

A SLIGHT MODIFICATION OF THE FIRST PHASE OF THE SIMPLEX ALGORITHM

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Received: June 2010 / Accepted: March 2012

Abstract: In this paper we give a modification of the first phase procedure for transforming the linear programming problem, given in the standard form to the canonical form, i.e., to the form with one feasible primal basis where standard simplex algorithm can be applied directly. The main idea of the paper is to avoid adding m artificial variables in the first phase. Instead, Step 2 of the proposed algorithm transforms the problem to the form with $m-1$ basic columns. Step 3 is then iterated until the m -th basic column is obtained, or it is concluded that the feasible set of LP problem is empty.

Keywords: Linear programming, simplex algorithm, canonical form, two phase simplex algorithm, new first phase simplex algorithm.

MSC: 90C05

1. INTRODUCTION

Let us consider the linear programming problem

$$\begin{aligned} \min \quad & \sum_{j=1}^n c_j x_j + z_0 \\ \sum_{j=1}^n a_{i,j} x_j &= b_i, \quad i=1, \dots, m \\ x_j &\geq 0, \quad j=1, \dots, n \end{aligned} \tag{1}$$

where $A = (a_{i,j})_{m \times n}$ is a given matrix, $b \in R^m$ and $c \in R^n$ are given vectors. Without loss of generality, we assume that $\text{rank } A = m$. The problem can be presented in the form of a table

$-z_0$	c_1	c_2	\dots	c_n
b_1	$a_{1,1}$	$a_{1,2}$	\dots	$a_{1,n}$
b_2	$a_{2,1}$	$a_{2,2}$	\dots	$a_{2,n}$
\vdots	\vdots	\vdots	\ddots	\vdots
b_m	$a_{m,1}$	$a_{m,2}$	\dots	$a_{m,n}$

having $m+1$ rows and $n+1$ columns. It is called the linear programming table (LP). We denote and $c_j = a_{0,j}, j = 1, \dots, n, b_i = a_{i,0}, i = 1, \dots, m$ and $z_0 = a_{0,0}$. Recall that we can apply the simplex algorithm to problem (1) when the corresponding table is a simplex table (ST), that is, it has m different basic columns

$$A_j = (a_{0,j}, a_{1,j}, \dots, a_{m,j}) = (0, \dots, 0, 1, 0, \dots, 0) \in R^{m+1}, j \in \{1, \dots, n\},$$

where 1 is not in the so-called 0-th place and $b_i \geq 0$, for all $i = 1, \dots, m$. Standard procedure ([3], p. 74-79, [1], p. 34-38) to obtain ST is the two-phase simplex algorithm. In the first phase we solve the auxiliary problem

$$\begin{aligned} \min \sum_{i=1}^m \omega_i \\ \sum_{i=1}^n a_{i,j} x_j + \omega_i = b_i, \quad i = 1, \dots, m, \\ x_j \geq 0, \quad \omega_i \geq 0, \quad j = 1, \dots, n, \quad i = 1, \dots, m. \end{aligned} \quad (2)$$

Variables $\omega_i, i = 1, \dots, m$, are slack or artificial variables. If the optimal value of $\sum_{i=1}^m \omega_i$ is equal to 0, we can pass to the second phase of simplex algorithm. Otherwise, we conclude that problem (1) does not have a feasible solution. Specification of the simplex algorithm and other modification one can find in [2].

Now we will give a modified first phase simplex algorithm to obtain the simplex table, or problem in canonical form. This modification is simpler than the classical first phase algorithm because it does not need artificial variables and allows change of the function $c^T x$ at the same time.

2. MODIFICATION OF THE FIRST PHASE

We begin with the table (1'). Put $t = 0$, that is $a_{i,j} = a_{i,j}^0$ for every i and j .

