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AN ALGORITHM FOR A SIMPLE CONSTRUCTION OF SUBOPTIMAL DIGITAL CONVEX POLYGONS

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Abstract. The relationship between the number of edges and the diameter of digital convex polygons was studied in the papers [6], [2], [3], [4]. This paper gives a linear algorithm (w.r.t. the number of vertices) for a simple approximate construction of optimal digital convex polygons, that is, those digital convex polygons, which have the smallest possible diameter for a given number of edges. The algorithm partly uses the efficient construction [2] of a special sequence of optimal digital convex polygons. It constructs in a simplified manner the suboptimal (with error tolerance equal to 1) digital convex polygons. The proofs of this suboptimality can be found in the paper [4].

1. INTRODUCTION

A *digital convex polygon* is a convex polygon, all the vertices of which have integer coordinates.

This paper presents an asymptotically optimal algorithm for the construction of optimal or suboptimal [4] digital convex n-gons, where n is a given natural number. The optimality of polygons is considered here exclusively in the following sense: the constructed n-gon should have the smallest possible diameter (the edge size of the minimal inscribed square with the edges parallel to the coordinate axes). The vertices of the constructed n-gons are listed in the positive ordering.

The construction is almost optimal in the sense that the achieved value for the diameter of the constructed polygon is at most for one greater than the theoretical minimum. The paper [3] contains an *exact* construction of *optimal* digital convex $2s$-gons. However, in [4], which considers the construction of optimal digital convex n-gons for arbitrary n, the emphasis is put on the simplicity of the construction.

In some cases (for n of the form $4s + 2$) the already known construction of optimal digital convex polygons is therefore replaced by a simpler one, which gives suboptimal digital convex polygons.

Time complexity of the proposed algorithm is linear w.r.t. the number of vertices of the required polygon. An efficient construction of the Farey sequence is used. The algorithm for construction of optimal digital convex $4s$-gons is an easy generalization of the algorithm proposed in [2] for the construction of digital convex polygons in a special sequence $P(t)$. The point of the algorithm proposed here is the efficient use of a family of so-called Basic polygons, which are introduced in [4] in order to cover the cases when the number n of edges is not divisible by 4. The algorithm also incorporates an efficient determination of the parameter t of the Farey sequence.

2. PRELIMINARIES

The *diameter* (in the sense of city block distance) of a digital convex polygon P is the maximal one among the values $|x_i - x_j|$ and $|y_i - y_j|$, where (x_i, y_i) and (x_j, y_j) are two arbitrary vertices of P.

Let $y_{\min}$ and $x_{\max}$ respectively denote the minimal y-coordinate and the maximal x-coordinate of the considered digital convex polygon P. The *SE-arc* (*southeast arc*) of P is the sequence of consecutive edges (V_i, V_{i+1}), $0 \le i \le k-1$, where:

— V_i denotes a vertex (x_i, y_i) of P

— $x_0 \le x_1$; $y_0 = y_1 = y_{\min}$; $x_{k-1} < x_{\max}$; $x_k = x_{\max}$.

The NE-arc, the NW-arc and the SW-arc of a digital convex polygon are defined in an analogous way.

Given an edge $e = [(x_1, y_1), (x_2, y_2)]$ of a digital convex polygon, the *edge slope* of e denotes the fraction:

$$\frac{|x_1 - x_2|}{|y_1 - y_2|} \text{ if } e \in \text{NE- or SW arc;} \quad \frac{|y_1 - y_2|}{|x_1 - x_2|} \text{ if } e \in \text{SE- or NW-arc,}$$

while *bd-length* of the edge e is the sum $|x_1 - x_2| + |y_1 - y_2|$.

If two digital convex polygons P_1 and P_2 have edge-slope-disjoint corresponding arcs, then there exists the uniquely determined third digital convex polygon P_3, called the *sum* of P_1 and P_2. Each arc of the polygon P_3 includes all the edges of the corresponding arcs of P_1 and P_2, sorted so that the convexity condition is preserved.

