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ON SOLVING STOCHASTIC MADM PROBLEMS

Ion VĂDUVA

University of Bucharest, Faculty of Maths and Computer Science, Romania
vaduva@fmi.unibuc.ro

Cornel RESTEANU

National Institute for Research and Development in Informatics, Bucharest Romania
resteanu@ici.ro

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Abstract: The paper examines a MADM problem with stochastic attributes. The transformation of a stochastic MADM problem into a cardinal problem is done by the standardization of the probability distribution of each attribute X and calculating the information of each attribute as Shannon's entropy or Onicescu's informational energy. Some well known (performant) methods to solve a cardinal MADM problem are presented and a method for combining results of several methods to give a final MADM solution is discussed.

Keywords: Multiple attribute decision making, stochastic MADM problems, stochastic entries, entropy, informational energy.

1. INTRODUCTION

A MADM (i.e. Mulltiple Attribute Decision Making) problem can be formulated as follows [2,4,6,12,14,15]: there are n decision alternatives to be taken and there are m criteria or attributes used to determine the best (optimun) alternative decision. In order to make a decision, a "*sense*" for selecting decisions is associated with each criterion, namely, the best decision is selected if its attribute has a minimum or a maximum value. The problem is to select the "best" decision alternative with respect to all the criteria combined with *sense* requirements.

The data of a MADM problem can be represented as in the following table [2,9]:

A **table** representing decision data.

	C_1	C_2	$\cdots$	C_M
A_1	a_{11}	a_{12}	$\cdots$	a_{1m}
A_2	a_{21}	a_{22}	$\cdots$	a_{2m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	
A_n	a_{n1}	a_{n2}	$\cdots$	a_{nm}
P	p_1	p_2	$\cdots$	p_m
$sense$	$sense_1$	$sense_2$	$\cdots$	$sense_m$

The entries $a_{ij}, 1 \le i \le n, 1 \le j \le m$ define the $n \times m$ *decision matrix*. The vector $P = (p_1, p_2, ..., p_m)'$ is usually a probability vector $(p_i > 0, \sum_{i=1}^{m} p_i = 1)$ specifying the "importance" of each criterion (i.e. P is a positive weights vector) and the vector $sense = (sense_1, sense_2, ..., sense_m)'$ (see the comments from above) specifies the requirements for selecting the best decision alternative.

The entries a_{ij} could be real numbers, linguistic qualificatives [9], logical values or any other elements from a specified ordered set.

There are various methods for solving a MADM problem, i.e. for determining an *order* of alternatives and then selecting the best decision alternative.

The nature of a method is given by the entries a_{ij} [2,4,11,15]. If a_{ij} are deterministic then the problem is *cardinal;* if a_{ij} are of a fuzzy nature then the problem is *fuzzy,* while if some $a_{ij}, i = 1, 2, ..., n$ are stochastic elements having known probability distributions, then the MADM problem is *stochastic.* Sometimes, the decision matrix can have a complex structure in the sense that its entries can have more indexes [2] as $a_{ijk}, 1 \le k \le d$. (The index k may refer to several human decidents involved in decision process). Such a problem is a MADM problem *with several decidents.*

In this paper we shall present a simple MADM method for stochastic entries of the decisin matrix, a case which is not often discussed in the literature.

2. PROCESSING OF STOCHASTIC ENTRIES

Assume that an entry a_{ij} is a real random variable X specified either as a discrete probability distribution in the form

$$X:\begin{pmatrix} a_1, & a_2, & \dots & a_k \\ p_1, & p_2, & \dots & p_k \end{pmatrix}, a_i \in R, \, p_i > 0, \quad \sum_{i=1}^{k} p_i = 1,$$

or as a continuous probability distribution given by its probability density function (p.d.f.) $f(x)$ in the form

$$X \to f(x), \quad x \in [a,b], -\infty < a < b < \infty, \quad \int_a^b f(x)dx = 1.$$

In other words the elements a_i, p_i in the discrete case or $a, b, f(x)$ in the continuous case are given, for a stochastic MADM problem. Many of the methods for solving MADM problems are reduced to solving cardinal (deterministic) problems. This idea is applied here for solving stochastic MADM problems in the sense that we first transform a stochastic problem into a cardinal one. More precisely, the stochastic decision matrix $\| a_{ij} \|$ is transformed into a cardinal (deterministic) matrix $\| h_{ij} \|$.

