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OPTIMAL CONTROL IN A FUZZY ENVIRONMENT

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Abstract. More realistic formulations of optimal control problems are proposed in the paper. Fuzzy sets are used to represent the impreciseness and non-reproducibility of the experimental data and subjectivity of the cost function(s) definition. The problem is transformed via Bellman-Zadeh's approach to the crisp (nonfuzzy) optimal control problem. Illustrative numerical examples are presented.

Key words and phrases: fuzzy mathematical programming, fuzzy optimal control, fuzzy decision making.

1. INTRODUCTION

Using fuzzy sets for a description of uncertainties in the transversality conditions was proposed recently [2]. In this paper the concept of fuzzy optimal control (FOC) is developed for the case of fuzzy systems model. The fuzzy model introduced here represents dynamic behaviour of the system by a nonlinear cylinder ("duct") with a variable radius in the model description. It allow as on to consider all experimental data with various degrees of acceptance and, as a result, to receive a more effective control sequence than in the deterministic model.

An approach to determine membership function of the model fuzzy sets including all experimental data is introduced. Zimmermann's approach is used to represent cost as a fuzzy set [5].

Both formulations of the FOC (with fuzzy transversality conditions and with fuzzy model) can be transformed via Bellman-Zadeh's approach to the well-known mathematical programming problem.

2. PROBLEM FORMULATION

A Conventional optimal control problem is stated as follows [4]:

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Find a control sequence u_k $(k = 0, \dots, N-1)$ which minimizes:

$$J = g(x_N) + \sum_{k=0}^{N-1} h(x_k, u_k) \qquad x \in E^n,\ u \in E^m \tag{1^a}$$

System dynamics is:

$$x_{k+1} = f(x_k, u_k) \qquad k = 0, \dots, N-1. \tag{1^b}$$

Initial condition x_0 is given and x_N satisfies transversality condition:

$$S(x_N) = 0, \qquad S \in E^q. \tag{1^c}$$

Fuzzy sets are used to describe the case when transversality condition (1^c) is not determined exactly and Zimmermann's approach [1] is applied to perform fuzzy minimization. The modification of the optimal control problem called FO is:

Find such a control sequence u_k $(k = 0, \dots, N-1)$ which minimizes not as strict imperative the fuzzy objective function:

$$J = g(x_N) + \sum_{k=0}^{N-1} h(x_k, u_k) \longrightarrow \min, \qquad x \in E^n,\ u \in E^m. \tag{2^a}$$

The system dynamics is

$$x_{k+1} = f(x_k, u_k) \qquad k = 0, \dots, N-1 \tag{2^b}$$

and fuzzy transversality condition:

$$S(x_N) \simeq 0. \tag{2^c}$$

By $\simeq$ we denote a fuzzy expression "approximately equal to" which means that:

a) a strict satisfaction of the of left side $(S(x_N))$ equal to the right side (zero) is the most appropriate $(\mu = 1)$:

b) small violations of the equation are acceptable to degree $(0 < \mu < 1)$;

c) great violations of the equation are not acceptable $(\mu = 0)$.

Here μ is a membership function.