A class $P(t)$, $t = 1, 2, \ldots$, of optimal digital convex polygons was introduced in [6]. The edge slopes of edges of each arc of the polygon $P(t)$ are all different fractions of the form q/p, where the natural numbers p and q are relatively prime and $p + q \le t$. In addition, the edge slope of the first edge in each arc of $P(t)$ is equal to $0/1$. The number of edges of $P(t)$ is denoted by $n(t)$. It is easy to show that

$$n(t) = 4 \sum_{s=1}^{t} \varphi(s), \qquad (***)$$

where $\varphi(s)$ denotes the number of integers between 1 and s which are relatively prime with s (the well-known Euler function from number theory; e.g. $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = \varphi(4) = 2$, $\varphi(5) = 4$).

Farey sequence of order t, (shortly $F(t)$, [5]), is the strictly increasing sequence, which includes all the fractions of the form b/a, where the integers a and b are relatively prime and $1 \leq b < a \leq t$. $F(5)$ is listed in Example 1, Section 5.

3. CONSTRUCTION

Suboptimal (in some cases optimal) digital convex n-gons can be represented as the sum (in the sense explained above) of three digital convex polygons, called Initial, Basic and Additional polygon respectively.

The number of edges of both Initial and Additional polygon is divisible by 4. Basic polygon has 9, 6 and 7 edges for n of the form $4s+1$, $4s+2$ and $4s+3$ respectively.

If the problem of determining a suboptimal digital convex n-gon is considered, then let t denote the integer such that the prolonged inequality $n(t-1) < n \leq n(t)$ is satisfied. The bd-lengths of edges of the Basic polygon may be smaller than t, equal to t and even greater than t. All the edges of Initial and Additional polygon have bd-lengths smaller than t, respectively equal to t. The edges in the latter two polygons are "packed" into quadruples of edges with the same edge slope in distinct arcs (the edge slopes used in these quadruples must not be used in any arc of the Basic polygon).

There exists a 1-1 correspondence between the elements of the Farey sequence and edge slopes of the polygon P. Namely, the mapping

$$b/a \rightarrow b/(a-b)$$

(which maps a member b/a of $F(t)$ to the corresponding edge slope q/p) is a bijection which preserves the ordering. Besides, note that the integers b and a are relatively prime if and only if the integers b and $a-b$ are as well.

In this way the listing of vertices of an arc becomes equivalent to the listing of members of the Farey sequence, but the latter listing (in increasing order) is possible in linear time [1]. Thus the sorting of vertices within each one of the arcs is avoided.

4. ALGORITHM

Input: a natural number n.

Output: an optimal digital convex n-gon P, that is, the one which is included into a digital square of a minimal edge size.

Stages of the algorithm:

1. Determining the natural number t so that n belongs to the half-open interval $(n(t-1), n(t)]$.

2. Determining the case (numerated by one of the integers $0, 1, \dots, 12$) in dependence of n, t and $n(t)$.
3. Generating each one of the four arcs of the polygon P by one pass through the Farey sequence $F(t)$.

4.1. Stage 1: Determining the Number t

In accordance with the expression (***), the number t, such that $n \in (n(t-1), n(t)]$ is determined by summing up the summands of the form $4 * \varphi(s)$, for $s = 1, 2, \dots$, until the sum (equal to $n(t)$) becomes greater than or equal to the number n.

On the other hand, the value of $\varphi(s)$ is determined by using the expression [5]:

$$\varphi(s) = s\left(1 - \frac{1}{p_1}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_v}\right), \qquad \text{where } s = p_1^{\alpha_1} \cdot \ldots \cdot p_v^{\alpha_v}$$

is a prime factorization of the number s.

4.2. Stage 2: Determining the Case

The cases are mostly determined on the basis of n mod 4 and t mod 4. If n mod 4 = 0 (case 0), then Basic polygon is not used. The remaining 12 cases are specified by the following table:

Case	n	t	Basic	Case	n	t	Basic
1	$4s+1$	$4k$	$B1$	7	$4s+2$	$2k$	$B5$
2	$4s+1$	$4k+1$	$B1$	8	$4s+2$	$2k+1$	$B5$
3	$4s+1$	$4k+1$	$B2$	9	$4s+3$	$4k$	$B6$
4	$4s+1$	$4k+2$	$B3$	10	$4s+3$	$4k+1$	$B7$
5	$4s+1$	$4k+3$	$B3$	11	$4s+3$	$4k+2$	$B8$
6	$4s+1$	$4k+3$	$B4$	12	$4s+3$	$4k+3$	$B9$

Cases 3 and 6 are used iff $n = n(t) - 3$ (the Basic polygons $B1$, respectively $B3$, do not match in these cases), although the Basic polygons $B2$, respectively $B4$, are valid for $n \geq n(t-1) + 5$. The alternative cases 2 and 5 respectively are used instead for $n \in [n(t-1) + 5, n(t) - 7]$.