The procedure consists of the following steps:

Step 1. A stochastic entry X is transformed into a *standardized* stochastic entry Y in the form

$$Y = \frac{X}{\sigma} \tag{2.1}$$

where $\sigma^2 = Var(X)$ is the variance of X (i.e. σ is the *standard deviation* of X).

In the discrete case we have

$$m = E[X] = \sum_{i=1}^{k} a_i p_i, \sigma^2 = E[(X-m)^2] = \sum_{i=1}^{k} p_i a_i^2 - m^2$$

and in the continuous case we have

$$m = E[X] = \int_a^b x f(x)dx, \sigma^2 = E[(X-m)^2] = \int_a^b x^2 f(x)dx - m^2,$$

assuming that m and σ^2 exist.

According to (2.1) the discrete distribution of Y is

$$Y:\begin{pmatrix} b_1, & b_2, & \dots, & b_k \\ p_1, & p_2, & \dots, & p_k \end{pmatrix}, \quad b_i = \frac{a_i}{\sigma}, \quad \alpha = b_1, \, \beta = b_k \tag{2.2}$$

and in the continuous case the probability density function of Y is

$$Y \to g(y) = \sigma f(\sigma y), \quad y \in [\alpha, \beta], \quad \alpha = a\sigma, \, \beta = b\sigma. \tag{2.2'}$$

For each distribution, the standardized *range* is calculated as $r = \beta - \alpha$.

Step 2. The next proposed step is to assign to a stochastic entry Y in the decision matrix, the *information* contained in the corresponding probability distribution. This information can be represented either by *Shannon's entropy* or by *Onicescu's informational energy* of Y.

In the discrete case the entropy of Y is

$$h = -\sum_{i=1}^{k} p_i \log p_i \tag{2.3}$$

and the informational energy is

$$e = \sum_{i=1}^{k} p_i^2. \tag{2.3'}$$

In the continuous case the entropy of Y is

$$h = -\int_{\alpha}^{\beta} g(y)\log g(y)\,dy = -\sigma \int_{\alpha\sigma}^{\beta\sigma} f(\sigma x)\log\big(\sigma f(\sigma x)\big)\,dx \tag{2.4}$$

and the informational energy is

$$e = \int_{\alpha}^{\beta} g^2(y)dy = \sigma^2 \int_{a\sigma}^{b\sigma} f^2(\sigma x)dx. \tag{2.4'}$$

Now, the decision matrix of the problem has elements in the form $\| h_{ij} \|$ or $\| e_{ij} \|$, i.e. it is a cardinal one.

One can also define the decision matrix as $\| \rho_{ij} \|$ or as $\| \rho_{ij} h_{ij} \|$ respectively $\| \rho_{ij} e_{ij} \|$. (ρ_{ij} is the range of the criterion j for the alternative i). If the range is ∞, then it is not used.

Note. *In the continuous case, the formulae (2.4) and (2.4') say that for a random variable X having the density $f(x)$ we have*

$$e = E[f(X)] \tag{2.4''}$$

$$h = -E[\log f(X)]. \tag{2.4'''}$$

In [10], the procedure to determine the probability density function $g(v)$ of $Y = f(X)$ is presented, namely

$$g(v) = -vA'(v), \quad A(v) = mes\{x \mid f(x) \geq v\}.(3.5) \tag{2.5}$$

For some particular continuous distributions the p.d.f of $f(X)$ is [10]:

for exponential distribution $Exp(\lambda)$ the p.d.f. is $g(v) = \dfrac{v}{\lambda}$ (i.e.uniform);

for the normal $N(0,1)$ distribution the p.d.f. is $g(v) = 2[-\log(\sqrt{2\pi v}]^{-1/2}$;

for the bivariate normal N (**0,** ***I***) (**I**=the $N(0,I)(I = the 2\times 2\, unit\, matrix)$, the p.d.f. is $g(v) = 2\pi,$ i.e. $f(X)$ is uniform on $[0, 2\pi]$.

Note that while $F(X)$ is always uniform on $[0,1]$, the previous examples show that $f(X)$ is not generally distributed as uniform.