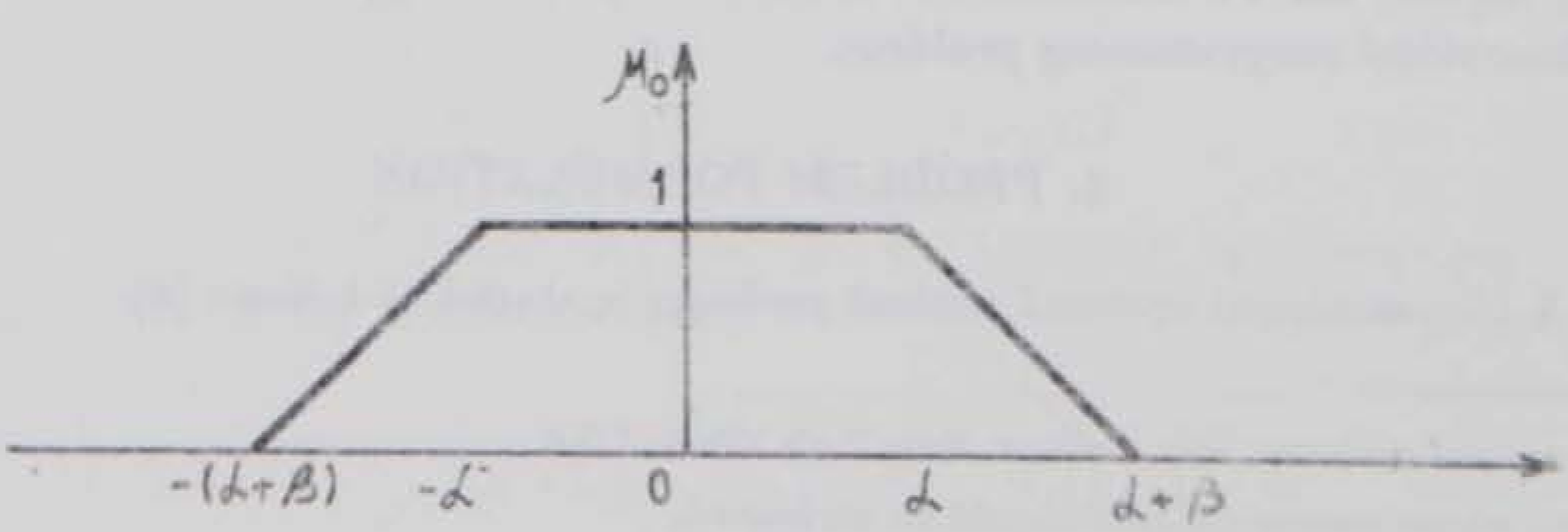

Fig. 1. Membership function of fuzzy transversality condition

If we use linear membership function of type Fig. 1:

$$\mu = \begin{cases} 1, & |S| > \alpha + \beta \\ 1 - \beta^{-1}(|S(x_N)| - \alpha), & \alpha \le |S| \le \alpha + \beta \\ 0, & |S| < \alpha \end{cases} \tag{3}$$

then (2^c) means that $|S(x_N)| < \alpha + \beta$ where $(\alpha + \beta)$ represents uncertainties of the formulation of (1^c) that really exist in practice.

