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# FUZZY OPTIMIZATION OF PRIMAL-DUAL PAIR USING PIECEWISE LINEAR MEMBERSHIP FUNCTIONS

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**Abstract:** Present paper improves the model of Bector and Chandra on duality in fuzzy linear programming by using non-linear membership functions. Numerical problem discussed by these authors has also been worked out through our non–linear model to demonstrate the obtain improvements.

**Keywords:** Linear primal-dual problems, Fuzzy environment, tolerance, Non-linear membership function.

MSC: 03E72, 49N15

## 1. INTRODUCTION

Theories in economics indicate that abundance of resources correlates inversely with their market values. Any study pertaining to resource allocation is therefore useful only when the resources are limited. Solutions of Linear Programming (LP) problems give answer to the allocation of resources through the solution of the primal problem [2]. The solution to the dual problem does the market valuation of the resources. In crisp environment, solution of primal (dual) implicitly provides the solution of dual (primal). In fuzzy environment, instead of optimization, the goal is to find optimal satisfaction of aspiration levels for both primal and dual problems. The concept of Fuzzy Linear Programming (FLP) was introduced by Zimmermann [5]. In FLP, both optimization of primal-dual pair and reaching aspiration value in each case as close as possible are important. The primal-dual theory can be considered as a theory of industry-market problems and has numerous economic, business and industrial interpretations and applications [1]. Quite a lot of literature is available on fuzzy linear programming, but there are comparatively very few works on fuzzy dual problems. The first work on duality of fuzzy linear programming was done by Rodder and Zimmermann [3], who worked on a problem of economics and derived their result on the basis of economic interpretation of dual variables. Bector and Chandra [1] introduced a fuzzy pair of primal-dual problems modifying the construction of the fuzzy dual model of Rodder-Zimmermann formulation.

Yang et al. [4] in their study of FLP found that their approach to deal with non-linear membership functions was straightforward and computationally efficient. Present paper improves the Bector and Chandra model of fuzzy primal-dual problem using non-linear membership functions. The numerical example of [1] has been done on the present model, which resulted in an improved value of $\eta$ in the pair $(\lambda, \eta)$ obtained in [1].

The rest of the paper is organized as follows: In section 2, main features of Bector-Chandra model are given for ready reference. Piecewise linear membership functions are introduced in section 3 and the corresponding crisp equivalent of the fuzzy primal-dual pair are defined. Section 4 establishes some weak duality results for the piecewise linear membership functions corresponding to their linear counterparts. In section 5, numerical example of [1] has been worked out according to the present model, and the results are compared with those of [1]. The conclusion is given in section 6.

## 2. BECTOR-CHANDRA MODEL FOR FUZZY DUAL

Let $R^n$ denote the n-dimensional Euclidean space and $R_+^n$ its non negative part.

For $x, c \in R^n, w, b \in R^m$ and the matrix $A \in R^m \times R^n$, the Linear Primal (LP) and Linear Dual (LD) problems are expressed in the vector forms:

(LP) Maximize $c^T x$

subject to: $Ax \leq b, x \geq 0$

and

(LD) Maximize $b^T w$

subject to: $A^T w \geq c, w \geq 0$

Bector and Chandra [1] gave the fuzzy versions of $LP$ and $LD$, respectively. In the sense of Zimmermann [5], it is described as below:

Find $x \in R^n$ such that

$$\begin{aligned} & c^T x \underset{\sim}{\geq} Z_0 \\ & Ax \underset{\sim}{\leq} b, x \geq 0 \end{aligned} \tag{1}$$

and

Find $w \in R^m$ such that

$$\begin{aligned} & b^T w \underset{\sim}{\leq} W_0 \\ & A^T w \underset{\sim}{\geq} c, w \geq 0 \end{aligned} \tag{2}$$

Here "$\underset{\sim}{\geq}$" and "$\underset{\sim}{\leq}$" are fuzzy versions for the usual symbols "$\geq$" and "$\leq$" respectively, having the linguistic interpretation as explained in [5]. Let $Z_0$, $W_0$ respectively denote the aspiration levels of the two objectives $c^T x$ and $b^T w$. Further assuming $p_0$ and $p_i\,(i=1,2,3,...,m)$ to be subjectively chosen positive constants representing the admissible tolerance values associated with the objective function and $m$ linear constraints of (1) respectively, the crisp equivalent of the fuzzy problem (1) in Bector and Chandra model [1, p319] is given as below:

