements are called final, accepting or "yes" states);
4. an initial state (or start state) s_0 in S;
5. a next-state function F from $S \times T$ into S.

Such an automaton M is denoted by: $M = (T, S, Y, s_0, F)$.

A state is said to be **accessible state** if it can be reached from the start state. A state $s \in S$ is called **sink state** if $f(s,x) = s$ for all $x \in T$.

The nondeterministic finite automaton is a variant of finite automaton with the following characteristic:

zero or more than one possible value may exist for state transition (in the deterministic finite automaton, the next possible state is uniquely determined).

More about finite state automata and machines can be found in [6].

Let p be a permutation of the set $N_n = \{1, 2, ..., n\}$.

A **restricted permutation** is a permutation p such that the positions of the marks after the permutation are restricted. It can be specified by an $n \times n$ zero-one matrix $A = (a_{ij})$ where:

- $a_{ij} = 1$, if the mark *j* is permitted to occupy the *i*-th place;
- $a_{ij} = 0$, otherwise.

We can say that $a_{ij} = 1$ if and only if it is allowed to be $p(i) = j$.

We introduce a technique based on finite state automata for counting the number of strongly restricted permutations of N_n satisfying the condition $p(i) - i \in I$ (for some set I), and another type of Lehmer's strongly restricted permutations when in a circle, each mark moves clockwise only but not more than k places (for the history of this kind of a problems see [1]).

More about restricted permutations can be found in [1], [3] and [7].

An ***n*-variation** of the set $N_{n+s} = \{1, 2, ..., n+s\}$ is any 1-to-1 mapping p from the set N_n into N_{n+s} . The restricted *n*-variation can be specified by an $n \times (n+s)$ zero-one matrix $A = (a_{ij})$ in which:

$A_{ij} = 1$ if and only if it is allowed to be $p(i) = j$.

2. MAIN RESULTS

For each type of restricted permutations, we construct a finite state automaton able to recognize and enumerate them. At the beginning, we consider restricted permutations satisfying the condition $-k \le p(i) - i \le r$ (for arbitrary natural numbers *k* and *r*). Suppose that we are in the process of constructing a restricted permutation and that the partially built permutation is

$$\begin{pmatrix} 1 & 2 & \dots & h-1 \\ p(1) & p(2) & \dots & p(h-1) \end{pmatrix}.$$

We have to select a value for $p(h)$.

First, we can notice that choosing the *h*-th element of the permutation, $p(h)$, depends only on the status of the element *h,* preceding *k* elements and the following *r* elements of the permutation under construction (status can be used or unused).

The states consist of $k+r+1$ slots corresponding to the preceding k elements $(h-k, h-k+1, ..., h-1)$, momentary element h and the following r elements of the permutation under construction $(h+1, h+2, ..., h+r)$. Some have already been filled (if corresponding element is used), and we need to fill one more, and then to drop the first slot off to give a new state (we will denote filled slots by 1 and vacant by 0). However, if the first slot is vacant, that is the one we need to fill now, because it is our last chance. There are lot of inaccessible states (more precisely accessible states have exactly *k* ones), so we can drop them away.

This approach has some similarities with the method by D. Blackwell described in [5] or [7], but the main difference is that Blackwell's method gives asymptotic behavior of the number of certain restricted permutations (the growth of that number is as the n-th power of the largest eigenvalue) and we give the exact number of restricted permutations. Also, he solved one particular case and we solved the problem in general.

We illustrate this with the following example.

Example 1. *Let us consider restricted permutations satisfying the condition* $-3 \le p(i)-i \le 4$ *(i.e. k=3 and r=4). Suppose that the partially built permutation is*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots \\ 4 & 1 & 7 & 3 & \dots \end{pmatrix}.$$

We have to select the fifth value, i.e. the momentary element h=5 or we can say that we are finding p(5).