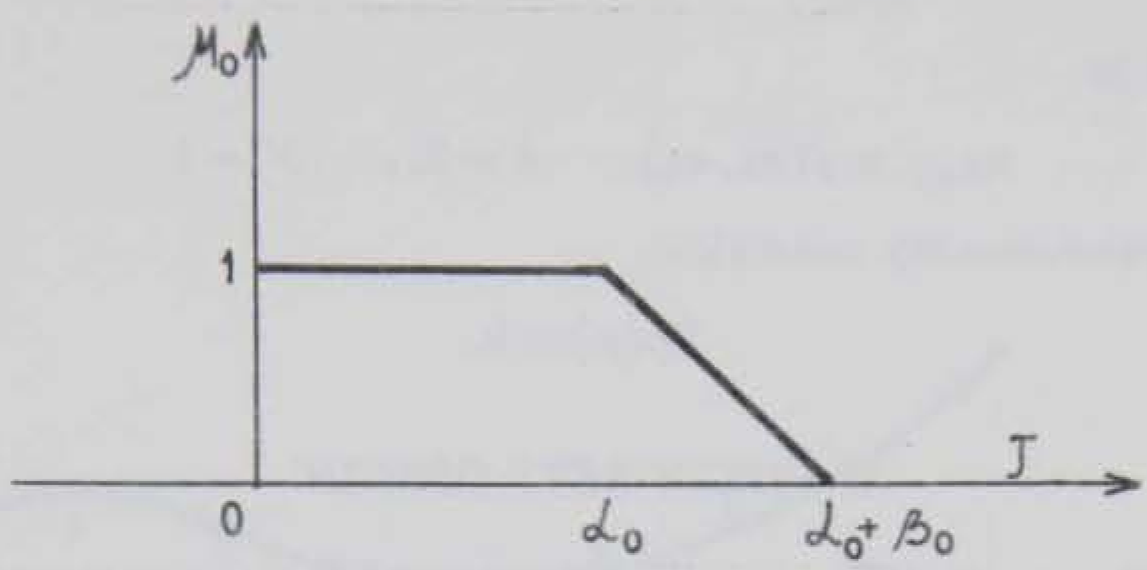

Fig. 2. Membership function of fuzzy objective condition

Here "min" in (2^a) denotes "as small as possible" represented, for example, by membership function (Fig. 2):

$$\mu_0 = \begin{cases} 1, & J < \alpha_0 \\ 1 - \beta_0^{-1}(J - \alpha_0), & \alpha_0 \le J \le \alpha_0 + \beta_0 \\ 0, & J > \alpha_0 + \beta_0 \end{cases} \tag{4}$$

In this FOC formulation we suppose strict equalities in a crisp sense of (2^b) but in real-life problems the dynamics of systems can't be described exactly by crisp equations because of really existing uncertainties. Structural uncertainties are reflected by the fact that abstraction and ignoring of some factors take place in the model building because of the complexity of the systems nature. Parameter uncertainties are the result of impreciseness of the identification procedures, especially for the nonlinear case. Hence, the following problem is more realistic:

Find such a control sequence u_k $(k = 0, \dots, N-1)$ which minimizes not as a strict imperative the fuzzy objective function:

$$J = g(x_N) + \sum_{k=0}^{N-1} h(x_k, u_k) \longrightarrow \min \qquad x \in E^n,\ u \in E^m \tag{5^a}$$

and the fuzzy model is:

$$x_{k+1} \simeq f(x_k, u_k) \qquad k = 0, \dots, N-1 \tag{5^b}$$

Transversality condition is:

$$S(x_N) = 0. \tag{5^c}$$

Combining the FOC problem and the above optimal control problem with the fuzzy model, we can formulate the optimal control problem under uncertainty in the model and in the transversality condition:

Find such a control sequence u_k ($k = 0, \dots, N-1$) which minimizes not as a strict imperative the fuzzy objective function:

$$J = g(x_N) + \sum_{k=0}^{N-1} h(x_k, u_k) \longrightarrow \min \qquad x \in E^n,\ u \in E^m \tag{6^a}$$

the fuzzy model is:

$$x_{k+1} \simeq f(x_k, u_k) \qquad k = 0, \dots, N-1 \tag{6^b}$$

and the fuzzy transversality condition:

$$S(x_N) \simeq 0. \tag{6^c}$$

3. FUZZY SETS DESIGN

Deterministic model (1^b) s a generalization of the experimental data on the basis of some identification procedure. We can represent experimental data as curves (Fig. 3). In some types of real systems, like biotechnological processes the reproducibility of the data is low, i.e. curves representing relations obtained in the same conditions are different and deviations between them are significant.