Algorithm:

Step 1. Solve the system $Ax = b$ by Gauss procedure with respect to arbitrary m variables, for example the first, that is x_1, \dots, x_m . We obtained the following LP:

$-z_0^1$	0	0	\dots	0	c_{m+1}^1	\dots	c_n^1
b_1^1	1	0	\dots	0	$a_{1,m+1}^1$	\dots	$a_{1,n}^1$
b_2^1	0	1	\dots	0	$a_{2,m+1}^1$	\dots	$a_{2,n}^1$
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\ddots	\vdots
b_m^1	0	0	\dots	1	$a_{m,m+1}^1$	\dots	$a_{m,n}^1$

(3)

If $b_i^1 \geq 0$ for every $i = 1, \dots, m$, we go to step 4.

If there is $b_i^1 < 0$ for some $i \in \{1, \dots, m\}$ we go to step 2.

Notice that after this step we have a table of LP with m basic columns but b_i do not have to be all nonnegative.

Step 2. Denote by $\tilde{I} = \{i | b_i^1 < 0, i = 1, \dots, m\}$. Let

$$|b_s^1| = \max \{ |b_i^1| | b_i^1 < 0, i = 1, \dots, m \} \quad (4)$$

In order to get the next table we will:

1. multiply the s -th row by -1 ;
2. add the s -th row multiplied by -1 to every row from $\tilde{I} \setminus s$.

We obtain the next LP table:

$-z_0^2$	0	0	...	0	...	0	c_{m+1}^2	...	c_l^2	...	c_n^2
b_1^2	1	0	...	0	...	0	$a_{1,m+1}^2$...	$a_{1,l}^2$...	$a_{1,n}^2$
b_2^2	0	1	...	-1	...	0	$a_{2,m+1}^2$...	$a_{2,l}^2$...	$a_{2,n}^2$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots
b_r^2	0	0	...	-1	...	0	$a_{r,m+1}^2$...	$a_{r,l}^2$...	$a_{r,n}^2$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots
b_s^2	0	0	...	-1	...	0	$a_{s,m+1}^2$...	$a_{s,l}^2$...	$a_{s,n}^2$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots
b_m^2	0	0	...	0	...	1	$a_{m,m+1}^2$...	$a_{m,l}^2$...	$a_{m,n}^2$

(5)

Here, for example, $2, r \in \tilde{I}$. The s -th row has not to be unique, but if there are more candidates we will choose one. Notice that this table has $m-1$ basic columns and all $b_i \geq 0$. Our goal is to get one more basic column, which will have 1 in the s -th row.

Step 3. If $a_{s,j}^t \leq 0$, our problem does not have a feasible solution and algorithm stops (because $b_s^t > 0$). Let us denote by $J^t = \{j | a_{s,j}^t > 0, j = 1, \dots, n\}$. For every $j \in J^t$, we find

$$\frac{b_{r_j}^t}{a_{r_j,j}^t} = \min \left\{ \frac{b_i^t}{a_{i,j}^t} \mid a_{i,j}^t > 0, i = 1, \dots, m \right\} \quad (6)$$

Notice that r_j is not unique. There are two cases.

1. If, for some $j \in J^t, r_j = s$, we will change the table using $a_{s,j}^t$ as the pivot element. It means: divide the s -th row by $a_{s,j}^t$ and add the s -th row multiplied by $-\frac{a_{i,j}^t}{a_{s,j}^t}$ to every other row $i = 0, 1, \dots, s-1, s+1, \dots, m$, in order to make the j -th column basic. Then we go to step 4.

2. For every $j \in J^t$ is $r_j \neq s$. In this case, we take $l \in J^t$ arbitrary and $a_{r_l,l}^t$ as the pivot element. The l -th column is made basic instead of the column having 1 in row r_l . After this, put $t=t+1$ and go to step 3.

Step 4. The obtained table is ST, that is, our problem is in canonical form and we apply the simplex algorithm directly.

We will explain this algorithm on the next linear programming problem. We give the corresponding LP table after Step 1.