The diameter ng of the sum of Initial and Basic polygon is equal to $n(t-1) + (n \bmod 4)$ in ten cases, except for the cases 3 and 6, when it is equal to $n(t-1)+5$. The required n-gon uses exactly $(n - ng)/4$ edges of the Additional polygon in each arc (this fact is used in the construction, in Stage 3). The value $n(t-1)$ (necessary for determining ng) can be easily determined at the end of Stage 1 as $n(t) - 4\varphi(t)$.

The list of Basic polygons is given in Table 1. The edge slopes od edges of SE-arc, NE-arc, NW-arc and SW-arc, in this order, are listed for each Basic polygon.

Table 1. The list of Basic polygons

$$B1: \quad \frac{0}{1}, \frac{k}{3k+1}, \frac{2k-1}{2k}, \quad \frac{0}{3}, \frac{2k}{2k-1}, \quad \frac{0}{3}, \frac{2k-1}{2k}, \frac{3k+1}{k}, \quad \frac{0}{2}$$

$$B2: \quad \frac{0}{1}, \frac{k+1}{3k+2}, \frac{2k}{2k+1}, \quad \frac{0}{2}, \frac{2k+1}{2k}, \quad \frac{0}{2}, \frac{2k}{2k+1}, \frac{3k+2}{k+1}, \quad \frac{0}{2}$$

$$B3: \quad \frac{0}{1}, \frac{k+1}{3k+2}, \frac{2k}{2k+1}, \quad \frac{0}{2}, \frac{2k+1}{2k}, \quad \frac{0}{2}, \frac{2k}{2k+1}, \frac{3k+2}{k+1}, \quad \frac{0}{2}$$

$$B4: \quad \frac{0}{1}, \frac{k+1}{3k+2}, \frac{2k+1}{2k+3}, \quad \frac{0}{1}, \frac{2k+3}{2k+1}, \quad \frac{0}{1}, \frac{2k+1}{2k+3}, \frac{3k+4}{k+1}, \quad \frac{0}{1}$$

$$B5: \quad \frac{0}{1}, \frac{k}{k+1}, \quad \frac{0}{2}, \quad \frac{0}{1}, \frac{k}{k+1}, \quad \frac{0}{2}$$

$$B6: \quad \frac{0}{1}, \quad \frac{0}{1}, \frac{3k+1}{k}, \quad \frac{0}{1}, \frac{2k}{2k+1}, \quad \frac{0}{2}, \frac{k}{3k+1}$$

$$B7: \quad \frac{0}{1}, \quad \frac{0}{1}, \frac{3k+2}{k+1}, \quad \frac{0}{1}, \frac{2k}{2k+1}, \quad \frac{0}{2}, \frac{k+1}{3k+2}$$

$$B8: \quad \frac{0}{2}, \quad \frac{0}{2}, \frac{3k+2}{k+1}, \quad \frac{0}{2}, \frac{2k+2}{2k+1}, \quad \frac{0}{1}, \frac{k+1}{3k+2}$$

$$B9: \quad \frac{0}{1}, \quad \frac{0}{1}, \frac{3k+4}{k+1}, \quad \frac{0}{1}, \frac{2k+2}{2k+3}, \quad \frac{0}{2}, \frac{k+1}{3k+4}$$

The edge slopes of edges within an arc are listed in the increasing order. The first edge of each arc has the edge slope of the form $0/p$.

The diameters of the Basic polygons $B1, B2, \dots, B9$ are in order:

$$5k+2,\ 5k+4,\ 5k+4,\ 5k+6,\ k+2,\ 3k+2,\ 3k+3,\ 3k+4,\ 3k+5.$$

The edge slopes of Basic polygon can be stored in a double array indexed by ordinal numbers of arcs and by ordinal numbers of edges within an arc. Such an array is generated depending on the case and the number k (obtained from t).