Now, the situation with the elements from 2 to 9 ($h-k=5-3=2$ *and* $h+r=5+4=9$) *has to be described.*

We use numbers 4, 7 and 3 (at their positions we put 1), and the others are unused (there are 0):

$$\begin{matrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix}$$

The state corresponding to this situation is 01100100. The first slot (corresponding to the number 2) is empty, and this is the last chance to fill it. So, $p(5)$ *must be equal to 2. Since* $p(5)-5=2-5=-3$*, we are going 3 elements to the left (our automaton can make this movement only if it comes to letter -3 from the alphabet* $T=\{-3,-2,-1,0,1,2,3,4\}$*).*

After that, the new partially built permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots \\ 4 & 1 & 7 & 3 & 2 & \dots \end{pmatrix}.$$

We have to select the sixth value, i.e. the momentary element $h=6$.

We have to describe the situation with the elements from 3 to 10 ($h-k=6-3=3$ *and* $h+r=6+4=10$).

The used numbers are 4, 7 and 3 (at their positions we put 1), and the others are unused (there are 0):

1	1	0	0	1	0	0	0
3	4	5	6	7	8	9	10

The state corresponding to this situation is 11001000. The first slot (corresponding to the number 3) is not empty, so $p(6)$ *can take any unused element: 5, 6, 8, 9 or 10 (corresponding letters are -1, 0, 2, 3 or 4).* ■

Theorem 1. *The set of the internal states S consists of the states that have exactly k ones and one more sink state, denoted by Q, corresponding to impossible state for this kind of the restricted permutations.*
The alphabet T is given by:

$$T = \{-k, -k+1, ..., -2, -1, 0, 1, 2, ..., r-1, r\}.$$

In this case, there is only one "yes" state – when first k slots are filled (in this case, we used all numbers less than h, so they made one permutation). In the second example, we will have more than one "yes" state.

The start state is the same as the only one "yes" state.

The next-state function $(for\ -k \le x \le r)$ *is defined by:*

- $F(S_1, x) = Q$ *if the first slot is vacant and* $x \ne -k$ *;*
- $F(S_1, x) = Q$ *if the* $(k+1+x)-th$ *slot is filled;*
- $F(S_1, x) = S_2$ *if the first slot is filled and* $(k+1+x)-th$ *slot is vacant.*

The state S_2 *is obtained from the state* S_1 *when* $(k+1+x)-th$ *slot is filled, then we drop the first slot off and put one empty slot at the end.*

The finite state automaton $M = (T, S, Y, s_0, F)$ *recognizes only the restricted permutations satisfying the condition* $-k \le p(i) - i \le r$ *.*

Proof of the Theorem 1: Filled and empty slots in each state are shifting to the left with the next-state function $F(S_1, x) = S_2$. It provides that each number in the permutation occurs no more than ones.

Only one "yes" state – when the first *k* slots are filled, provide that all numbers smaller than a momentary element are used, so they made one permutation.

Restraint $-k \le x \le r$ for the alphabet T leads to the fact that permutations satisfy condition $-k \le p(i) - i \le r$. □

In the next theorem, we are going further by giving generalizations of the restricted permutations defined by the previous theorem. We will consider restricted permutations satisfying the condition $p(i) - i \in I$ for all $i \in N_n$, where I is some finite subset of the set $\{-k, -k+1, ..., -2, -1, 0, 1, 2, ..., r-1, r\}$. Than, we will construct the corresponding automaton M. We will omit the proof of the Theorem 2, as it is essentially the same as the proof of the Theorem 1.

Theorem 2. *The set of the internal states S, only one "yes" state, the start state and the next-state function* F $(for\ x \in I)$ *are the same as in the Theorem 1.*

The alphabet T *is different:* $T = I$.

The finite state automaton $M = (T, S, Y, s_0, F)$ *recognizes only restricted permutations satisfying the condition* $p(i) - i \in I$. □

In [2], Baltić gave some of the applications of the finite state automata in the enumeration of the restricted permutations, and here we are going further. We will use the finite state automata for the enumeration of the restricted n-variation**s** of the set $N_{n+s} = \{1, 2, ..., n+s\}$ satisfying the condition $p(i) - i \in I$. The case of restricted variations differs only in the "yes" states. There can be more than one "yes" state.

Theorem 3. *The alphabet T, the set of the internal states S, the start state and the next-state function* F $(for\ x \in I)$ *are the same as in the Theorem 2.*

The "yes" states are all those states with exactly k slots filled where all slots from (k+s+1)-th to (k+r+1)-th are vacant.