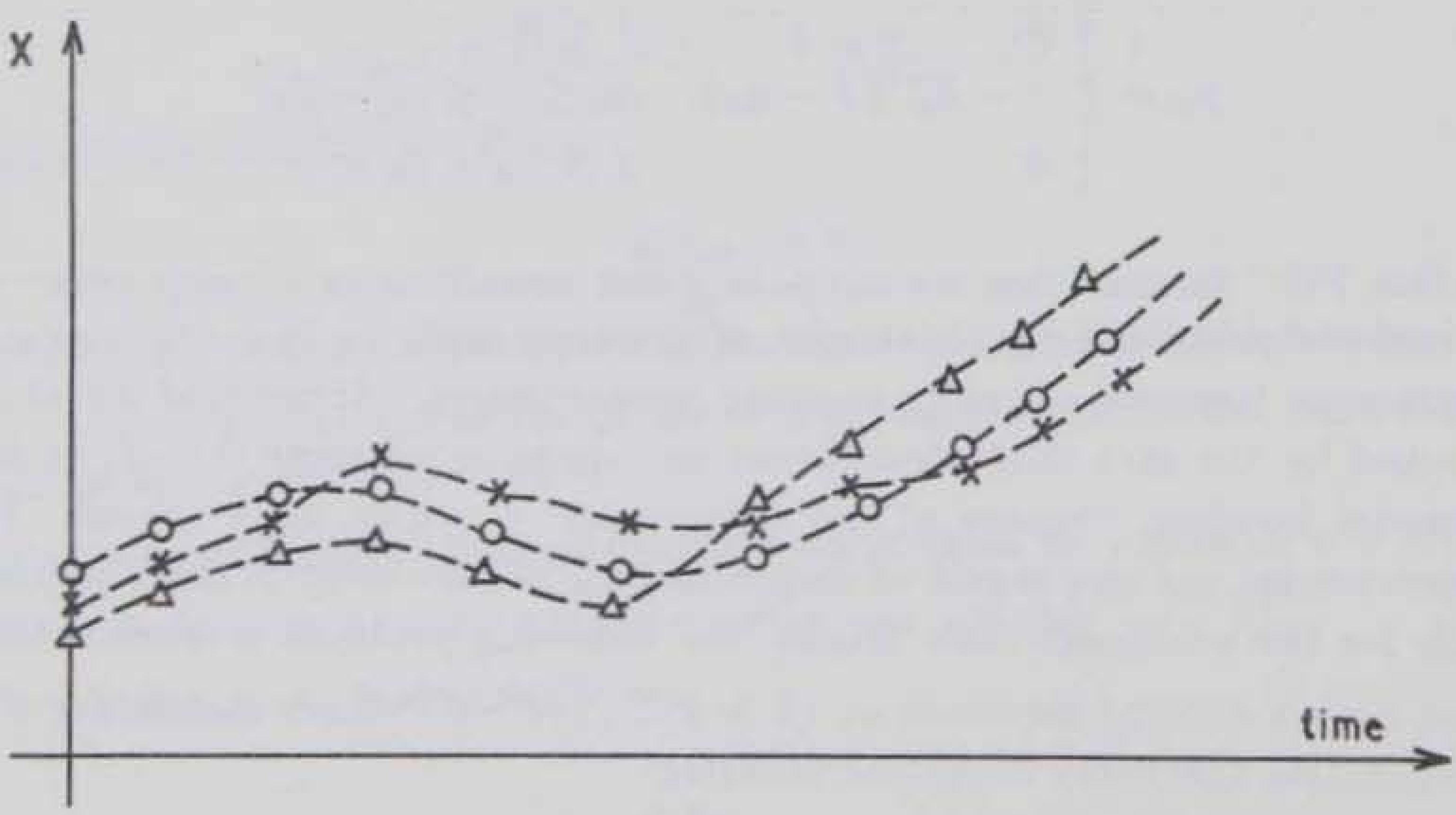

Fig. 3. Combination of experimental data.
(o — experiment No. 1; x — experiment No. 2; Δ — experiment No. 3)

Deterministic model of type (1^b) depicted (for a fixed control vector) a curve trajectory which is as close as possible to one or to all experimental data. If the measure of identification is:

$$J = \sum_{k=0}^{N} (x_k - x_k^e)^2/(x_k^e) \longrightarrow \min \tag{7^a}$$

then model curve is close to the experimental data (Fig. 4). When the measure of identification is:

$$J = \sum_{i=1}^{r} \sum_{k=0}^{N} (x_k - x_k^{ei})^2/(x_k^{ei}) \longrightarrow \min \tag{7^b}$$

where r is the number of experiments, model curve is close to all experimental data (Fig. 5).

