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EFFICIENCY AND DUALITY IN NONSMOOTH MULTIOBJECTIVE FRACTIONAL PROGRAMMING INVOLVING η-PSEUDOLINEAR FUNCTIONS

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Abstract: In this paper, we shall establish necessary and sufficient conditions for a feasible solution to be efficient for a nonsmooth multiobjective fractional programming problem involving η – pseudolinear functions. Furthermore, we shall show equivalence between efficiency and proper efficiency under certain boundedness condition. We have also obtained weak and strong duality results for corresponding Mond-Weir subgradient type dual problem. These results extend some earlier results on efficiency and duality to multiobjective fractional programming problems involving pseudolinear and η – pseudolinear functions.

Keywords: Multiobjective fractional programming, nonsmooth programming, pseudolinearity, η – pseudolinearity, duality.

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1. INTRODUCTION

In the optimization theory, convexity and its different generalizations play an important role. Mangasarian [16] introduced the concept of pseudoconvex functions as a generalization of the convex functions. Chew and Choo [5] introduced a new class of functions both pseudoconvex and pseudoconcave known as pseudolinear functions. Chew and Choo [5] obtained first and second order characterizations for pseudolinear functions. Komlosi [11] and Rapcsak [22] studied and characterized higher order pseudolinear functions. Rueda [23] introduced the concept of $\eta-$pseudolinear functions and Ansari et al. [2] have derived several characterizations for $\eta-$pseudolinear functions. Recently, Giorgi and Rueda [10] have extended some results of Chew and Choo [5] and obtained the conditions for a feasible solution to be efficient and properly efficient. Ansari and Rezaei [1] introduced the concept of $\eta-$pseudolinear functions for nondifferentiable locally Lipschitz functions, and characterized $\eta-$pseudolinear functions and related programs.

In fractional programming, we study the optimization problems in which objective functions are ratios of two functions. These problems arise in different areas of modern research such as economics [4], information theory [17], engineering design [26], and heat exchange networking [29]. Schaible [24] and Bector [3] have studied duality in fractional programming for single objective functions. Duality in multiobjective fractional programming problems involving generalized convex functions have been widely studied by Chandra et al. [7], Egudo [8], Mukherjee and Rao [20], Weir [27] and Kuk et al. [15]. For more references and further details please see; Mishra and Giorgi [18]. Kaul et al. [12] have studied Mond-Weir-type of dual for the multiobjective fractional programming problems involving pseudolinear and $\eta-$pseudolinear functions and derived various duality results. Duality for nonsmooth multiobjective fractional programming problems involving generalized convex functions have been studied by Kim [13], Kuk [14], Stancu-Minasian et al. [25] and Nobakhtian [21].

In the present paper, we shall establish the necessary and sufficient conditions for a feasible solution to be efficient for a nonsmooth multiobjective fractional programming problem involving $\eta-$pseudolinear functions. Furthermore, we show the equivalence between efficiency and proper efficiency under certain boundedness condition. We also derive weak and strong duality results for corresponding Mond-Weir subgradient type dual problem. The results of this paper extend several results of Chew and Choo [5], Kaul et al. [12] and Giorgi and Rueda [10] to the nonsmooth case.

2. DEFINITIONS AND PRELIMINARIES

Let R^n be the $n-$dimensional Euclidean space and $\langle .,. \rangle$ denotes the usual inner product in R^n. Let $K \subseteq R^n$ be a nonempty set. Let $\eta : K \times K \to R^n$ be a map.

The following conventions for vectors in R^n will be adopted:

1. $x = y$ if and only if $x_i = y_i$, for all $i = 1, 2, ..., n$;
2. $x > y$ if and only if $x_i > y_i$, for all $i = 1, 2, ..., n$;

3. $x \geqq y$ if and only if $x_i \geqq y_i$, for all $i = 1, 2, ..., n$;
4. $x \geq y$ if and only if $x_i \geqq y_i$, for all $i = 1, 2, ..., n$ but $x \neq y$.

Definition 2.1 [6]. *A function $f : K \to R$ is said to be Lipschitz (of rank M) near x, if there exists a nonnegative scalar M and a positive constant ε, such that*

$$|f(x') - f(x'')| \leq M \|x' - x''\| \text{ for all } x', x'' \in x + \varepsilon B,$$

where $x + \varepsilon B$ is open ball of radius ε, about x. The function f is said to be Lipschitz on K, if the above condition is satisfied for all $x \in K$.

Definition 2.2 [6]. Let $f : K \to R$ be locally Lipschitz at a given point $x \in K$. The Clarke's generalized directional derivative of f at $x \in K$ in the direction of a vector $v \in K$, denoted by $f^{\circ}(x; v)$ is defined by

$$f^{\circ}(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

We know that the usual one sided directional derivative of f at $x \in K$ in the direction of a vector $v \in K$, denoted by $f'(x; v)$ is defined by

$$f'(x; v) = \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t},$$

whenever the limit exists. Obviously, $f^{\circ}(x; v) \geq f'(x; v)$.