$$\begin{aligned} &\text{Maximize } \lambda \\ \text{subject to: } &(\lambda-1)p_0 \le c^T x - Z_0, \\ &(\lambda-1)p_i \le b_i - A_i x, i=1,2....m \\ &\lambda \le 1 \text{ and } x,\lambda \ge 0 \end{aligned} \tag{3}$$

where $A_i$ and $b_i$ are the $i^{th}$ row of matrix $A$ and $i^{th}$ component of vector $b(i=1,2,...,m)$, respectively. Similarly for $q_0$ and $q_j\,(j=1,2,...,n)$ being the corresponding values for (2), the crisp equivalent for the problem (2) is given as below:

$$\begin{aligned} &\text{Minimize}(-\eta) \\ \text{subject to: } &(\eta-1)q_0 \le W_0 - b^T w, \\ &(\eta-1)q_j \le A_j^T w - c_j,\ j=1,2........n, \\ &\eta \le 1 \text{ and } w,\eta \ge 0 \end{aligned} \tag{4}$$

Pair (3)-(4) is termed as fuzzy pair of primal–dual linear programming in [1].

## 3. FUZZY MODEL WITH PIECEWISE LINEAR MEMBERSHIP FUNCTIONS

Although linear membership functions permit an easy conversion of the FLP problem into a crisp linear programming problem, yet in many cases membership functions are best represented by non-linear functions. Yang et.al. [4] have provided a representation of concave (and non-concave) non-linear membership functions approximated by two and three linear segments. Our approach in this work is to approximate the linear membership functions of [1] by two linear segments. We use the following notations for problems (1) and (2):

- $\mu_0^P(x)/\mu_0^D(w)$ : Membership functions for Primal/Dual corresponding to objective function;
- $\mu_{0k}^P(x)/\mu_{0k}^D(w)$ : Membership functions for $k^{th}(k=1,2)$ linear segment of Primal/Dual corresponding to objective function;
- $\mu_i^P(x)/\mu_j^D(w)$ : Membership functions for Primal/Dual corresponding to *m/n* constraints $i=1,2,...,m$ and $j=1,2,...n$

We start with the following linear membership functions of Bector-Chandra model.

$$\mu_0^P(x)=\begin{cases}1 & ,c^T x\geq Z_0,\\ 1-\dfrac{Z_0-c^T x}{p_0} & ,Z_0-p_0\leq c^T x<Z_0,\\ 0 & ,Z_0-p_0\geq c^T x,\end{cases} \tag{5}$$

$$\mu_i^P(x)=\begin{cases}1 & ,A_i x\leq b_i,\\ 1-\dfrac{A_i x-b_i}{p_i} & ,b_i<A_i x\leq b_i+p_i,\\ 0 & ,A_i x>b_i+p_i.\end{cases} \tag{6}$$

Let us now approximate $\mu_0^P(x)$ by two linear segments as shown in Figure 1, and define the membership functions corresponding to each segment:

$$\mu_{01}^P(x)=\begin{cases}1 & ,c^T x\geq Z_{01},\\ 1-\dfrac{Z_{01}-c^T x}{p_{01}} & ,Z_{01}-p_{01}\leq c^T x\ <Z_{01},\\ 0 & ,otherwise,\end{cases} \tag{7a}$$

and

$$\mu_{02}^P(x)=\begin{cases}1 & ,c^T x\geq Z_{01},\\ 1-\dfrac{Z_{02}-c^T x}{p_{02}} & ,Z_{02}-p_{02}\leq c^T x\ <Z_{02},\\ 0 & ,\text{otherwise},\end{cases} \tag{7b}$$

where $p_{01}+p_{02}>p_0$ and $Z_{01}<Z_{02}=Z_0$

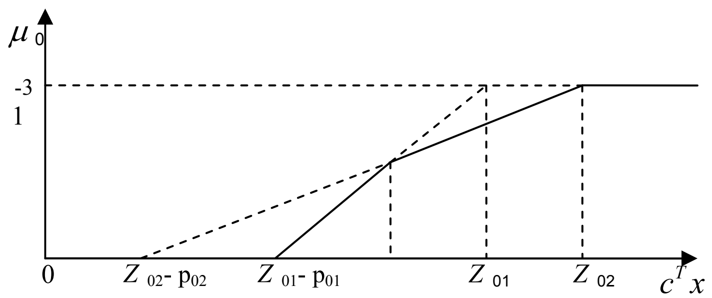


**Figure 1:** Linear membership function in two line segments

Using $\lambda = \min\{\mu_{01}^P, \mu_{02}^P, \mu_1^P, \ldots.\mu_m^P\}$ [6, p245], the crisp equivalent of the fuzzy problem (1) becomes:

$$\begin{aligned}
&\text{Maximize } \lambda\\
&\text{subject to: } \lambda \le 1 - \frac{Z_{01} - c^T x}{p_{01}}\\
&\qquad \lambda \le 1 - \frac{Z_{02} - c^T x}{p_{02}}\\
&\qquad \lambda \le 1 - \frac{A_i x - b_i}{p_i},\ i = 1, 2, 3, \ldots, m\\
&\qquad \lambda \le 1 \text{ and } x, \lambda \ge 0
\end{aligned} \tag{8}$$