Densities $g(v)$ can be used to calculate informational energy and entropy.

For instance, by direct calculation [5], one can obtain informational energy for particular continuous distributions, namely:

for the normal $N(m,\sigma)$ distribution we have $e = \frac{1}{2\sqrt{\pi}\sigma}$;

for the exponential $Exp(\lambda)$ distribution we have $e = \frac{\lambda}{2}$;

for standard Cauchy distribution we have $e = \frac{1}{2\pi}$.

If the calculation of e or h is easier with (2.4'') and (2.4''') then the p.d.f. of $f(X)$ should be used, as an alternative to (2.4) and (2.4').

In the next section we will present some known performant methods for solving cardinal MADM problems [2,4,12,15].

3. SOME CARDINAL MADM METHODS

Two of the best accepted MADM methods [9] are SAW (Simple Additive Weighting) and TOPSIS (Technique for Order Preference by Similarity to Ideal Solution). We will present these methods in the following lines.

3.1. Simple Additive Weighting

Let us consider $a = \| a_{ij} \|$ the decision matrix, $a_{ij} \in R$. (If some attribute takes initially discrete *linguistic* values, they will be transformed conventionally into some real numbers (e.g. marks)). In the following we assume that $a_{ij} \neq 0$. If for some j there is an $a_{ij} < 0$, then all the elements in the column j are translated with the same positive constant h, i.e. $a_{ij} := a_{ij} + h$, such as all a_{ij} become positive numbers.

The SAW method consists in the following steps [2,9]:

Step 1. Normalize elements of the decision matrix a, obtaining the matrix $R = \| r_{ij} \|$, by one of the folowing alternative procedures:

a) *Vectorial normalization.* This normalization uses one of the formulae

$$r_{ij} = \frac{a_{ij}}{\sqrt{\sum_{j=1}^{m} a_{ij}^2}} \quad or \quad r_{ij} = \frac{a_{ij}}{\sum_{i=1}^{m} a_{ij}}. \tag{3.1}$$

b) *Normalization by linear transformations.* For the criterion j of maximum the formula is used

$$r_{ij} = \frac{a_{ij}}{a_i^{max}}, a_i^{max} = \max_{1 \le j \le m} a_{ij} \tag{3.2}$$

and for the criterion j of minimum the formula is used

$$r_{ij} = 1 - \frac{a_{ij}}{a_i^{max}}. \qquad (3.2')$$

One can also use the following normalization formulae

$$r_{ij} = \frac{a_i^{max} - a_{ij}}{a_i^{max} - a_i^{min}}, a_i^{min} = \min_j a_{ij} \qquad (3.3)$$

for a maximum criterion and

$$r_{ij} = \frac{a_{ij} - a_i^{min}}{a_i^{max} - a_i^{min}} \qquad (3.3')$$

for a minimum criterion.

c) Normalization by alternative linear transformations. This normalization uses formulae

$$r_{ij} = \frac{a_{ij}}{a_i^{max}} \qquad (3.4)$$

for a maximum criterion and

$$r_{ij} = \frac{1}{a_{ij}} \max_j \frac{1}{a_{ij}} = \frac{a_i^{min}}{a_{ij}} \qquad (3.4')$$

for a minimum criterion.

As far as normalization is concerned, note that $0 < r_{ij} \leq 1$.

Note also that the requirement that all a_{ij} are positive is not compulsory for (3.3) and (3.3') and in all other normalization formulae only the conditions $a_i^{max} \neq 0, a_i^{min} \neq 0$. are necessary.

Step 2. Calculate the function $f : A \mapsto R$ (A is the finite set of alternatives) as

$$f_i = f(A_i) = \frac{\sum_{j=1}^{m} p_j r_{ij}}{\sum_{i=1}^{m} p_j}, \quad 1 \leq i \leq n \qquad (3.5)$$

where p_j are positive weights representing the relative importance of criteria. (If p_j are probabilities then $\sum_{j=1}^{m} p_j = 1$).

Step 3. Order the values f_i obtaining the ordered sequence $f_{(1)} < f_{(2)} < ... < f_{(n)}$. The best alternative is $A_{(n)}$ corresponding to $f_{(n)}$.

In general, the result of SAW does not depend on the normalization technique.