Definition 2.3 [6]. *Let $f : K \to R$ be locally Lipschitz at a given point $x \in K$. The Clarke's generalized subdifferential of f at $x \in K$, denoted by $\partial^c f(x)$ is defined by*

$$\partial^c f(x) = \left\{ \xi \in \square^n : f^{\circ}(x, v) \geq \langle \xi, v \rangle, \text{ for all } v \in R^n \right\}.$$

Definition 2.4 [19]. *Let x be an arbitrary point of K. The set K is said to be invex at x with respect to η, if for all $y \in K$,*

$$x + t\eta(y, x) \in K, \quad \text{for all } t \in [0,1].$$

The set K is said to be invex with respect to η if K is invex with respect to η for all $x \in K$.

Condition C [19]. *Let $K \subseteq R^n$ be an invex set with respect to η, then η is said to satisfy the condition C, if*

1. $\eta(x, x + t\eta(y, x)) = -t\eta(y, x)$
2. $\eta(y, x + t\eta(y, x)) = (1 - t)\eta(y, x)$.

Yang et al. [28] have shown that

$$\eta(x+t\eta(y,x),x)=t\eta(y,x). \tag{1}$$

Definition 2.5 [1]. *A locally Lipschitz function* $f:K\subseteq R^n\to R$ *is said to be*

1. pseudoinvex with respect to η if for all $x,y\in K$ and for some $\xi\in\partial^c f(x)$ $\langle\xi,\eta(y,x)\rangle\geq 0$ implies $f(y)\geq f(x)$.
2. pseudoincave with respect to η if for all $x,y\in K$ and for some $\xi\in\partial^c f(x)$ $\langle\xi,\eta(y,x)\rangle\leq 0$ implies $f(y)\leq f(x)$.
3. $\eta-$pseudolinear if f is both pseudoinvex and pseudoincave with respect to the same η.

Lemma 2.1 [1]. *Let* K *be an invex set with respect to* $\eta:K\times K\to R^n$, *that satisfies the condition C. Then* f *is pseudolinear with respect to* η *on* K *if and only if there exists a function* $p:K\times K\to R^+$ *such that for all* $x,y\in K$ *and for some* $\xi\in\partial^c f(x)$,

$$f(y)=f(x)+p(x,y)\langle\xi,\eta(y,x)\rangle.$$

Definition 2.6 [6]. *A real valued function* f *is said to be regular at* x *if for all* $v\in R^n$, *one sided directional derivative* $f'(x;v)$ *exists and* $f'(x,v)=f^{\circ}(x,v)$.

Lemma 2.2 [6]. *Let* f *and* g *be Lipschitz near* x *and suppose that* $g(x)\neq 0$. *Then* $\frac{f}{g}$ *is Lipschitz near* x *and*

$$\partial^c\left(\frac{f}{g}\right)(x)\subset\frac{g(x)\partial^c f(x)-f(x)\partial^c g(x)}{(g(x))^2}.$$

If in addition $f(x)\geqq 0$, $g(x)>0$ and if f and $-g$ are regular at x, then equality holds and $\frac{f}{g}$ is regular at x.

Lemma 2.3. Let f and g are two $\eta-$pseudolinear functions defined on an open invex subset K of R^n with the same proportional function $p(x,y)$ and $g(x)>0$ for every x in K. If f and $-g$ are regular on K and η satisfies the condition C on K. Then $\frac{f}{g}$ is also $\eta-$pseudolinear with respect to a new proportional function

$$\overline{p}(x,y)=\frac{g(x)p(x,y)}{g(y)}.$$

Proof: *Since f and g are $\eta-$pseudolinear with respect to the same proportional function p, it follows that for any x and y in K,*

$$f(y) = f(x) + p(x,y)\langle \xi, \eta(y,x)\rangle, \text{ for some } \xi \in \partial^c f(x) \tag{2}$$

$$g(y) = g(x) + p(x,y)\langle \zeta, \eta(y,x)\rangle, \text{ for some } \zeta \in \partial^c g(x). \tag{3}$$

Now using the Lemma 2.2, we have for some $\xi \in \partial^c f(x)$ and $\zeta \in \partial^c g(x)$

$$p(x,y)\left\langle \partial^c\left(\frac{f(x)}{g(x)}\right), \eta(x,y)\right\rangle = p(x,y)\left\langle \frac{g(x)\xi - f(x)\zeta}{(g(x))^2}, \eta(y,x)\right\rangle$$

$$= p(x,y)\left(\frac{g(x)\langle \xi, \eta(y,x)\rangle - f(x)\langle \zeta, \eta(y,x)\rangle}{(g(x))^2}\right)$$

$$= \frac{g(y)}{g(x)}\left(\frac{f(y)}{g(y)} - \frac{f(x)}{g(x)}\right),$$

by using (2) and (3).
Thus

$$\frac{f(y)}{g(y)} = \frac{f(x)}{g(x)} + \frac{g(x)p(x,y)}{g(y)}\left\langle \partial^c\left(\frac{f(x)}{g(x)}\right), \eta(y,x)\right\rangle,$$

which implies that $\frac{f}{g}$ is $\eta-$pseudolinear with respect to the proportional function $\overline{p}(x,y) = \frac{g(x)p(x,y)}{g(y)}$.