Example. Find the canonical form of a linear programming problem which corresponds to the next table T^1 (obtained after applying Gauss procedure):

-10	0	0	0	0	0	2	3	-5	-50
-2	1	0	0	0	0	1	-6	3	-4
1	0	1	0	0	0	-4	2	3	2
-3	0	0	1	0	0	2	-2	7	-1
4	0	0	0	1	0	3	0	-14	-2
-1	0	0	0	0	1	0	0	-2	3

We go to step 2. Here $|-3| = \max\{|-2|, |-3|, |-1|\}$ and the principal row is $s = 3$. The next table is T^2 :

-10	0	0	0	0	0	2	3	-5	-50
1	1	0	-1	0	0	-1	-4	-4	-3
1	0	1	0	0	0	-4	2*	3	2
3	0	0	-1	0	0	-2	2	-7	1
4	0	0	0	1	0	3	0	-14	-2
2	0	0	-1	0	1	-2	2	-9	4

$J^2 = \{7, 9\}$ because $a_{3,7}^2 = 2 > 0$ and $a_{3,9}^2 = 1 > 0$. We find

$$\frac{1}{2} = \min\left\{\frac{1}{2}, \frac{3}{2}, \frac{2}{2}\right\} \text{ for } j=7, \quad \frac{1}{2} = \min\left\{\frac{1}{2}, \frac{3}{1}, \frac{2}{4}\right\} \text{ for } j=9.$$

Since $r_7 = 2$ and $r_9 = 2$ or $r_9 = 5$, all different from 3, we take $a_{2,7}^2 = 2$ as the pivot element. After pivoting we get table T^3 :

$-\frac{23}{2}$	0	$-\frac{3}{2}$	0	0	0	8	0	$-\frac{19}{2}$	-53
3	1	2	-1	0	0	-9	0	2	1
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	-2	1	$\frac{3}{2}$	1
2	0	-1	-1	0	0	2	0	-10	-1
4	0	0	0	1	0	3	0	-14	-2
1	0	-1	-1	0	1	2*	0	-12	4

Here is $J^3 = \{6\}$ and $\frac{1}{2} = \min\left\{\frac{2}{2}, \frac{4}{3}, \frac{1}{2}\right\}$. The pivot is $a_{5,6}^3 = 2$. The table T^4 is:

$-\frac{31}{2}$	0	$\frac{5}{2}$	4	0	-4	0	0	$\frac{77}{2}$	-61
$\frac{15}{2}$	1	$-\frac{5}{2}$	$-\frac{11}{2}$	0	$\frac{9}{2}$	0	0	-52	10
$\frac{3}{2}$	0	$-\frac{1}{2}$	-1	0	1	0	1	$-\frac{21}{2}$	3
1	0	0	0	0	-1	0	0	2*	-3
$\frac{5}{2}$	0	$\frac{3}{2}$	$\frac{3}{2}$	1	$-\frac{3}{2}$	0	0	4	-5
$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	0	-6	1

$J^4 = \{8\}$, $\frac{1}{2} = \min\left\{\frac{1}{2}, \frac{5}{8}\right\}$, $r_8 = 3$ and the pivot is $a_{3,8}^4 = 2$. Since the pivot is in the third row, next table will be the simplex table:

$-\frac{139}{4}$	0	$\frac{5}{2}$	4	0	$\frac{61}{4}$	0	0	0	$-\frac{13}{4}$
$\frac{67}{2}$	1	$-\frac{5}{2}$	$-\frac{11}{2}$	0	$-\frac{43}{2}$	0	0	0	-68
$\frac{27}{4}$	0	$-\frac{1}{2}$	-1	0	$\frac{17}{4}$	0	1	0	$-\frac{51}{4}$
$\frac{1}{2}$	0	0	0	0	$-\frac{1}{2}$	0	0	1	$-\frac{3}{2}$
$\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	0	0	0	1
$\frac{7}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{5}{2}$	1	0	0	-8

The last table is simplex table. The basic solution which corresponds to it is feasible solution but it is not optimal. Now we can apply the simplex algorithm directly.