The finite state automaton $M = (T, S, Y, s_0, F)$ *recognizes only restricted n-variations of the set* $N_{n+s} = \{1, 2, ..., n+s\}$ *satisfying the condition* $p(i) - i \in I$.

Proof of the Theorem 3: All the states where slots from $(k+s+1)$-th to $(k+s+1)$-th are vacant generate an n-variation of the set $N_{n+s} = \{1, 2, ..., n+s\}$.

Other parts of the proof of Theorem 3 are the same as used in proof of Theorem 1, so we omit them. □

3. THE FIRST EXAMPLE

We will consider restricted permutations satisfying the condition $p(i) - i \in \{-2, 0, 2\}$ for all $i \in N_n$, and construct the corresponding automaton M.

It is a special case of permutation described in the previous section, but with more restrictions. Everything is the same but the alphabet differs: $T = \{-2, 0, 2\}$.

Internal states are $S = \{a, b, c, d, e, f, Q\}$. The following table shows the correspondence between the internal states and the filled and vacant slots:

Table 1: The internal states

States	a	b	c	d	e	f	Q
Slots	11000	10100	10010	01100	01010	00110	

The initial state and the only one "yes" state is a.

For example, if the state of automaton is c it is corresponding to slots 10010. Input -2 may be viewed as causing a change in the state of the automaton from c to Q, because the first slot is filled. Input 0 cause change from c to d: from the state c=10010, we get 10110 when we fill momentary element (that's 1 on the third slot), and after dropping the first slot off and putting one empty slot at the end, we finally get $d = 01100$). Input 2 cause change from c to f (from the state c = 10010, we get 10011 and after that f = 00110). From the state f = 00110, we need to fill 0 at the first slot, so $F(f,-2) = d$ and $F(f,0) = Q$ and $F(f,2) = Q$.

The next-state function $F = (S, x)$ is presented by Table 2:

Table 2: The next-state function $F = (S, x)$

	States S						
Inputs x	a	b	c	d	e	f	Q
-2	Q	Q	Q	a	b	d	Q
0	a	Q	d	Q	Q	Q	Q
2	c	e	f	Q	Q	Q	Q

The automaton M can be shown using its state diagram, shown in Figure 1.

There are inaccessible states b and e, so we can omit them. Also, for more clarity, we can omit edges leading to sink state Q and the state Q – leading to a new automaton M_2 . In a fact, M_2 is nondeterministic automaton (there is zero possible value for some state transition; if we look at their graphs – the graph of M_2 is the subgraph of the graph of M). It is presented in Figure 2.

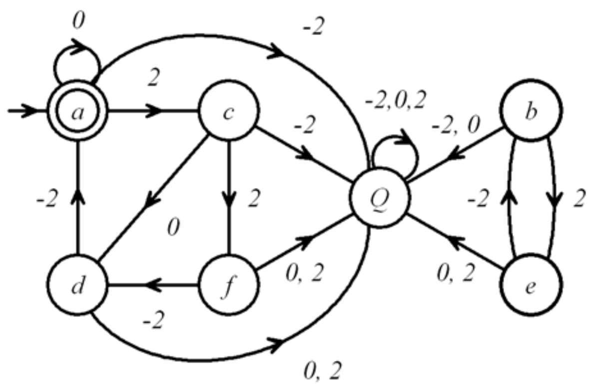

Figure 1: The automaton M

The state diagram of the automaton M_2 represented in Figure 2 is an oriented graph. In the terms of Graph Theory, the construction of the restricted permutation corresponds to forming the closed walk of length n from vertex a to vertex a .

Let us denote the number of the walks of length n from vertex v to vertex a with $v(n)$. Without loss of generality, we can take $a(0) = 1$ and $b(0) = c(0) = d(0) = e(0) = f(0) = 0$.