Throughout the paper we assume that $\eta : K \times K \to R^n$ satisfies the condition C.

We consider the following nonsmooth multiobjective fractional programming problem:

$$\text{(NMFP)} \quad \min\left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_k(x)}{g_k(x)}\right)$$

subject to $h_j(x) \le 0,\ j \in \{1,2,\ldots,m\}$,

where $f_i : K \to R,\ g_i : K \to R,\ i \in I = \{1,2,\ldots,k\};\ h_j : K \to R,\ j \in J = \{1,2,\ldots,m\}$ are locally Lipschitz functions on a nonempty invex subset $K \subseteq R^n$. Now on, we assume that the functions f and $-g$ are regular on K and $f(x) \geqq 0,\ g(x) > 0$ for all $x \in R^n$.

Let $P = \{x \in K : h_j(x) \le 0, j \in J\}$ and $J(x) = \{j \in J : h_j(x) = 0\}$, for some $x \in K$ denote the set of all feasible solutions for (NMFP) and active constraints index set at $x \in K$ respectively.

Definition 2.7. *A feasible solution* $\bar{x}$ *is said to be an efficient (Pareto optimal) solution for (NMFP), if there exist no* $x \in P$ *such that*

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\bar{x})}{g_i(\bar{x})}, \qquad \text{for all } i = 1, 2, ..., k$$

$$\text{and } \frac{f_r(x)}{g_r(x)} < \frac{f_r(\bar{x})}{g_r(\bar{x})}, \qquad \text{for some } r.$$

Definition 2.8 [9]. *A feasible solution* $\bar{x}$ *is said to be properly efficient solution for (NMFP) if it is efficient and if there exists a scalar* $M > 0$, *such that for each* i,

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \leq M\left[\frac{f_r(\bar{x})}{g_r(\bar{x})} - \frac{f_r(x)}{g_r(x)}\right],$$

for some r such that

$$\frac{f_r(x)}{g_r(x)} > \frac{f_r(\bar{x})}{g_r(\bar{x})} \text{ whenever } x \in P \text{ with } \frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})}.$$

3. EFFICIENCY

In this section, we shall establish the necessary and sufficient conditions for a feasible solution to be efficient and show the equivalence between efficiency and proper efficiency under certain boundedness condition.

Now, we shall prove the following theorem analogous to Proposition 3.2 of Chew and Choo [5] for the problem (NMFP) by the use of Lemma 2.1, Lemma 2.2 and Lemma 2.3.

Theorem 3.1. *Let* $\bar{x}$ *be a feasible solution for the problem (NMFP). Let the functions* f_i *and* g_i *be* η*-pseudolinear on invex set* $K \subseteq R^n$ *with respect to same proportional function* $p_i\left(i = 1, 2, ..., k\right)$ *and* h_j *be* η*-pseudolinear on* K *with proportional function* q_j *for* $j \in J\left(\bar{x}\right)$. *Then* $\bar{x}$ *is an efficient solution of (NMFP), if and only if there exist* $\lambda_i > 0,\ i = 1, 2, ..., k;\ \mu_j \geq 0,\ j \in J(\bar{x})$ *and for some* $\xi_i \in \partial^c\left(\frac{f_i(\bar{x})}{g_i(\bar{x})}\right)$ *and* $\zeta_j \in \partial^c h_j(\bar{x})$, *such that*

$$\sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j \in J(\bar{x})} \mu_j \zeta_j = 0. \tag{4}$$

Proof: Suppose λ_i and μ_j exist and satisfy the given condition and (4). Let $\bar{x}$ is not efficient. Therefore, there exists a point $y \in K$, such that

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} \geq \frac{f_i(y)}{g_i(y)}, \text{ for all } i$$

and

$$\frac{f_r(\bar{x})}{g_r(\bar{x})} > \frac{f_r(y)}{g_r(y)}, \text{ for some } r.$$

Then

$$0 \geq \sum_{j \in J(\bar{x})} \frac{\mu_j}{\bar{q}_j(\bar{x}, y)} (h_j(y) - h_j(\bar{x}))$$

$$= \sum_{j \in J(\bar{x})} \mu_j \left\langle \zeta_j, \eta(y, \bar{x}) \right\rangle, \text{ where } \zeta_j \in \partial^c h_j(\bar{x})$$

$$= -\sum_{i=1}^{k} \lambda_i \left\langle \xi_i, \eta(y, \bar{x}) \right\rangle$$

$$= -\sum_{i=1}^{k} \frac{\lambda_i}{\bar{p}_i(\bar{x}, y)} \left(\frac{f_i(y)}{g_i(y)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \right) > 0, \text{ by Lemma 2.3,}$$

which leads to a contradiction.