Similarly, in the dual problem (2), $\mu_0^D(w)$ is replaced by following membership functions.

$$\mu_{01}^D(w) = \begin{cases} 1 & , b^T w \le W_{01}, \\ 1 + \dfrac{W_{01} - b^T w}{q_{01}} & , W_{01} < b^T w \le W_{01} + q_{01}, \\ 0 & , \text{otherwise,} \end{cases} \tag{9a}$$

$$\mu_{02}^D(w) = \begin{cases} 1 & , b^T w \le W_{02}, \\ 1 + \dfrac{W_{02} - b^T w}{q_{02}} & , W_{02} < b^T w \le W_{02} + q_{02}, \\ 0 & , \text{otherwise,} \end{cases} \tag{9b}$$

where $q_{01} + q_{02} > q_0$ and $W_{01} > W_{02} = W_0$.

Further, letting $\eta = \min\{\mu_{01}^D, \mu_{02}^D, \mu_1^D, \ldots.\mu_n^D\}$, the crisp equivalent for the dual problem is

$$\begin{aligned}
&\text{Minimize } (-\eta)\\
&\text{subject to: } \eta \le 1 + \frac{W_{01} - b^T w}{q_{01}},\\
&\qquad \eta \le 1 + \frac{W_{02} - b^T w}{q_{02}},\\
&\qquad \eta \le 1 + \frac{A_j^T w - c_j}{q_j},\ j = 1, 2, 3 \ldots\ldots n,\\
&\qquad \eta \le 1, \quad and \quad \eta, w \ge 0
\end{aligned} \tag{10}$$

The pair (8)-(10) is termed as fuzzy pair of primal–dual linear programming problems in our model.

## 4. SOME WEAK DUALITY RESULTS

**Standard weak duality theorem:** Let $x$ and $w$ be any feasible solutions to problems (LP) and (LD) respectively, then

$$c^T x \le b^T w \tag{11}$$

**Modified weak duality theorem:** Let $(x, \lambda)$ be feasible for (3) and $(w, \eta)$ be feasible for (4), then

$$(\lambda - 1)p^T w + (\eta - 1)q^T x \le (b^T w - c^T x) \tag{12}$$

where $p^T = (p_1, p_2, ..., p_m)$ and $q^T = (q_1, q_2, ..., q_n)$.

The modified weak duality theorem has been proved by Bector and Chandra [1, p 320] for the fuzzy environment, where they considered membership functions to be linear. Obviously (12) reduces to (11) for $\lambda = \eta = 1$. Further, since (12) does not involve tolerance values $(p_0\ and\ q_0)$ associated with the objective functions, it can be concluded that any change in the membership functions for the objective functions as suggested in (7a), (7b), (9a) and (9b), will not effect the result. This justifies the following remark.

**Remark 1:** The modified weak duality theorem is valid even if $(x, \lambda)$ and $(w, \eta)$ are feasible solutions for (8) and (10), respectively.

The inequality [1, (28)], relating the relative difference of aspiration level $Z_0$ of $c^T x$ and $W_0$ of $b^T w$ in terms $p_0\ and\ q_0$ respectively, can be modified in the form of the following result.

**Theorem 1.** *Let $(x, \lambda)$ be the feasible solution for non-linear primal (8) and $(w, \eta)$ be feasible solution for non-linear dual (10), then*

$$(\lambda - 1)(p_{01} + p_{02}) + (\eta - 1)(q_{01} + q_{02}) \le 2(c^T x - b^T w) + \{(W_{01} + W_{02}) - (Z_{01} + Z_{02})\} \tag{13}$$

**Proof:** From the first two inequalities of (8) and (10) we have

$$\begin{aligned} &(\lambda - 1)(p_{01} + p_{02}) \le 2c^T x - (Z_{01} + Z_{02}), \\ &(\eta - 1)(q_{01} + q_{02}) \le (W_{01} + W_{02}) - 2b^T w \end{aligned} \tag{14}$$

Adding the two inequalities in (14), the proof is complete.

We now extend the result [1, Corollary 1] to the non-linear case.