4.3. Generating the arcs of the required polygon

The construction of the required digital convex polygon P is separated into four independent constructions of its arcs (SE-, NE-, NW-and SW-arc in turn). Each arc is constructed by using only one pass through the Farey sequence $F(t)$. First edge of an arc of P is obtained by *writing down* the first edge of the corresponding arc A of the Basic polygon. The sequence $F(t)$ is initialized afterwards.

The following scheme is used for the general step of the construction of an arc of P:

— Construct the following member b/a of the sequence $F(t)$ (this construction is based on the recursive connection given in [5]; an implementation of a more general construction is described in [1]).

— Determine the corresponding edge slope q/p with $q := p$, $p := a - b$.

— If the edge slope q/p is *acceptable*, then *register* the corresponding edge.

We proceed with a more detailed description of the boolean function *acceptable* and the procedure *register*.

We primarily give the explanations for the cases 1–12. As above, let A denote the current arc of the Basic polygon.

A necessary condition for the value TRUE of *acceptable* is that one of the following two conditions holds with the edge slope q/p:

a) q/p is not used in the Basic polygon

b) q/p is used in A.

The condition b) is also sufficient. The same statement holds for the condition a) whenever $p + q < t$; the Initial polygon consists of exactly such edges.

However, when the condition a) is accompanied with $p + q = t$, then the value of a counter, denoted by c and initialized by 0, is increased by 1. In accordance with the above remark on the Additional polygon, the final requirement for acceptance in that case is that $c \leq (n - ng)/4$.

A particular attention should be paid to those edge slopes q_u/p_u, $u = 1, 2, \ldots, w(A)$, of the arc A of the Basic polygon, which satisfy that $p_u + q_u > t$; they are sorted in increasing order and stored in an auxiliary vector V (the initial value of u is set to 1).

These edge slopes will never be addressed by some members of $F(t)$, but they should nevertheless be included into the polygon P. The proper moment for inserting the corresponding edges into the arc of P which is being constructed — must not be missed. The following loop is consequently activated with the procedure *register* before *writing down* the edge with the edge slope q/p:

While $q_u/p_u < q/p$, and $u \leq w(A)$, the edge with the edge slope q_u/p_u is *written down* and the value of the counter u is increased by 1.

The only condition for acceptance in Case 0 is: $p + q < t$ or ($p + q = t$ and $c \leq (n - ng)/4$).

The coordinates (x_0, y_0) of the first vertex of the SE-arc of P are given in advance. Given a current vertex (x_i, y_i), an edge with the edge slope q/p is *written down* by producing the next vertex (x_{i+1}, y_{i+1}) in accordance with the connections

$$x_{i+1} = x_i + x_{\text{dif}} \quad \text{and} \quad y_{i+1} = y_i + y_{\text{dif}},$$

where the pair $(x_{\text{dif}}, y_{\text{dif}})$ is equal to $(+p, +q)$, $(-q, +p)$, $(-p, -q)$, $(+q, -p)$, within the SE-arc, NE-arc, NW-arc, SW-arc respectively.

5. EXAMPLES

EXAMPLE 1. Given $n = 35$, it is calculated in Stage 1 that $t = 5$. It is further determined in stage 2 that $k = 1$; Basic = $B7$ (Fig. 2); Case = 10; $ng = 27$.

The construction of the required 35-gon P in Stage 3, uses the sequence F(5):

$$1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5.$$

This sequence is bijected to the sequence

$$1/4, 1/3, 1/2, 2/3, 1/1, 3/2, 2/1, 3/1, 4/1,$$

which includes all the edge slopes q/p of edges of an arc of the polygon $P(5)$ in increasing order.

The coordinates of the initial vertex of P are arbitrarily taken to be $(0, 0)$. This vertex is the common vertex of the SW-arc and the SE-arc. The vertices of P are generated and listed in the positive orientations (SE-arc, NE-arc, NW-arc and SW-arc are passed, in this order). The common vertex of two neighboring arcs is followed by the denotation -----.