Conversely, we assume that $\bar{x}$ is an efficient solution. Therefore, for $1 \leq r \leq k$, for some $\xi_i \in \partial^c \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} \right)$ and $\zeta_j \in \partial^C h_j(\bar{x})$ the system of inequalities:

$$\left.\begin{array}{l} \left\langle \zeta_j, \eta(x, \bar{x}) \right\rangle \leq 0; \ j \in J(\bar{x}) \\ \left\langle \xi_i, \eta(x, \bar{x}) \right\rangle \leq 0; \ i = 1, 2, \ldots, r-1, r+1, \ldots, k \\ \left\langle \xi_r, \eta(x, \bar{x}) \right\rangle < 0; \text{ for some } \xi_r \in \partial^C \left(\frac{f_r(\bar{x})}{g_r(\bar{x})} \right) \end{array}\right\} \tag{5}$$

has no solution $x \in K$. For if x is a solution of the system and $y = \bar{x} + t\eta(x, \bar{x})$ $(0 < t \leq 1)$, then for $j \in J(\bar{x})$ and using (1), it follows that

$$\left\langle \zeta_j, \eta(y, \bar{x}) \right\rangle = t \left\langle \zeta_j, \eta(x, \bar{x}) \right\rangle \leq 0, \text{ where } \zeta_j \in \partial^c h_j(\bar{x})$$

and thus we get

$$h_j(y) \leq h_j(\bar{x}) = 0.$$

If $j \notin J(\bar{x})$ then $h_j(\bar{x}) < 0$ and so $h_j(y) < 0$, when t is sufficiently small. Therefore for small t, y is a point of K. Using the Lemma 2.3 and (5), we obtain

$$\frac{f_i(y)}{g_i(y)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} = t\bar{p}_i(\bar{x}, y)\left\langle \xi_i, \eta(x, \bar{x})\right\rangle \le 0, \quad i \ne r$$

and

$$\frac{f_r(y)}{g_r(y)} - \frac{f_r(\bar{x})}{g_r(\bar{x})} < 0 .$$

This contradicts the choice of $\bar{x}$.
Hence system (4) has no solution in the nonempty invex set K.

Applying Farkas' Lemma [16], there exist $\lambda_{r_i} \ge 0$, $\mu_{r_j} \ge 0$ and $\xi_i \in \partial^c\left(\frac{f_i(\bar{x})}{g_i(\bar{x})}\right)$ $\zeta_j \in \partial^c h_j(\bar{x})$ such that

$$\sum_{i \ne r} \lambda_{r_i} \xi_i + \xi_r = -\sum_{j \in J(\bar{x})} \mu_{r_j} \zeta_j . \tag{6}$$

Summing (6) over r, we get (4) with

$$\lambda_i = 1 + \sum_{r \ne i} \lambda_{r_i}, \quad \mu_j = \sum_{r=1}^{k} \mu_{r_j} .$$

This completes the proof.

Definition 3.1 [5]. *A feasible point $\bar{x}$ for the problem (NMFP) is said to satisfy the boundedness condition, if the set*

$$\left\{ \frac{\bar{p}_i(\bar{x}, x)}{\bar{p}_j(\bar{x}, x)} \mid x \in K, \frac{f_i(\bar{x})}{g_i(\bar{x})} > \frac{f_i(x)}{g_i(x)}, \frac{f_j(\bar{x})}{g_j(\bar{x})} < \frac{f_j(x)}{g_j(x)} \right\} \tag{7}$$

is bounded from above, where $\bar{p}_i(\bar{x}, y) = \frac{g_i(\bar{x})}{g_i(y)} p_i(\bar{x}, y)$.

The following theorem extends the Proposition 3.5 of Chew and Choo [5] and the Proposition 2 of Giorgi and Rueda [10] to the nonsmooth η – pseudolinear case.

Theorem 3.2. *Every efficient solution of the problem (NMFP) involving η – pseudolinear functions, satisfying the boundedness condition is a properly efficient solution of the problem (NMFP).*

Proof: Let $\bar{x}$ be an efficient solution. Then from Theorem 2.1, it follows that there exist $\lambda_i > 0$ and $\mu_j \geq 0$, such that

$$\sum_{i=1}^{k} \lambda_i \xi_i = -\sum_{j \in J(\bar{x})} \mu_j \zeta_j .$$

Therefore, for any feasible x, we have

$$\sum_{i=1}^{k} \lambda_i \left\langle \xi_i, \eta(x,\bar{x}) \right\rangle = -\sum_{j \in J(\bar{x})} \mu_j \left\langle \zeta_j, \eta(x,\bar{x}) \right\rangle .$$

We observe that for any $x \in K$,

$$\sum_{i=1}^{k} \lambda_i \left\langle \xi_i, \eta(x,\bar{x}) \right\rangle \geq 0. \tag{8}$$

Otherwise, we would arrive at a contradiction as in the first part of Theorem 3.1.