After Step 2 we have $b_i^2 \geq 0$ and $b_s^2 > 0$. At first, we show that this algorithm keeps $b_i^t \geq 0$ and $b_s^t > 0$ for $t \geq 2$.

Theorem 1. *Applying the step 3.2 we get the LP table T^t where $b_i^t \geq 0$ and $b_s^t > 0$ for $t \geq 2$ and $b_s^{t+1} \leq b_s^t$. Applying this algorithm we obtain the canonical form of a given problem or we conclude that this problem does not have a feasible solution.*

Proof: Suppose that $J^t \neq \emptyset$ and we have not arrived at step 4. The claim is true for $t = 2$ and suppose that is true for all tables before T^t . It means $b_i^{t-1} \geq 0$ and $b_s^{t-1} > 0$. Denote by $a_{r_i,l}^{t-1} \geq 0$ pivot at step 3. It is $r_j \neq s$ for every $j \in J^{t-1}$. That is

$$\frac{b_{r_i}^{t-1}}{a_{r_i,l}^{t-1}} = \min \left\{ \frac{b_i^{t-1}}{a_{i,l}^{t-1}} \mid a_{i,l}^{t-1} > 0, \quad i = 1, \dots, m \right\} \quad (6')$$

It holds:

$$b_{r_i}^t = \frac{b_{r_i}^{t-1}}{a_{r_i,l}^{t-1}} \geq 0,$$

$$b_s^t = b_s^{t-1} - \frac{a_{s,l}^{t-1}}{a_{r_i,l}^{t-1}} b_{r_i}^{t-1} = a_{s,l}^{t-1} \left(\frac{b_s^{t-1}}{a_{s,l}^{t-1}} - \frac{b_{r_i}^{t-1}}{a_{r_i,l}^{t-1}} \right) > 0,$$

because $a_{s,l}^{t-1} > 0, a_{r_i,l}^{t-1} > 0, (6')$ and $r_i \neq s$. For $i \neq s$, we have

$$b_i^t = b_i^{t-1} - \frac{a_{i,l}^{t-1}}{a_{r_i,l}^{t-1}} b_{r_i}^{t-1} \geq 0,$$

if $a_{i,l}^{t-1} \leq 0$. If $a_{i,l}^{t-1} > 0$ then

$$b_i^t = a_{i,l}^{t-1} \left(\frac{b_i^{t-1}}{a_{i,l}^{t-1}} - \frac{b_{r_i}^{t-1}}{a_{r_i,l}^{t-1}} \right) \geq 0.$$

The second part of the theorem is evident. Let us notice that this algorithm keeps the number of basic columns if the pivot is not in the s -th row. If it is, then the number of basic columns augments for one and there will be m basic columns.

In order to avoid cycling, we can modify this algorithm. Instead of finding minimum, we find lexicographical minimum. Recall that we say that vector $x \in R^n$ is lexicographically greater than 0 if $x \neq 0$ and the first coordinate different from 0 is positive.

Modified Algorithm. This algorithm differs from previous in step 3. Notice that all rows are lexicographically positive after step 2. If a row with $b_i^2 = 0$ is not positive, we will make it positive by changing the hierarchy of columns. After this we suppose that all the rows except zeroth are lexicographically positive.

Step 3'. If $a_{s,j}^t \leq 0$, our problem has not a feasible solution and algorithm stops. Let us denote by $J^t = \{j | a_{s,j}^t > 0, j = 1, \dots, n\}$. For every $j \in J^t$, we find

$$\frac{b_{r_j}^t}{a_{r_j,j}^t} = \min \left\{ \frac{b_i^t}{a_{i,j}^t} \mid a_{i,j}^t > 0, i = 1, \dots, m \right\} \quad (6)$$

1. If, for some $j \in J^t$, $r_j = s$, we will change the table using $a_{s,j}^t$ as the pivot element. After this we go to step 4.