**Theorem 2.** *Let $(\overline{x}, \overline{\lambda})$ and $(\overline{w}, \overline{\eta})$ be feasible solutions of (8) and (10), respectively that satisfy:*

(i) $(\overline{\lambda} - 1)p^T \overline{w} + (\overline{\eta} - 1)q^T \overline{x} = (b^T \overline{w} - c^T \overline{x})$,

(ii) $(\overline{\lambda} - 1)(p_{01} + p_{02}) + (\overline{\eta} - 1)(q_{01} + q_{02}) = 2(c^T \overline{x} - b^T \overline{w}) + \{(W_{01} + W_{02}) - (Z_{01} + Z_{02})\}$

(iii) $Z_{01} + Z_{02} \le W_{01} + W_{02}$,

then $(\overline{x}, \overline{\lambda})$ is optimal to (8) and $(\overline{w}, \overline{\eta})$ is optimal to (10).

**Proof:** Let $(x,\lambda)$ and $(w,\eta)$ be some feasible solutions of (8) and (10), respectively. Then, by Remark 1, we have

$$(\lambda-1)p^T w+(\eta-1)q^T x \le (b^T w-c^T x) \tag{15}$$

From (15) and the hypothesis (i) of Theorem 2, we have

$$(\lambda-1)p^T w+(\eta-1)q^T x-(b^T w-c^T x) \le (\overline{\lambda}-1)p^T \overline{w}+(\overline{\eta}-1)q^T \overline{x}-(b^T \overline{w}-c^T \overline{x}) \tag{16}$$

From hypothesis (i) and (16), it can be implied that $(\overline{x},\overline{\lambda},\overline{w},\overline{\eta})$ is optimal to the following problem whose maximum is zero.
Maximize $\{(\lambda-1)p^T w+(\eta-1)q^T x-(b^T w-c^T x)\}$

subject to: $(\lambda-1)p_{01} \le c^T x-Z_{01}$,

$(\lambda-1)p_{02} \le c^T x-Z_{02}$,

$(\eta-1)q_{01} \le W_{01}-b^T w$,

$(\eta-1)q_{02} \le W_{02}-b^T w$,

$(\lambda-1)p_i \le c_i-A_i x$,

$(\eta-1)q_j \le A_j^T w-c_j$,

$\lambda \le 1, \eta \le 1$,

$x,w,\lambda,\eta \ge 0$.

Multiplying (i) by 2 and then adding to (ii), we get

$$2(\overline{\lambda}-1)p^T \overline{w}+2(\overline{\eta}-1)q^T \overline{x}+(\overline{\lambda}-1)(p_{01}+p_{02})+(\overline{\eta}-1)(q_{01}+q_{02})+\{(Z_{01}+Z_{02}) -(W_{01}+W_{02})\}=0$$

Since each term of the above expression is non-positive due to hypothesis (iii) and the fact that $\overline{\lambda},\overline{\eta} \le 1$, it must therefore be separately equal to zero. Hence,

$$\begin{aligned}(\overline{\lambda}-1)(p_{01}+p_{02})&=0\\ (\overline{\eta}-1)(q_{01}+q_{02})&=0\end{aligned} \tag{17}$$

Further, since $p_{01},p_{02}$, $q_{01},q_{02}>0$, we also have

$$\begin{aligned}&(\lambda-1)p_{01} \le 0, (\lambda-1)p_{02} \le 0\\ &(\eta-1)q_{01} \le 0, (\eta-1)q_{02} \le 0\end{aligned} \tag{18}$$

From (17) and (18), it is obvious that

$$(\lambda-1)p_{01} \le (\overline{\lambda}-1)p_{01},\ (\lambda-1)p_{02} \le (\overline{\lambda}-1)p_{02}$$
$$(\eta-1)q_{01} \le (\overline{\eta}-1)q_{01},\ (\eta-1)q_{02} \le (\overline{\eta}-1)q_{02}$$

Above inequalities imply that $\lambda \le \overline{\lambda}$ and $-\eta \ge -\overline{\eta}$. This proves the theorem.

**Remark 2:** Following the arguments of [1, Remark 3], there may not be any direct or converse duality theorems between (8) and (10), and if (8) [or (10)] has an optimal solution, then (10)[or(8)] certainly has a feasible solution. Further, the optimal values of two objective functions of (8) and (10) may not be equal in general**.**

## 5. NUMERICAL EXAMPLE

Consider the following primal-dual problem [1, p 323].

(P) Maximize $2x$, subject to $x \underset{\sim}{\leq} 1, x \geq 0,$

and

(D) Minimize $w$, subject to $w \underset{\sim}{\geq} 2$, $w \geq 0$ .