Since the set defined by (7) is bounded above, therefore following set is also bounded from above:

$$\left\{ (k-1)\frac{\lambda_j \bar{p}_i(\bar{x},x)}{\lambda_i \bar{p}_j(\bar{x},x)} \mid x \in K, \frac{f_i(\bar{x})}{g_i(\bar{x})} > \frac{f_i(x)}{g_i(x)}, \frac{f_j(\bar{x})}{g_j(\bar{x})} < \frac{f_j(x)}{g_j(x)}, 1 \leq i, j \leq k \right\}. \tag{9}$$

Let $M > 0$ be a real number that is an upper bound of the set defined by (9). Now, we shall show that $\bar{x}$ is a properly efficient solution of the problem (NMFP). Assume that there exist r and $x \in K$, such that

$$\frac{f_r(x)}{g_r(x)} < \frac{f_r(\bar{x})}{g_r(\bar{x})}.$$

Then,

$$\left\langle \xi_r, \eta(x,\bar{x}) \right\rangle < 0, \text{ for some } \xi_r \in \partial^c \left(\frac{f_r(\bar{x})}{g_r(\bar{x})} \right). \tag{10}$$

Let us define

$$-\lambda_s \left\langle \xi_s, \eta(x,\bar{x}) \right\rangle = \max \left\{ \lambda_i \left\langle \xi_i, \eta(x,\bar{x}) \right\rangle \mid \left\langle \xi_i, \eta(x,\bar{x}) \right\rangle > 0 \right\}. \tag{11}$$

Using (8), (10) and (11), we get

$$\lambda_r \left\langle \xi_r, \eta(x,\bar{x}) \right\rangle \leq (k-1)(-\lambda_s \left\langle \xi_s, \eta(x,\bar{x}) \right\rangle).$$

Therefore,

$$\left(\frac{f_r(x)}{g_r(x)} - \frac{f_r(\bar{x})}{g_r(\bar{x})} \right) \leq (k-1)\frac{\lambda_s \bar{p}_r(\bar{x},x)}{\lambda_r \bar{p}_s(\bar{x},x)} \left(\frac{f_s(\bar{x})}{g_s(\bar{x})} - \frac{f_s(x)}{g_s(x)} \right).$$

Using the definition of M, we get

$$\left(\frac{f_r(x)}{g_r(x)}-\frac{f_r(\overline{x})}{g_r(\overline{x})}\right)\leq M\left(\frac{f_s(\overline{x})}{g_s(\overline{x})}-\frac{f_s(x)}{g_s(x)}\right).$$

Hence, $\overline{x}$ is a properly efficient solution of the problem (NMFP).

4. DUALITY

For the nonsmooth multiobjective fractional programming problem (NMFP), we consider the following Mond-Weir subgradient type dual problem:

(NMFD) Maximize $\left(\frac{f_1(u)}{g_1(u)},\frac{f_2(u)}{g_2(u)},...,\frac{f_k(u)}{g_k(u)}\right)$

subject to $0\in\sum_{i=1}^{k}\lambda_i\partial^c\left(\frac{f_i(u)}{g_i(u)}\right)+\partial^c\left(\mu^t h(u)\right)$

$\mu^t h(u)\geq 0,$

$\mu\geqq 0,$

$\lambda_i>0,\ i=1,2,...,k.$

Theorem 4.1 (Weak Duality). *Let y be a feasible solution for (NMFP) and (u,λ,μ) be a feasible solution for the (NMFD) such that f_i and g_i are $\eta-$pseudolinear functions with respect to the same proportional function p_i $(i=1,2,...,k)$ and $\mu^t h$ is $\eta-$pseudolinear with respect to q, then the following cannot hold*

$$\frac{f_i(y)}{g_i(y)}\leq\frac{f_i(u)}{f_i(u)},\ i=1,2,...,k.$$

Proof: We assume that the above inequality is satisfied. Using Lemma 2.1 and Lemma 2.3, the $\eta-$pseudolinearity of $\frac{f_i}{g_i}$ on K with respect to proportional function $\overline{p}_i$ implies that

$$\overline{p}_i(y,u)\langle\xi_i,\eta(y,u)\rangle\leqq 0,\ \text{for all}\ i=1,2,...,k\ \text{and for some}\ \xi_i\in\partial^c\left(\frac{f_j(u)}{g_j(u)}\right)$$

$$\overline{p}_j(y,u)\langle\xi_j,\eta(y,u)\rangle<0,\ \text{for some}\ j\ \text{and}\ \xi_j\in\partial^c\left(\frac{f_j(u)}{g_j(u)}\right).$$

Since $\overline{p}_i(y,u)>0,$ for all $i=1,2,...,k,$ we get

$$\langle\xi_i,\eta(y,u)\rangle\leqq 0,\ \text{for all}\ i=1,2,...,k\ \text{and for some}\ \xi_i\in\partial^c\left(\frac{f_i(u)}{g_i(u)}\right)$$

$$\left\langle \xi_j, \eta(y,u) \right\rangle < 0, \text{ for some } j \text{ and } \xi_j \in \partial^c \left(\frac{f_j(u)}{g_j(u)} \right).$$

Since $\lambda_i > 0$, for each $i = 1, 2, ..., k,$ we get

$$\sum_{i=1}^{k} \lambda_i \left\langle \xi_i, \eta(y,u) \right\rangle < 0. \tag{12}$$

As y is feasible in (NMFP) and (u, λ, y) in (NMDP), it follows that

$$\mu^t h(y) \leqq 0$$

and

$$\mu^t h(u) \geqq 0.$$

Using η – pseudolinearity of $\mu^t h,$ we get

$$q(y,u) \left\langle \zeta, \eta(y,u) \right\rangle \leqq 0, \text{ for some } \zeta \in \partial^c (\mu^t h(u)).$$

Since $q(y,u) > 0,$ we get

$$\left\langle \zeta, \eta(y,u) \right\rangle \leqq 0. \tag{13}$$

From (12) and (13), we get a contradiction to the first dual constraint of the problem (NMFD).

