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CHARACTERIZING OPTIMALITY IN NONCONVEX OPTIMIZATION: ADDENDUM

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Abstract. We draw attention to the curious fact that a necessary condition for global optimality requires a uniqueness assumption to hold at a local optimum.

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1. INTRODUCTION

This is a self-contained addendum to the recent paper [2]. We show that a necessary condition for global optimality requires an additional assumption in order to hold at a local optimum. In the process we correct a flaw in the proof of [2, Theorem 2.2].

2. GLOBAL AND LOCAL OPTIMALITY CONDITIONS

Consider a (generally nonconvex) program

$$(P) \qquad \begin{array}{ll} \text{Min} & f^0(z) \\ \text{s.t.} & \\ & f^i(z) \le 0, \qquad i \in \mathcal{P} = \{1, \dots, m\} \end{array}$$

where all functions $f^0, f^i : \mathbb{R}^n \to \mathbb{R}$, $i \in \mathcal{P}$, are assumed continuous. We assume that, after some splitting $z = (x, \theta)$, $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^p$, $n + p = N$, the functions $f^0(\cdot, \theta), f^i(\cdot, \theta) : \mathbb{R}^n \to \mathbb{R}$ are convex for every $\theta \in \mathbb{R}^p$. Such a program (P) is said to be *convexifiable-by-a-splitting* and it is written in the form

$$(P, \theta) \qquad \begin{array}{ll} \text{Min} & f^0(x, \theta) \\ \text{s.t.} & \\ & f^i(x, \theta) \le 0, \qquad i \in \mathcal{P}. \end{array}$$

Note that, for every fixed θ, (P, θ) is a convex program. Many problems, including all linear and convex programs, enjoy this property. Following the notation from [2], we denote, for every $\theta \in \mathbb{R}^p$,

$$\begin{aligned} F(\theta) &= \{x \in \mathbb{R}^n : f^i(x,\theta) \le 0,\ i \in \mathcal{P}\} \\ \mathcal{F} &= \{\theta \in \mathbb{R}^p : F(\theta) \neq \emptyset\} \\ \mathcal{P}^= &= \{i \in \mathcal{P} : x \in F(\theta) \Rightarrow f^i(x,\theta) = 0\} \\ \mathcal{P}^<(\theta) &= \mathcal{P} \setminus \mathcal{P}^=(\theta). \end{aligned}$$

Around a candidate for optimality $z^* = (\tilde{x}(\theta^*), \theta^*)$, where $\tilde{x}(\theta^*)$ is an optimal solution of the program (P, θ^*), we denote

$$\begin{aligned} F_*^=(\theta) &= \{x \in \mathbb{R}^n : f^i(x,\theta) \le 0,\ i \in \mathcal{P}^=(\theta^*)\} \\ \mathcal{L}_*^<(z,u) &= f^0(z) + \sum_{i \in \mathcal{P}^<(\theta^*)} u_i f^i(z) \\ c &= \operatorname{card} \mathcal{P}^<(\theta^*) \\ \mathbb{R}_+^c &= \{u \in \mathbb{R}^c : u \ge 0\}. \end{aligned}$$

A necessary condition for global optimality of z^* follows.

2.1. THEOREM. *Consider a convexifiable-by-a-splitting program (P) and its feasible point $z^* = (\tilde{x}(\theta^*), \theta^*)$, where $\tilde{x}(\theta^*)$ is an optimal solution of the convex program (P, θ^*). Assume that the point-to-set mapping F is lower semicontinuous at θ^* relative to $\mathcal{F}$. If z^* is a globally optimal solution of (P) then there exists a vector function $\tilde{u} = \tilde{u}(\theta)$ such that*

$$\mathcal{L}_*^<(z^*, u) \le \mathcal{L}_*^<(z^*, \tilde{u}(\theta^*)) \le \mathcal{L}_*^<(z, \tilde{u}(\theta)) \tag{2.1}$$

for every

$$z \in \{F_*^=(\theta), \mathcal{F} \cap N(\theta^*)\} \tag{2.2}$$

where $N(\theta^*)$ is some neighbourhood of θ^*, and for every non-negative $u \in \mathbb{R}_+^c$.

PROOF. This result was proved in [2, Theorem 2.2]. However, $N(\theta^*)$ was erroneously omitted in (2.2). The intersection with $N(\theta^*)$ is required, because the inclusion $\mathcal{P}^<(\theta^*) \subset \mathcal{P}^<(\theta)$, used in the proof, holds locally.

It is curious that the above result does not hold at a local optimum. The proof from [2, Theorem 2.2] fails for this case, because *local* optimality of z^* is not contradicted by the assumption in its proof that $K_1(\theta) \cap K_2 \neq \emptyset$. The following would serve as a counterexample.

2.2. EXAMPLE. Consider

$$\begin{aligned} \text{Min} \quad & f^0 = z_1 z_2^2 \\ \text{s.t.} \quad & f^1 = z_1 - 1 \le 0 \\ & f^2 = -z_1 - 1 \le 0. \end{aligned}$$

The program is convexifiable-by-a-splitting $x = z_1$, $\theta = z_2$. Point $z_1^* = 1$, $z_2^* = 0$ is a local minimum. Mapping F is trivially lower semicontinuous, but there is no $\bar{u} \geq 0$ satisfying (2.1) for every z on the set (2.2).

However, if z^* is locally optimal *and the optimal solution* $\bar{x}(\theta)$ is unique, then it has been shown in [1] that $\bar{u} \geq 0$, satisfying (2.1) and (2.2) exists. This result is stated for the sake of completeness.

2.3. THEOREM. *Consider a convexfiable-by-a-splitting program* (P) *and its feasible point* $z^* = (\bar{x}(\theta^*), \theta^*)$, *where* $\bar{x}(\theta^*)$ *is a unique optimal solution of the convex program* (P, θ^*). *Assume that the point-to-set mapping* F *is lower semicontinuous at* θ^* *relative to* $\mathcal{F}$. *If* z^* *is a locally optimal solution of* (P), *then there exists a vector function* $\bar{u} = \bar{u}(\theta)$ *such that* (2.1) *holds for every* z *from* (2.2), *where* $N(\theta^*)$ *is some neighbourhood of* θ^*, *and for every non-negative* $u \in \mathbb{R}_+^c$.

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