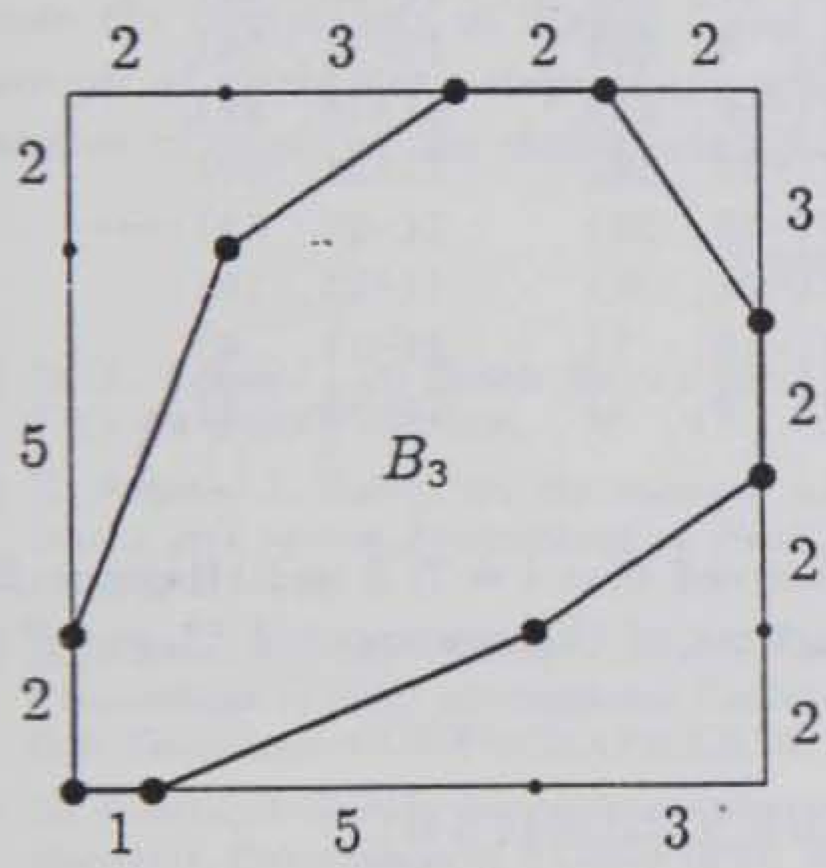

Fig. 1. $B3$ for $k = 1$

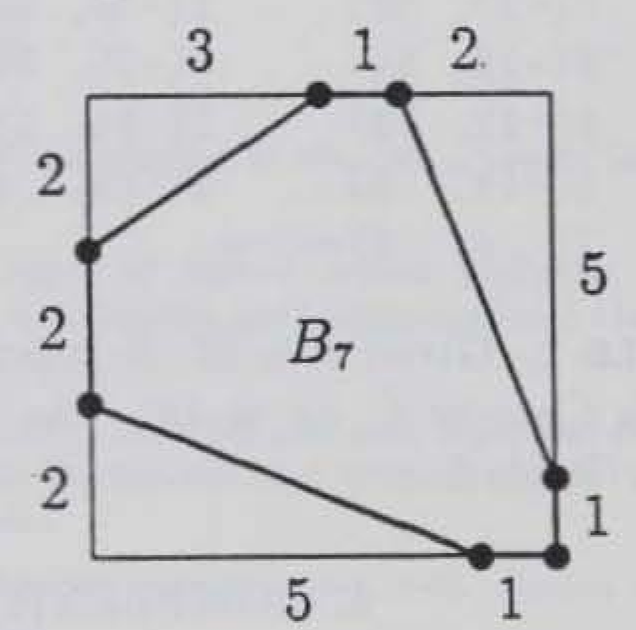

Fig. 2. $B7$ for $k = 1$

The end vertices of edges, the edge slopes of which belong to Initial and Additional polygon. are denoted by the letters "I" and "A" respectively. When the edge slopes used in Basic polygon are considered, four different letters are used instead: "B" for the beginning edge of the arc, "S", "E" and "G" for the edges with edge slopes smaller than, equal to and greater than t respectively. Note that the letter "S" is not present with Example 1, while the letter "G" is not present with the Example 2.

The list of vertices of the constructed 35-gon P is given in Table 2.