In the following theorems, we have weakened the conditions of η – pseudolinearity on the objective and constraint functions.

Theorem 4.2 (Weak Duality). *If y is a feasible solution for the problem (NMFP) and (u, λ, μ) is a feasible solution for the problem (NMFD) involving η-pseudolinear functions, such that $\sum_{i=1}^{k} \lambda_i \frac{f_i}{g_i}$ is η – pseudolinear with respect to p and $\mu^t h$ is η – pseudolinear with respect to q, then the following inequality cannot hold*

$$\frac{f_i(y)}{g_i(y)} \leq \frac{f_i(u)}{g_i(u)}, \; i = 1, 2, ..., k.$$

Proof: Assume that the above inequality is satisfied. Since $\lambda_i > 0,$ for each i=1, 2, ..., k, we get

$$\sum_{i=1}^{k} \lambda_i \frac{f_i(y)}{g_i(y)} < \sum_{i=1}^{k} \lambda_i \frac{f_i(u)}{g_i(u)}.$$

Using the $\eta-$pseudolinearity of $\sum_{i=1}^{k}\lambda_i \frac{f_i}{g_i}$ inequality (3) of Theorem 4.1 is obtained. Rest of the proof is on the lines of the proof of Theorem 4.1.

Theorem 4.3. *Let us assume that $\bar{x}$ is a feasible solution for the problem (NMFP) and $(\bar{u},\bar{\lambda},\bar{\mu})$ is a feasible solution for the problem (NMFD) involving η-pseudolinear functions, such that*

$$\frac{f_i(\bar{x})}{g_i(\bar{x})}=\frac{f_i(\bar{u})}{g_i(\bar{u})},\ i=1,2,...,k. \tag{14}$$

If for all feasible solutions $(\bar{u},\bar{\lambda},\bar{\mu})$ of (NMFD), $\sum_{i=1}^{k}\lambda_i \frac{f_i}{g_i}$ is $\eta-$pseudolinear with respect to p and $\mu' h$ is η-pseudolinear with respect to q then $\bar{x}$ is a properly efficient solution for (NMFP) and $(\bar{u},\bar{\lambda},\bar{\mu})$ is a properly efficient solution for (NMFD).

Proof: Let us assume that $\bar{x}$ is not an efficient solution of (NMFP), then there exists some $y\in K$, such that

$$\frac{f_i(y)}{g_i(y)}\le\frac{f_i(\bar{x})}{g_i(\bar{x})},\ i=1,2,...,k.$$

Now by the given assumption $\frac{f_i(\bar{x})}{g_i(\bar{x})}=\frac{f_i(\bar{u})}{g_i(\bar{u})},\ i=1,2,...,k,$ we arrive at a contradiction to the Theorem 4.2. Hence $\bar{x}$ is an efficient solution for (NMFP). Proceeding in the same way we can prove that $\left(\bar{u},\bar{\lambda},\bar{\mu}\right)$ is an efficient solution for (NMFD).

Now assume that $\bar{x}$ is an efficient solution of (NMFP). Therefore for every scalar $M>0$, there exists some $x^*\in K$ and an index i, such that

$$\frac{f_i(\bar{x})}{g_i(\bar{x})}-\frac{f_i(x^*)}{g_i(x^*)}>M\left(\frac{f_k(x^*)}{g_k(x^*)}-\frac{f_k(\bar{x})}{g_k(\bar{x})}\right),$$

for all k satisfying

$$\frac{f_k(x^*)}{g_k(x^*)}>\frac{f_k(\bar{x})}{g_k(\bar{x})},$$

whenever

$$\frac{f_i(x^*)}{g_i(x^*)}<\frac{f_i(\bar{x})}{g_i(\bar{x})}.$$

Therefore, the difference $\left(\frac{f_i(\bar{x})}{g_i(\bar{x})}-\frac{f_i(x^*)}{g_i(x^*)}\right)$ can be made arbitrarily large and hence for $\bar{\lambda} > 0$, we get the following inequality

$$\sum_{i=1}^{k}\bar{\lambda}_i\frac{f_i(\bar{x})}{g_i(\bar{x})}-\frac{f_i(x^*)}{g_i(x^*)}>0. \tag{15}$$