2. For every $j \in J^t$ is $r_j \neq s$. In this case, we find for every $j \in J^t$

$$\frac{V_{r_j}^t}{a_{r_j,j}^t} = \text{lex min} \left\{ \frac{V_i^t}{a_{i,j}^t} \mid a_{i,j}^t > 0, i = 1, \dots, m \right\}$$

take $l \in J^t$ arbitrary and $a_{r_l,l}^t$ as pivot element. The l -th column is made basic instead of column which has 1 in row r_l . After this, put $t = t + 1$ and go to step 3.

Theorem 2. Let us suppose that after step 2 all the rows except zeroth are lexicographically positive. Applying modified step 3'.2 we get the LP table T^t where all the rows except zeroth are lexicographically positive. We obtain the canonical form or after a finite number of steps, we conclude that the given linear programming problem does not have a feasible solution.

Proof: Similar as in previous Theorem 1, we can prove that all the rows remain lexicographically positive. The s -th row of our algorithm is, at the same time, the objective function for the first phase of the classical two-phase algorithm and the ordinary row. To see this, let us compare this algorithm with the classical first phase of the two-phase simplex algorithm applied after step 2. After step 2 we need only one artificial variable, that is, we consider the next problem:

$$\begin{aligned} & \min \omega_s \\ & \sum_{j=1}^n a_{i,j}^2 x_j = b_i^2, \quad i = 1, \dots, s-1, s+1, \dots, m, \\ & \sum_{j=1}^n a_{s,j}^2 x_j + \omega_s = b_s^2, \\ & x_j \geq 0, \quad \omega_s \geq 0, \quad j = 1, \dots, n. \end{aligned} \quad (2')$$

The following simplex table (after adding the s -th row multiplied by -1 to the 0 -th row) corresponds to this problem:

$$\begin{array}{c|cccccccccc}
 -b_s^2 & 0 & 0 & \cdots & 1 & \cdots & 0 & -a_{s,m+1}^2 & \cdots & -a_{s,l}^2 & \cdots & -a_{s,n}^2 & 0 \\
 \hline
 b_1^2 & 1 & 0 & \cdots & 0 & \cdots & 0 & a_{1,m+1}^2 & \cdots & a_{1,l}^2 & \cdots & a_{1,n}^2 & 0 \\
 b_2^2 & 0 & 1 & \cdots & -1 & \cdots & 0 & a_{2,m+1}^2 & \cdots & a_{2,l}^2 & \cdots & a_{2,n}^2 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
 b_r^2 & 0 & 0 & \cdots & -1 & \cdots & 0 & a_{r,m+1}^2 & \cdots & a_{r,l}^2 & \cdots & a_{r,n}^2 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
 b_s^2 & 0 & 0 & \cdots & -1 & \cdots & 0 & a_{s,m+1}^2 & \cdots & a_{s,l}^2 & \cdots & a_{s,n}^2 & 1 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
 b_m^2 & 0 & 0 & \cdots & 0 & \cdots & 1 & a_{m,m+1}^2 & \cdots & a_{m,l}^2 & \cdots & a_{m,n}^2 & 0
 \end{array} \quad (5')$$

Notice that the zeroth row is the s -th row (except the last coordinate) multiplied with -1. The rest of the proof follows because lexicographic modification of the simplex algorithm guarantees that we will arrive after a finite number of steps to the minimum value of the objective function or conclude that the given problem does not have a feasible solution.

3. CONCLUSION

The proposed algorithm avoids adding m artificial variables in the first phase. Instead, Step 2 of the algorithm transforms the problem to the form with $m-1$ basic columns. Step 3 is then iterated until the m -th basic column is obtained, or it is concluded that the feasible set of LP problem is empty. In order to prove that algorithm terminates in a finite numbers of iterations, we used lexicographic modification of Step 3.

We think that such modified first phase algorithm has advantage because we, at the same time, look for the last basic column and change the objective function. Also this algorithm does not need artificial variables. It is almost one phase algorithm.

Acknowledgement. This research was supported by Serbian Ministry for Education and Science, Grant No.174033 "Graph Theory and Mathematical Programming with Applications to Chemistry and Computing"

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