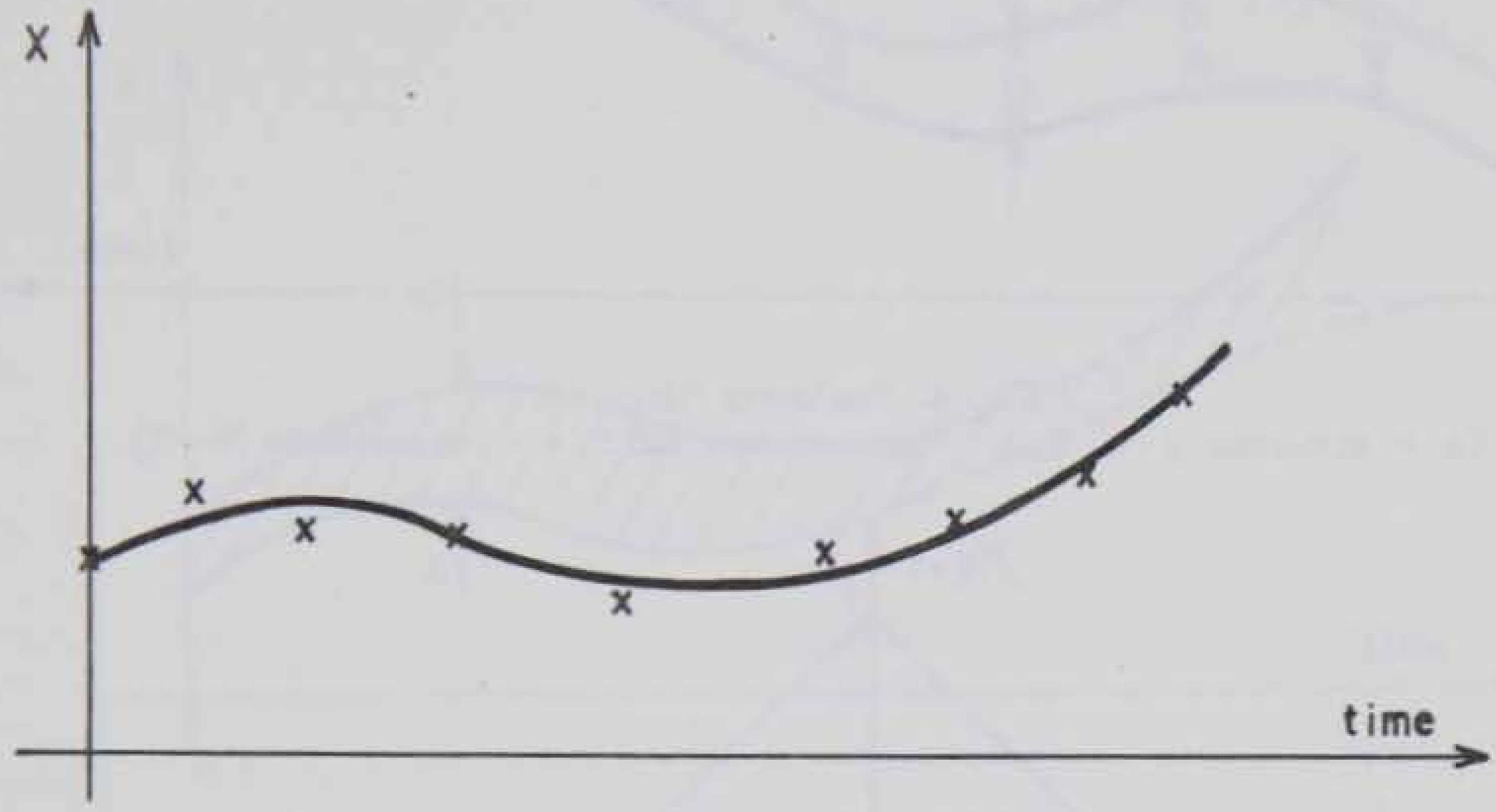

Fig. 4. Deviation between model and experimental data.
(× — experimental data; – — model curve)