Since x^* is a feasible solution for the problem (NMFP) and $(\bar{u},\bar{\lambda},\bar{\mu})$ is a feasible solution for the problem (NMFD), we get

$$h(x^*)\leqq 0 \tag{16}$$

$$\sum_{i=1}^{k}\bar{\lambda}_i\xi_i+\zeta=0, \text{where } \xi_i\in\partial^c\left(\frac{f_i(\bar{u})}{g_i(\bar{u})}\right) \text{ and } \zeta\in\partial^c(\bar{\mu}^t h(\bar{u})) \tag{17}$$

$$\bar{\mu}^t h(\bar{u})\geqq 0 \tag{18}$$

$$\bar{\mu}\geqq 0 \tag{19}$$

$$\bar{\lambda}_i>0,\ i=1,2,...,k. \tag{20}$$

Using (16), (18) and (19), we get

$$\bar{\mu}^t h(x^*)\leqq\bar{\mu}^t h(\bar{u}).$$

Since $\bar{\mu}^t h$ is $\eta-$pseudolinear with respect to q, we obtain that

$$q(\bar{u},x^*)\left\langle\zeta,\eta(x^*,\bar{u})\right\rangle\leqq 0, \text{for some } \zeta\in\partial^c(\bar{\mu}^t h(\bar{u})).$$

As $q(\bar{u},x^*)>0$, it follows that

$$\left\langle\zeta,\eta(x^*,\bar{u})\right\rangle\leqq 0, \text{ for some } \zeta\in\partial^c(\bar{\mu}^t h(\bar{u})).$$

Using (17), we obtain

$$\left\langle\sum_{i=1}^{k}\bar{\lambda}_i\xi_i,\eta(x^*,\bar{u})\right\rangle\geqq 0, \text{ where } \xi_i\in\partial^c\left(\frac{f_i(\bar{u})}{g_i(\bar{u})}\right).$$

Using η-pseudolinearity of $\sum_{i=1}^{k}\bar{\lambda}_i\frac{f_i}{g_i}$ with respect to p, it follows that

$$\sum_{i=1}^{k}\bar{\lambda}_i\frac{f_i(x^*)}{g_i(x^*)}-\sum_{i=1}^{k}\bar{\lambda}_i\frac{f_i(\bar{u})}{g_i(\bar{u})}\geqq 0. \tag{21}$$

Using (14) in (21), we get

$$\sum_{i=1}^{k} \bar{\lambda}_i \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \frac{f_i(x_0)}{g_i(x_0)} \right) \leqq 0,$$

which is a contradiction to (15). Hence, $\bar{x}$ is a properly efficient solution for (NMFP).

Now we assume that $(\bar{u}, \bar{\lambda}, \bar{\mu})$ is not properly efficient solution for (NMFP). Therefore, for every scalar $M > 0$, there exist a feasible point (u^*, λ^*, μ^*) in (NMFD) and an index i such that

$$\frac{f_i(u^*)}{g_i(u^*)} - \frac{f_i(\bar{u})}{g_i(\bar{u})} > M \left(\frac{f_k(\bar{u})}{g_k(\bar{u})} - \frac{f_k(u^*)}{g_k(u^*)} \right),$$

for all k satisfying

$$\frac{f_k(u^*)}{g_k(u^*)} < \frac{f_k(\bar{u})}{g_k(\bar{u})},$$

whenever

$$\frac{f_i(u^*)}{g_i(u^*)} > \frac{f_i(\bar{u})}{g_i(\bar{u})}.$$

Therefore, the difference $\left(\frac{f_i(u^*)}{g_i(u^*)} - \frac{f_i(\bar{u})}{g_i(\bar{u})} \right)$ can be made arbitrarily large and hence for $\lambda > 0$, we get

$$\sum_{i=1}^{k} \bar{\lambda}_i \left(\frac{f_i(u^*)}{g_i(u^*)} - \frac{f_i(\bar{u})}{g_i(\bar{u})} \right) > 0 \tag{22}$$

Since $\bar{x}$ and $(\bar{u}, \bar{\lambda}, \bar{\mu})$ are feasible solutions for (NMFP) and (NMFD) respectively, it follows as in the first part of the theorem that

$$\sum_{i=1}^{k} \bar{\lambda}_i \left(\frac{f_i(u_0)}{g_i(u_0)} - \frac{f_i(\bar{u})}{g_i(\bar{u})} \right) \leqq 0,$$

which is a contradiction to (22). Thus, $(\bar{u}, \bar{\lambda}, \bar{\mu})$ is a properly efficient solution for (NMFD).

The proof of the following theorem can be given on the lines of the proof of Theorem 3.4 of Giorgio and Rueda [10] in the light of the above Theorem 4.3.

Theorem 4.4 (Strong Duality). *Let $\bar{x}$ be an efficient solution for the problem (NMFP). Then there exist $\bar{\lambda} \in R^k$, $\bar{y} \in R^m$, such that $(\bar{x}, \bar{\lambda}, \bar{y})$ is feasible solution for (NMFD).*

Further, if for all feasible solutions (u, λ, μ) *for (NMFP),* $\sum_{i=1}^{k} \lambda_i \frac{f_i}{g_i}$ *is* η*-pseudolinear with respect to* p *and* $\mu^t h$ *is* $\eta-$*pseudolinear with respect to* q*, then* $(\bar{x}, \bar{\lambda}, \bar{y})$ *is properly efficient solution for (NMFD).*

5. CONCLUSIONS

In this paper, we have obtained conditions under which a feasible solution is an efficient solution and established that under certain boundedness condition an efficient solution is properly efficient solution for (NMFP) involving η-pseudolinear functions. We further formulated Mond-Weir subgradient type of dual for (NMFP) and derived several weak and strong duality results. The results presented in this paper extend several results of Chew and Choo [5], Giorgi and Rueda [10] and Kaul et al. [12] to the nonsmooth case.