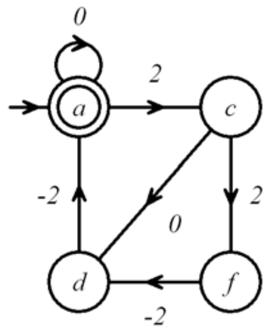

Figure 2: Nondeterministic automaton M_2

The number of the restricted permutations of the set N_n satisfying $p(i)-i \in \{-2,0,2\}$ is equal to $a(n)$. Notice that each of the closed walks consists of walks $a-a$, $a-c-d-a$ or $a-c-f-d-a$. This conclusion leads us to the bijection between the restricted permutations satisfying the condition $p(i)-i \in \{-2,0,2\}$ for all $i \in N_n$ and compositions of the number n into elements from the set $\{1,3,4\}$.

For example, $a(5)=6$, we have 6 permutations:

12345, 12543, 14325, 14523, 32145, 34125

and the corresponding 6 compositions:

$1+1+1+1+1,\quad 1+1+3,\quad 1+3+1,\quad 1+4,\quad 3+1+1,\quad 4+1.$

From the state diagrams of automata M and M_2, we can get a system of the recurrence equations (we discuss M_2 because it leads to a more simple system, with less number of equations). Notice, ones more, that the number of the restricted permutations of the set N_n is equal to the number of the closed walks of length n from vertex a to vertex a. The next-state function $F=(S,x)$ given in Table 2 translates directly into the system of recurrences.

If the first element of permutation is equal to 1, it corresponds to input 0 in automaton M, that causes no change of automaton's state. If the first element of permutation is equal to 3, it corresponds to input 2 in automaton M, that causes change from a to c. This observation leads us to conclusion that number of the closed walks of length n from vertex a to vertex a is equal to the sum of the number of the closed walks of length $n-1$ from vertex a to vertex a and the number of the closed path of length $n-1$ from vertex c to vertex a, i.e.

$$a(n+1)=a(n)+c(n).$$

Similarly, we come to the following system of recurrence equations:

$$a(n+1) = a(n) + c(n) \tag{1}$$

$$c(n+1) = d(n) + f(n) \tag{2}$$

$$d(n+1) = a(n) \tag{3}$$

$$f(n+1) = d(n) \tag{4}$$

with the initial conditions

$a(0) = 1$, $c(0) = 0$, $d(0) = 0$ and $f(0) = 0$.

From the equation (3), we find $d(n) = a(n-1)$. Putting that into the equation (4) gives $f(n) = a(n-2)$. Then, from the equation (2), we find $c(n) = a(n-2) + a(n-3)$. That leads us to homogeneous linear recurrence equation with constant coefficients:

$$a(n+1) = a(n) + a(n-2) + a(n-3),$$

with initial conditions $a(0) = 1$, $a(1) = 1$, $a(2) = 1$, $a(3) = 2$.

The auxiliary equation is $t^4 = t^3 + t + 1$. It can be transformed into the form: $(t^2+1)(t^2-t-1) = 0$ and its solutions are:

$$t_1 = i, t_2 = -i, t_3 = \frac{1+\sqrt{5}}{2} \text{ and } t_4 = \frac{1-\sqrt{5}}{2}.$$

So, $a(n) = C_1 t_1^n + C_2 t_2^n + C_3 t_3^n + C_4 t_4^n$. Using the initial conditions, we find:

$$C_1 = \frac{2-i}{10},\ C_2 = \frac{2+i}{10},\ C_3 = \frac{3+\sqrt{5}}{10},\ C_4 = \frac{3-\sqrt{5}}{10}.$$

Since

$$\frac{2-i}{10} \cdot i^n + \frac{2+i}{10} \cdot (-i)^n = \frac{2-i}{10} \cdot (\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}) + \frac{2+i}{10} \cdot (\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}) = \frac{2}{5} \cdot \cos\frac{n\pi}{2} + \frac{1}{5} \cdot \sin\frac{n\pi}{2}$$

and

$$\frac{3+\sqrt{5}}{10} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{3-\sqrt{5}}{10} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right) = \frac{1}{5} L(n+2)$$

(where $L(n)$ denotes n-th element of Lucas sequence given with $L(n+1) = L(n) + L(n-1),\ L(1) = 1,\ L(2) = 3$), we have:

$$a(n) = \frac{1}{5}\left(L(n+2) + 2 \cdot \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right).$$