Consider Fig. 2. Taking $Z_{01} = 1/2$ and $Z_{02} = 1$, we get $p_{01} = 3/2$ and $p_{02} = 4$, $p_1 (= 2)$ remains the same as in [1]. Using (8), the crisp equivalent of the fuzzy problem (P) becomes:

$$\begin{aligned} &\text{Maximize } \lambda \\ &\text{subject to: } \lambda \leq 1 - \frac{\frac{1}{2} - 2x}{\frac{3}{2}}, \ \lambda \leq 1 - \frac{1-2x}{4}, \\ &\qquad 2\lambda + 2x \leq 3, \ \lambda \leq 1 \text{ and } x, \lambda \geq 0 \end{aligned} \tag{19}$$

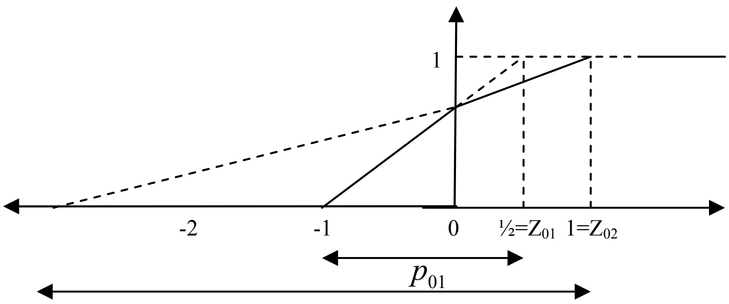


**Figure 2:** Piecewise linear membership functions for problem (P)

Referring to Figure 3 and using $W_{01}$=1.70, $W_{02}$=1, $q_{01}$=0.30, $q_{02}$=1.75 and $q_1$= 3 in (10), the crisp equivalent of fuzzy problem (D) can be obtained as following.

Minimize $(-\eta)$

$$\text{subject to: } \eta \le 1 + \frac{1.70 - w}{.30}, \quad \eta \le 1 + \frac{1-w}{1.75}, \quad 3\eta - w \le 1, \quad \eta \le 1 \text{ and } \eta, w \ge 0. \tag{20}$$

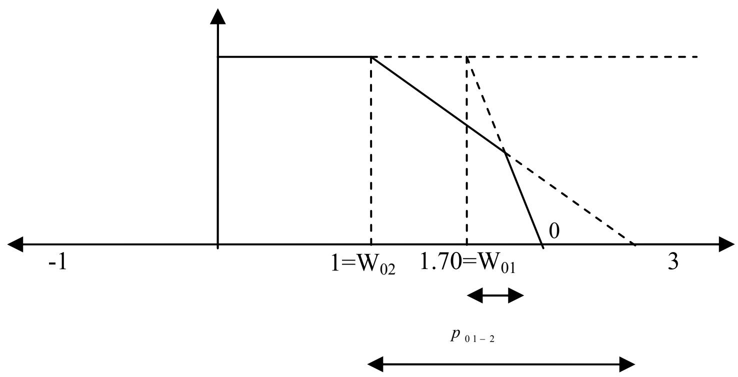


**Figure 3:** Piecewise linear membership functions for problem (D)

The optimum solutions $(\bar{\lambda}, \bar{\eta})$ of fuzzy primal-dual pair (P-D) through (19) and (20) are $\bar{\lambda} = 1$ and $\bar{\eta} = 0.79$ leading to $\bar{x} = 0.50$ and $\bar{w} = 1.37$. The optimum pair of results $(\bar{\lambda}, \bar{\eta})$ for the same problem from Bector- Chandra model were $\bar{\lambda} = 1$ and $\bar{\eta} = 0.75$ leading to $\bar{x} = 0.50$ and $\bar{w} = 1.25$. It can be observed that the maximum value of $\lambda$ has been fully achieved as 1 in both models, but the minimum value of $-\eta$ has been improved in our model from -0.75 to -0.79. This implies that in (D), the aspiration for $w \underset{\sim}{\geq} 2$ could be achieved up to $w$ = 1.37 by our approach, contrary to $w$ = 1.25 in [1].

## 6. CONCLUSION

A fuzzy primal-dual model is worked out in this paper using non-linear membership functions and thereby improving a similar model given by Bector and Chandra [1]. Some weak duality results in fuzzy environment corresponding to piecewise linear membership functions are also established. Working numerically on the example of [1], it is demonstrated that the fuzzy primal-dual model using piecewise linear membership functions is capable of giving better optimization. The problem is still open to possible extension to a class of more general non-linear membership functions.

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