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REFERENCES

[1] Ansari, Q. H., and Rezaei, M., "Generalized pseudolinearity", *Optim. Lett.*, 6(2012) 241-251.

[2] Ansari, Q. H., Schaible, S., and Yao, J. C., "η – Pseudolinearity", *Riviste Mat. Sci. Econom. Soc.,* 22 (1999) 31-39.

[3] Bector, C. R., "Duality in nonlinear fractional programming", *Zeitzchrift Flir Oper. Res.*, 17 (1973) 183-193.

[4] Cambini, A., Castagnoli, E., Martein, L., Mazzoleni, P., and Schaible, S. (eds.), *Generalized Convexity and Fractional Programming with Economic Applications*, *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, Berlin, New York, 1990.

[5] Chew, K. L., and Choo, E. U., "Pseudolinearity and efficiency", *Math. Program.,* 28 (1984) 226-239.

[6] Clarke, F. H., *Optimization and Nonsmooth Analysis,* John Wiley and Sons, New York, 1983.

[7] Chandra, S., Craven, B. D., and Mond, B., "Vector-valued Lagrangian and multiobjective fractional programming duality", *Numer. Funct. Anal. Optim*., 11 (1990) 239-254.

[8] Egudo, R. R., "Multiobjective fractional duality", *Bull. Austral. Math. Soc.,* 37 (1988) 367-388.

[9] Geoffrion, A. M., "Proper efficiency and the theory of vector maximization"*, J. Math. Anal. Appl.*, 22 (1968) 618-630.

[10] Giorgi, G., and Rueda, N. G., "η-Pseudolinearity and efficiency", *Inter. J. Optim. Theory Methods Appl.,* 1 (2009) 155-159.

[11] Komlosi, S.: "First and second-order characterizations of pseudolinear functions", *Eur. J. Oper. Res.*, 67 (1993) 278-286.

[12] Kaul, R. N., Suneja, S. K., and Lalitha, C. S., "Duality in pseudolinear multiobjective fractional programming", *Indian J. Pure Appl. Math.,* 24 (1993) 279-290.

[13] Kim, D. S., "Nonsmooth multiobjective fractional programming with generalized invexity", *Taiwanese. J. Math.,* 10 (2006) 467-478.
[14] Kuk, H., "Duality for nonsmooth multiobjective fractional programming with V-ρ-invexity", *JKSIAM*, 4 (2000) 1-10.
[15] Kuk, H., Lee, G. M., and Kim, D. S., "Nonsmooth multiobjective programs with V-ρ- invexity", *Indian J. Pure Appl. Math.,* 29 (1998) 405-412.
[16] Mangasarian, O. L., *Nonlinear Programming*, McGraw-Hill, New-York, 1969.
[17] Meister, B., and Oettli, W., "On the capacity of a discrete constant channel", *Inform. Control*, 11 (1998) 341-351.
[18] Mishra, S. K., and Giorgio, G., *Invexity and Optimization*, Springer-Verlag, Berlin, Heidelberg, 2009.
[19] Mohan, S. R., and Neogi, S. K., "On invex sets and preinvex functions", *J. Math. Anal. Appl.,* 189 (1994) 901-908.
[20] Mukherjee, R. N., and Rao, C. P., "Multiobjective fractional programming under generalized invexity", *Indian J. Pure Appl. Math.*, 27 (1996) 1175-1183.
[21] Nobakhtian, S., "Optimality and duality for nonsmooth multiobjective fractional programming with mixed constraints", *J. Global Optim.*, 41 (2008) 103-115.
[22] Rapscsak, T. "On pseudolinear functions", *Eur. J. Oper. Res*, 50 (1991) 353-360.
[23] Rueda, N. G., "Generalized convexity in nonlinear programming", *J. Inform. Optim. Sci.,* 10 (1989) 395-400.
[24] Schaible, S., "Fractional programming", in: R. Horst, and P. M. Pardalos, (eds.), *Handbook of Global Optimization*, Kluwer Academic Publishers, 1995, 495-608.
[25] Stancu-Minasian, I. M., Dogaru, G., and Stancu, A. M., "Duality for multiobjective fractional programming problems involving d -type- I -set - n functions", *Yugoslav J. Oper. Res.*, 19 (2009) 63-73.
[26] Tsai, J. F., "Global optimization of nonlinear fractional programming in engineering design", *Eng. Optim.,* 37(2005) 399-409.
[27] Weir, T., "A duality theorem for a multiobjective fractional optimization problem", *Bull. Austral. Math. Soc.,* 34 (1986) 415-425.
[28] Yang, X. M., Yang, X. Q., and Teo, K. L., "Criteria for generalized invex monotonicities", *Eur. J. Oper. Res.,* 164 (2005) 115-119.
[29] Zamora, J. M., and Grossmann, I. E., "MINLP model for heat exchanger networks", *Comput. Chem. Eng.,* 22 (1998) 367-384.