There is also one more combinatorial interpretation of such restricted permutations. Any permutation is a product (composition) of an identical permutation ε

and some transpositions. Transposition τ_{ij} is a permutation of two elements i and j – it is an exchange of two elements (i and j) of an ordered list with all other elements staying the same. The restricted permutations satisfying the condition $p(i)-i \in \{-2,0,2\}$ for all $i \in N_n$ are products of transpositions with no common element, i.e. in $\tau_{ij} \circ \tau_{kl}$ all four elements i, j, k, l are distinct. This leads us to bijection between the restricted permutations satisfying the condition $p(i)-i \in \{-2,0,2\}$ for all $i \in N_n$ and the number of all subsets of the set N_{n-2}, which do not contain two elements whose difference is equal to 2. More about these properties of the restricted permutations can be found in [1], [2] and [3].

For example, $a(5)=6$, and we have 6 permutations:

12345, 12543, 14325, 14523, 32145, 34125.

They can be represented as:

ε, τ_{35}, τ_{24}, $\tau_{24} \circ \tau_{35}$, τ_{13}, $\tau_{13} \circ \tau_{24}$.

Corresponding subsets of the set $N_3 = \{1,2,3\}$ are:

$\varnothing$, $\{3\}$, $\{2\}$, $\{2,3\}$, $\{1\}$, $\{1,2\}$.

This property holds in general:

The number of all subsets of set N_{n-k} which do not contain two elements whose difference is k is equal to the number of the restricted permutations of N_n satisfying the condition $p(i)-i \in \{-k,0,k\}$ for all $i \in N_n$.

Gerald E. Bergum and Verner E. Hoggat enumerated the subsets which do not contain two elements whose difference is 2 (see [4]), but we are the first who gives connections with other combinatorial structures.

4. THE SECOND EXAMPLE

Now we will generalize Theorem 1 for the circular case - which is surely an interesting one. Similarly, we may ask for the number of the permutations of the set N$_n$ such that $p(i)-i \equiv 0,1,...,k \pmod{n}$. This has the effect of allowing $p(n) \in \{1,2,...,k\}$, $p(n-1) \in \{1,2,...,k-1\}$, and so on, $p(n-k+1)=1$.

We need some auxiliary slots to correspond to the vacant positions from among the first k – which could be filled by late elements, and to accept all these needed to be filled. These slots will be presented red with underline formatting.

We will consider restricted permutations satisfying the condition $p(i)-i \equiv 0,1,2 \pmod{n}$ for all $i \in N_n$. This problem is well known. You can find the solution based on the Transfer-matrix method in Example 4.7.7 in [8], but we will use different approach: we construct the corresponding automaton M and there from, we find the system of the recurrence equations, which leads to the solution.

The alphabet in this case also differs: $T = \{\mathrm{I}, \mathrm{II}, 0, 1, 2\}$. The inputs I and II denote filling vacant slots at the end of permutation (I corresponds to $p(i)=1$, where $i=n$ or $i=n-1$, and II corresponds to $p(i)=2$, where $i=n$).

Internal states are $S = \{a,b,c,d,e,f,g,h,i,j,k,l,m,Q\}$.

The following table shows the correspondence between the internal states and the filled and vacant slots:

Table 3: The internal states

States	a	b	c	d	e	f	g
Slots	00000	10000	01100	00010	11000	10100	10010
States	h	i	j	k^1	l^1	m^0	Q
Slots	01100	01010	00110	11000	10100	11000	

The states *e*, *k* and *m* have the same positions of filled and vacant slots. That states differs because of the following reason. If we have input II, the permutation under construction should end in that step. If we have input I, the permutation under construction should end in that step or in the next one. The same situation is with states *f* and *l*. The states *a*, *b*, *c* and *d* denote the beginning states (corresponding to first step of construction permutation). All other states have exactly two slots filled.

The initial state is *a*.

The set of the "yes" states is $\mathrm{Y} = \{a,b,e,k,m\}$. It consists of states that have 0, 1 or 2 filled slots (we took *a* as the "yes" state because it is common to say that there is 1 permutation of length 0; fact that *b* is a "yes" state corresponds to permutation 1).