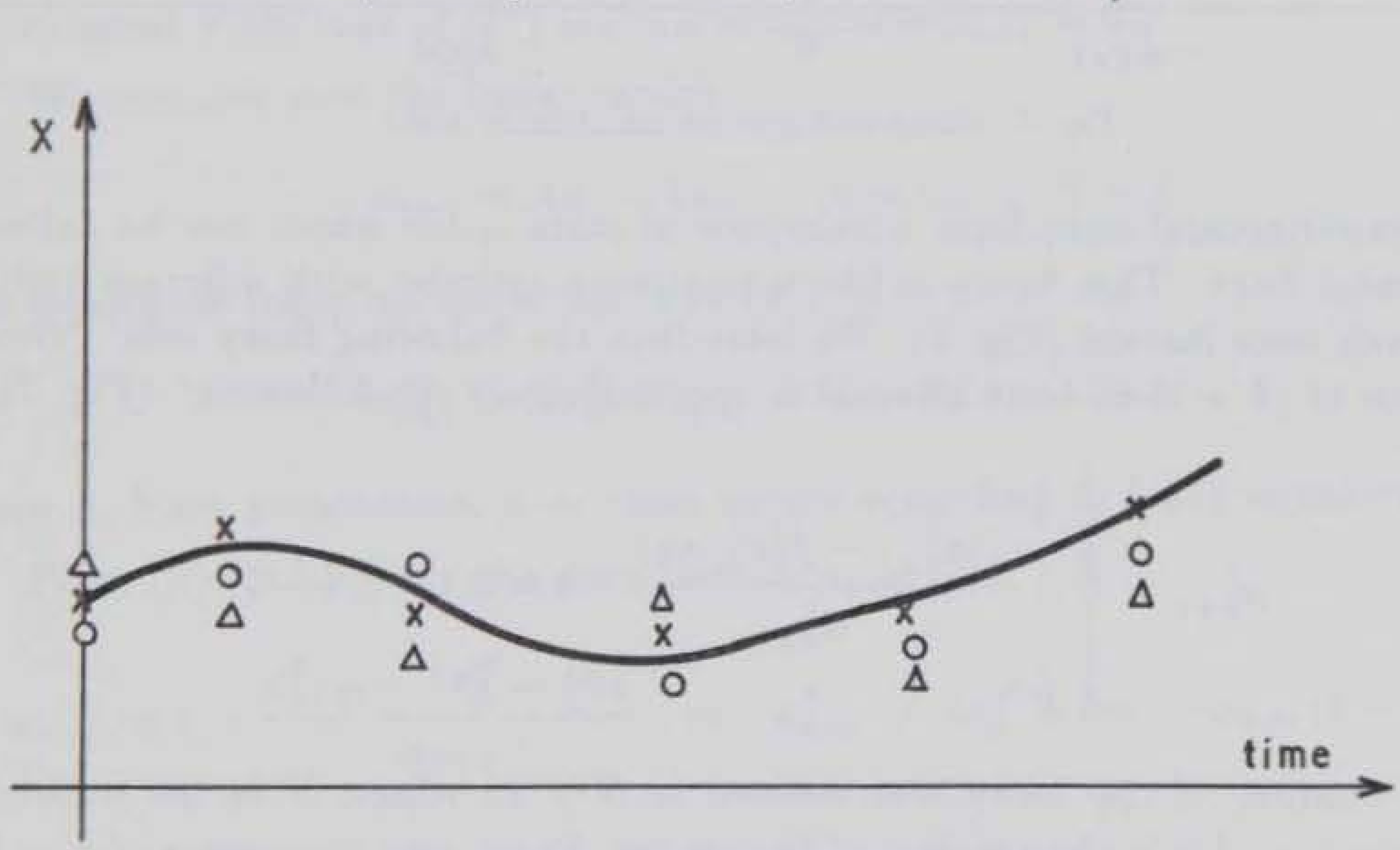

Fig. 5. Deviation between model and all experimental data.
(o — experiment No. 1; × — experiment No. 2; × — experiment No. 3; – — model curve)

Models according to (7^a) and (7^b) do not represent well the real system behaviour. Ignoration of deviations of experimental data from model curve can lead to non-optimality of the control strategy. A successful tool for a more realistic description of system dynamics are fuzzy sets.