3.2. Technique for Order Preference by Similarity to Ideal Solution

This method consists of the following steps [2,4, 9,12,15]:

Step 1. Normalize the decision matrix a by obtaining the normalized matrix R (as in Step 1 of the SAW method).

Step 2. Build up the weighted normalized matrix $V = \| v_{ij} \|$ where $v_{ij} = p_j r_{ij}, 1 \le i \le n, 1 \le j \le m.$

Step 3. Build up the ideal positive solution V^+ and the ideal negative solution V^- defined as

$$V^+ = (v_1^+, v_2+, ..., v_m^+), V^- = (v_1^-, v_2^-, ..., v_m^-)$$

where

$$v_j^+ = \begin{cases} \max_i v_{ij} & \text{if the criterion } j \text{ is a maximum one} \\ \min_i v_{ij} & \text{if the criterion } j \text{ is a minimum one} \end{cases} \tag{3.6}$$

$$v_j^- = \begin{cases} \min_i v_{ij} & \text{if the criterion } j \text{ is a maximum one} \\ \max_i v_{ij} & \text{if the criterion } j \text{ is a minimum one} \end{cases} \tag{3.6'}$$

Step 4. Calculate the distances between weighted normalized entries v_{ij} and each of the ideal solutions (one uses the Euclideean distance), namely

$$D^+ = \sqrt{\sum_{j=1}^{m} (v_{ij} - v_j^+)^2}, \quad D^- = \sqrt{\sum_{i=1}^{m} (v_{ij} - v_j^-)^2}. \tag{3.7}$$

Step 5. Calculate the relative closeness to the ideal solution for each alternative as

$$Q_i = \frac{D_i^+}{D_i^+ + D_i^-}. \tag{3.8}$$

Note that $0 < Q_i < 1$.

Step 6. Order the values of Q_i obtaining $Q_{(1)} \le Q_{(2)} \le ... \le Q_{(n)}$.

The best alternative is $A_{(n)}$ corresponding to $Q_{(n)}$.

4. COMBINING SEVERAL SOLUTIONS

If we apply both stochastic MADM methods (i.e based on entropy or on energy), we may obtain different solutions (may be even different orderings of alternatives!). Finally we are interested in obtaining one solution by combining the two. The following procedure is proposed so as to give a unique solution.

Step 1. Formulate a new MADM problem with two criteria corresponding to the two solutions. The MADM decision matrix is an $n \times 2$ with the elements $\alpha_{ij}, 1 \le i \le n, 1 \le j \le 2$ defined as

$$\alpha_{ij} = \begin{cases} f_i & if \quad j = 1 \\ Q_i & if \quad j = 2 \end{cases} \tag{4.1}$$

where $f_i, Q_i, 1 \le i \le n$ are the values calculated by SAW, respectively TOPSIS methods. The weights assigned to the criteria could be 0.5 each, or could be specified by the decident.

Next, apply one of the cardinal methods presented in the previous section; here we propose the SAW method.

Step 2. Perform a normalization according to one of the procedures specified in the SAW method, obtaining the matrix $R = \| r_{ij} \|, 1 \le i \le n, 1 \le j \le 2$.

Step 3. Calculate $F_i = F(A_i)$ by formula (3.5),

Step 4. Order the values F_i obtaining $F_{(1)} < F_{(2)} < ... < F_{(n)}$. The best solution is the alternative corresponding to $A_{(n)}$.

Note. If for an initial MADM problem there are m solutions obtained, perhaps by m cardinal methods, and these solutions derive from the values of some functions $g_j, 1 \le j \le m$ (of the type f or Q from the previous section), then a new MADM cardinal problem can be formulated as in Step 1 of this section, with the decision matrix $D = \| d_{ij} \|, 1 \le i \le n, 1 \le j \le m$ defined as

$$d_{ij} = g_j(A_i). \tag{4.6}$$

Then, an algorithm (such as SAW) can be applied, obtaining the final *combined* solution.

Finally, we note that multivariate stochastic attributes could also be used; in this case, multivariate distributions are considered as attributes. Entropy and informational energy [5] are easily defined and calculated. The method based on $e = E[f(X)]$ and $h = -E[\log f(X)]$ where X is a two dimensional vector, can use the results from [10] (mentioned above), for bivariate normal and Cauchy distributions.

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