The next-state function $F = (S,x)$ is given by Table 4:

Table 4: The next-state function $F = (S,x)$

	Inputs x				
States S	I	II	0	1	2
a	Q	Q	b	c	d
b	Q	Q	e	f	g
c	k	Q	Q	h	i
d	l	Q	h	Q	j
e	Q	Q	e	Q	Q
f	Q	m	Q	f	g
g	Q	Q	f	Q	Q
h	k	Q	Q	h	i
i	Q	Q	h	Q	Q
j	l	Q	Q	Q	j
k	Q	Q	m	Q	Q
l	Q	m	Q	Q	Q
m	Q	Q	Q	Q	Q
Q	Q	Q	Q	Q	Q

Table 4 translates directly into the system of recurrences:

$$a(n+1) = b(n) + c(n) + d(n)$$

$$b(n+1) = e(n) + f(n) + g(n)$$

$$c(n+1) = k(n) + h(n) + i(n)$$

$$d(n+1) = l(n) + h(n) + j(n)$$

$$e(n+1) = e(n)$$

$$f(n+1) = m(n) + f(n) + g(n)$$

$$g(n+1) = f(n)$$

$$h(n+1) = k(n) + h(n) + i(n)$$

$$i(n+1) = h(n)$$

$$j(n+1) = l(n) + j(n)$$

$$k(n+1) = m(n)$$

$$l(n+1) = m(n)$$

$$m(n) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

with initial conditions:

$a(0) = b(0) = e(0) = k(0) = m(0) = 1$ and

$c(0) = d(0) = f(0) = g(0) = h(0) = i(0) = j(0) = l(0) = 0$.

It is easy to show that for $n \geq 2$, we have:

$$k(n) = l(n) = m(n) = 0, e(n) = j(n) = 1, f(n) = i(n) = F(n),$$
$$g(n) = F(n-1), c(n) = h(n) = F(n-1), b(n) = d(n) = F(n) + 1$$

($F(n)$ denotes n-th element of Fibonacci sequence given with $F(n+1) = F(n) + F(n-1), F(1) = 1, F(2) = 1$) , there from, we have:

$a(n) = 2 + F(n-1) + F(n+1)$ for $n \geq 3$.

From the well known fact about Fibonacci and Lucas numbers, we have:

$F(n-1) + F(n+1) = L(n)$,

where $L(n)$ denotes n-th element of Lucas sequence.
This conclusion leads us to:

$$a(n) = \begin{cases} 1, & n = 0,1 \\ 2, & n = 2 \\ 2 + L(n), & n \geq 3 \end{cases}$$

Similarly, but with much more hard calculations, we can solve the same problem for $k = 3$. Solution in this case is given by:

$$a(n) = 2 + 6T(n) + 8T(n-1) + 2T(n-2)\ , \quad \text{for } n \geq 3\ .$$

($T(n)$ denotes n-th element of Tribonacci sequence given with $T(n+1) = T(n) + T(n-1) + T(n-2), T(0) = 0, T(1) = 1, T(2) = 1$).

5. THE THIRD EXAMPLE

We will consider restricted variations satisfying the condition $p(i) - i \in \{-1, 2\}$ for all $i \in N_n$ and with $s = 2$ (n-variation**s** of the set N_{n+2}). We will construct the corresponding automaton M.

Now, the alphabet is $T = \{-1, 2\}$.

Internal states are $S = \{a, b, c, d, e, f, g, Q\}$.

The initial state is a. All states except Q are "yes" states.

The following table shows the correspondence between the internal states and the filled and vacant slots. Also, the next-state function $F = (S, x)$ is given by Table 5:

Table 5: The internal states and the next-state function $F = (S, x)$

	States S							
Slots	1000	0100	0010	1010	1100	0110	1110	
Inputs x	a	b	c	d	e	f	g	Q
-1	Q	a	b	Q	Q	e	Q	Q
2	c	d	f	f	d	g	g	Q

The nondeterministic automaton M can be presented by its state diagram as in Figure 3.