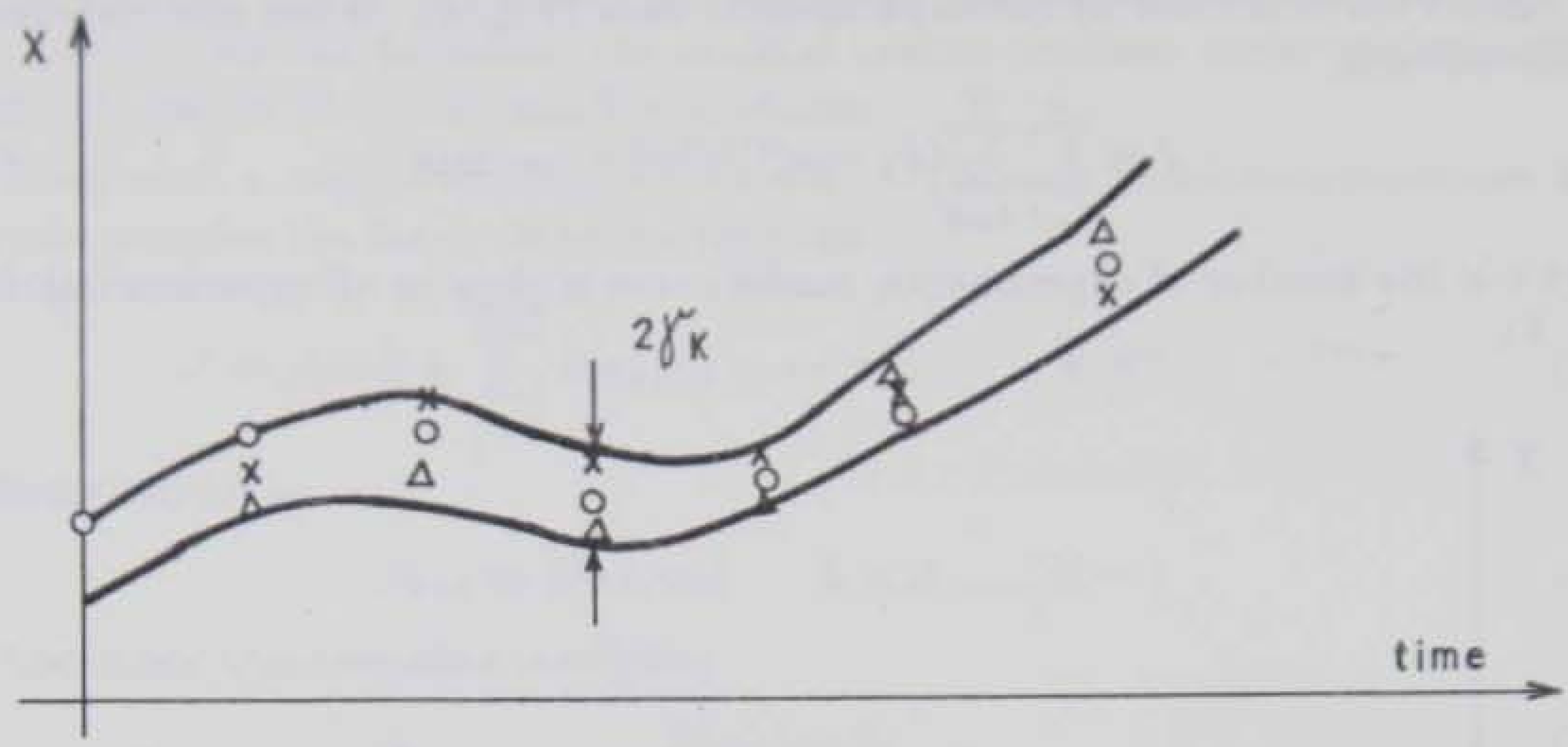

Fig. 6. Nonlinear "duct".
(o — experiment No. 1; x — experiment No. 2; x — experiment No. 3)