Table 2. The list of vertices of the constructed 35-gon P

B(1, 0)	A(5, 1)	I(8, 2)	I(10, 3)
I(11, 4)	A(13, 7)	I(14, 9)	I(15, 12)-----
B(15, 13)	A(14, 17)	I(13, 20)	G(11, 25)
I(10, 27)	I(9, 28)	A(6, 30)	I(4, 31)
I(1, 32)-----	B(0, 32)	A(-4, 31)	I(-7, 30)
I(-9, 29)	E(-12, 27)	I(-13, 26)	A(-15, 23)
I(-16, 21)	I(-17, 18)-----	B(-17, 16)	A(-16, 12)
I(-15, 9)	I(-14, 7)	I(-13, 6)	A(-10, 4)
I(-8, 3)	G(-3, 1)	I(0, 0)-----	

Table 3. The list of vertices of the constructed 57-gon P

B(1, 0)	A(7, 1)	I(12, 2)	I(16, 3)
I(19, 4)	E(24, 6)	I(26, 7)	S(29, 9)
A(33, 12)	I(34, 13)	I(36, 16)	I(37, 18)
I(38, 21)	I(39, 25)	I(40, 30)-----	B(40, 32)
A(39, 38)	I(38, 43)	I(37, 47)	I(36, 50)
I(35, 52)	S(33, 55)	A(30, 59)	I(29, 60)
I(26, 62)	I(24, 63)	I(21, 64)	I(17, 65)
I(12, 66)-----	B(10, 66)	A(4, 65)	I(-1, 64)
I(-5, 63)	I(-8, 62)	I(-10, 61)	S(-13, 59)
A(-17, 56)	I(-18, 55)	I(-20, 52)	I(-21, 50)
E(-23, 45)	I(-24, 42)	I(-25, 38)	I(-26, 33)-----
B(-26, 31)	A(-25, 25)	I(-24, 20)	I(-23, 16)
I(-22, 13)	I(-21, 11)	A(-18, 7)	I(-17, 6)
I(-14, 4)	I(-12, 3)	I(-9, 2)	I(-5, 1)
I(0, 0)-----			

EXAMPLE 2. Given $n = 57$, it is primarily derived that $t = 7$; $k = 1$, Basic $= B3$ (Fig. 1); Case $= 5$; $ng = 49$. The list of vertices of the constructed 57-gon P is given in Table 3.

6. COMPLEXITY OF THE ALGORITHM

THEOREM 1. *The algorithm given in Section 3 is asymptotically optimal.*

PROOF. The following asymptotic estimation for the number $n(t)$ has been derived in [2]:

$$n(t) = \frac{12t^2}{\pi^2} + O(t \log t).$$

Since $n(t-1) < n \le n(t)$, the number of edges of the constructed polygon P is of the same order of magnitude ($O(t^2)$). The presented construction of (optimal or suboptimal) digital convex polygon P is asymptotically optimal in the sense that the number of elementary steps of the construction is also of order $O(t^2)$. Such a conclusion can be derived by analyzing the stages of the algorithm:

STAGE 1. Factorization of a natural number s requires $O(\sqrt{s})$ elementary steps [5]. Calculating $\varphi(s)$ on the basis of the formula given in 4.1. requires v additional elementary steps, where v is obviously bounded from above by $\log_2(s)$ and consequently by $\sqrt{s}$. It follows that calculating $\varphi(s)$ for $s = 1, 2, \ldots, t$, and consequently the calculating of $n(t)$ and t itself requires $O(t\sqrt{t})$ elementary steps.

STAGE 2. Distinguishing the cases on the basis of n mod 4, t mod 4 and comparing n with already calculated $n(t)-3$ can be obviously performed in constant time.

STAGE 3. Given a member of the Farey sequence, the calculation of the next member is performed in a constant time [1]. On the other hand, the necessary calculations concerning q/p and related to each member of the Farey sequence can be also performed in a constant time; they include only the search of edges of a fixed Basic polygon (the number of edges in that polygon is bounded from above by 10 in all the cases).

The sequence $F(t)$ is passed four times during the generation. Thus the number of elementary steps used in Stage 3. is asymptotically equal to the 4-fold number of members of the sequence $F(t)$. The latter number has been estimated as $3t^2/\pi^2 + O(t\log t)$ ([5], Theorems 330 and 331).

Since the complexity of Stages 1 and 2, is smaller than $O(t^2)$, it follows that the number of elementary steps of the whole algorithm is asymptotically equal to the number of edges of the constructed polygon. □

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