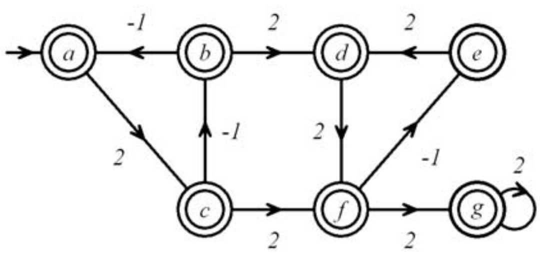

Figure 3: The automaton M

The next-state function $F=(S,x)$ given by the Table 5 (or from the state diagram of automaton M which appears in the Figure 3) translates directly into the system of recurrences:

$$a(n+1)=c(n)$$

$$b(n+1)=a(n)+d(n)$$

$$c(n+1)=b(n)+f(n)$$

$$d(n+1)=f(n)$$

$$e(n+1)=d(n)$$

$$f(n+1)=e(n)+g(n)$$

$$g(n+1)=g(n)$$

with initial conditions

$$a(0)=1, b(0)=1, c(0)=1, d(0)=1, e(0)=1, f(0)=1 \text{ and } g(0)=1.$$

It is easy to show that we have recurrence equation:

$$f(n+3)=f(n)+1, \text{ with initial conditions } f(0)=1, f(1)=2, f(2)=2.$$

Solving this recurrence, we find

$$f(n)=\left[\frac{n+5}{3}\right]$$

(where $[x]$ denotes the greatest integer less than or equal to a number x).

This conclusion with recurrence $a(n+3)=a(n)+f(n+1)+f(n-1)$ (obtained from recurrences $a(n+2)=b(n)+f(n)$ and $b(n+2)=a(n+1)+f(n)$) leads us to:

$$a(n)=\begin{cases} k^2, & n=3k-3 \\ k^2, & n=3k-2 \\ k\cdot(k+1), & n=3k-1 \end{cases}$$

This is a shifted sequence A008133 at [9].

6. FOURTH EXAMPLE

We will consider restricted variations satisfying the condition $p(i)-i\in\{-2,0,2\}$ for all $i\in N_n$ and with $s=1$ (n-variation**s** of the set N_{n+1}). We construct the corresponding automaton M.

Now, the alphabet is $T = \{-2, 0, 2\}$.

Internal states are $S = \{a, b, c, d, e, f, g, h, i, Q\}$.

The initial state is *a*. States *a*, *c* and *g* are “yes” states.

The following table shows the correspondence between the internal states and the filled and vacant slots. Also, the next-state function $F = (S, x)$ is given by Table 6:

Table 6: The internal states and the next-state function $F = (S, x)$

	States S									
Slots	110**00**	100**10**	011**00**	001**10**	110**10**	011**10**	111**00**	101**10**	111**10**	
Inputs x	a	b	c	d	e	f	g	h	i	Q
-2	Q	Q	a	c	Q	g	Q	Q	Q	Q
0	a	c	Q	Q	g	Q	Q	Q	Q	Q
2	b	d	e	f	h	i	e	f	i	Q

The nondeterministic automaton M can be presented by its state diagram as in Figure 4.

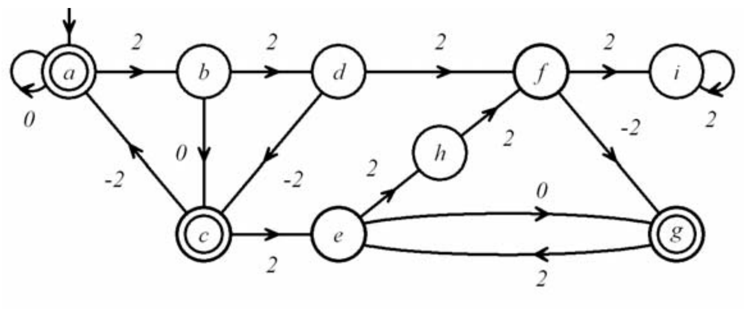

Figure 4: The automaton M

There is sink state i, so we can omit it (also the corresponding recurrence).

The next-state function $F = (S, x)$ given by the Table 6 translates directly into the system of recurrences:

$$a(n+1) = a(n) + b(n)$$

$$b(n+1) = c(n) + d(n)$$

$$c(n+1) = a(n) + e(n)$$

$$d(n+1) = c(n) + f(n)$$

$$e(n+1) = g(n) + h(n)$$

$$f(n+1) = g(n)$$

$g(n+1)=e(n)$

$h(n+1)=f(n)$

with initial conditions

$a(0)=1, b(0)=0, c(0)=1, d(0)=0, e(0)=0, f(0)=0, g(0)=1$ and $h(0)=0$.

For a sequence which is denoted by a lower case letter, we will denote the corresponding generating function by the same upper case letter ($a(n)\to A(z)$, $b(n)\to B(z)$, and so on). We find the following system:

$$\frac{A(z)-1}{z}=A(z)+B(z)$$

$$\frac{B(z)}{z}=C(z)+D(z)$$

$$\frac{C(z)-1}{z}=A(z)+E(z)$$

$$\frac{D(z)}{z}=C(z)+F(z)$$

$$\frac{E(z)}{z}=G(z)+H(z)$$

$$\frac{F(z)}{z}=G(z)$$

$$\frac{G(z)-1}{z}=E(z)$$

$$\frac{H(z)}{z}=F(z)$$

This is the system of linear equations (variables are $A(z), B(z), \ldots, F(z)$) and part of its solution that we are interested in is:

$$A(z)=\frac{1+z^3}{1-z-z^2-2z^4+2z^5+z^6+z^7+z^8}$$

From the denominator of this generating function $1-z-z^2-2z^4+2z^5+z^6+z^7+z^8$, we can find the linear recurrence equation
$a(n+8)-a(n+7)-a(n+6)-2a(n+4)+2a(n+3)+a(n+2)+a(n+1)+a(n)=0$, i.e.

$$a(n+8)=a(n+7)+a(n+6)+2a(n+4)-2a(n+3)-a(n+2)-a(n+1)-a(n).$$

The number of n-variation**s** of the set $N_{n+1}, a(n)$ satisfying the condition $p(i)-i \in \{-2,0,2\}$ for all $i \in N_n$ is determined by its generating function $A(z)$:

Table 7: The number of n-variation**s** of the set N_{n+1}, $a(n)$, satisfying $p(i)-i \in \{-2,0,2\}$

n	0	1	2	3	4	5	6	7	8	9	10	…
$a(n)$	1	1	2	4	8	12	21	35	60	96	160	…

This is the sequence A217694 at [9].

7. CONCLUSION

The finite state automata are powerful tool for generating some combinatorial structures. Tracing the generation, it leads us to the system of recurrence equations. The usage of finite state automata encompasses the other known techniques for counting restricted permutations, i.e. as expanding permanents (in the sense of the computational complexity), Stanley Transfer-matrix method [7] and Baltić's technique from [1] (it has less equations in the system of the recurrence equations then Stanley Transfer-matrix method and can easily manage with the circular case, or with some additional cases such as parity of the permutation). By this approach, we have solved several hard combinatorial problems.

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REFERENCES

[1] Baltić, V., "On the number of certain types of strongly restricted permutations", *Applicable Analysis and Discrete Mathematics*, 4(1) (2010) 119-135.
[2] Baltić, V., "Applications of the finite state automata in the enumerative combinatorics", *Proceedings of XXXVI Symposium on Operational Research*, Ivanjica, 2009, 55-158.
[3] Baltić, V., and Saković, M., "The counting of even and odd restricted permutations with the finite state automata ", *Proceedings of XXXIX Symposium on Operational Research*, Tara, 2012, 155-158.
[4] Bergum, G.E., and Hoggat, V.E, "A combinatorial problem involving recursive sequences and tridiagonal matrices", *The Fibonacci Quarterly*, 16 (1978) 113-118.
[5] Cvetkovic, D.M., Doob, M., and Sachs, H., *Spectra of Graphs*, Theory and Aplication, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.
[6] Grimaldi, R.P., *Discrete and Combinatorial Mathematics - An Applied Introduction*, Addison-Wesley, 1994.
[7] Lehmer, D.H., "Permutations with strongly restricted displacements", *Combinatorial Theory and its Applications II*, North-Holland, Amsterdam-London, 1970, 755-770.
[8] Stanley, R.P., *Enumerative Combinatorics*, *Volume 2,* Cambridge University Press, Cambridge, 1999.
[9] *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org/