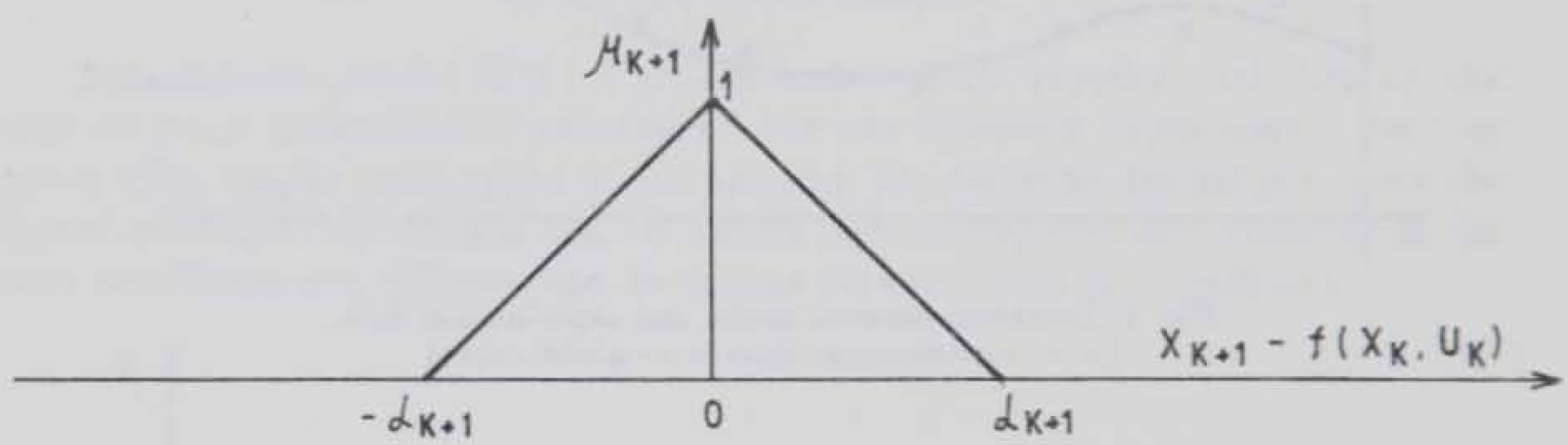

Fig. 7. Membership function of fuzzy state

All experimental data form a subspace of state space which can be called an experimental duct. This figure is like a nonlinear cylinder with different radiuses (γ_k) at each time instant (Fig. 6). We introduce the following fuzzy sets: "state of the system at $(k+1)$-th time interval is approximately the following" (Fig. 7):

$$\mu^j_{k+1} = \begin{cases} 1 & \\ 1 - \dfrac{x^j_{k+1} - f(x^j_k, u_k)}{\alpha^j_{k+1}} & k = 0, \dots, N-1 \\ 0 & j = 1, \dots, n \end{cases} \tag{8}$$

The number of the fuzzy sets defined is $N \times n$, where N is the number of time-intervals and n is the number of the states. Fuzzy sets parameters (α^j_{k+1}) are different for each k and j, i.e. $\alpha_{k+1} \in E^n$. The experimental duct (Fig. 8) can be represented by fuzzy model (5^b) and respectively by membership functions (8). The duct intersection is depicted i Fig. 8 for the $(k+1)$-th time-interval. Fuzzy set "model is approximately the following" (5^b) means that:

a) most appropriate is strict equality of (1^b): $\mu_{k+1} = 1$

b) small violations of (1^b) are acceptable with degree $(0 < \mu_{k+1} < 1)$;

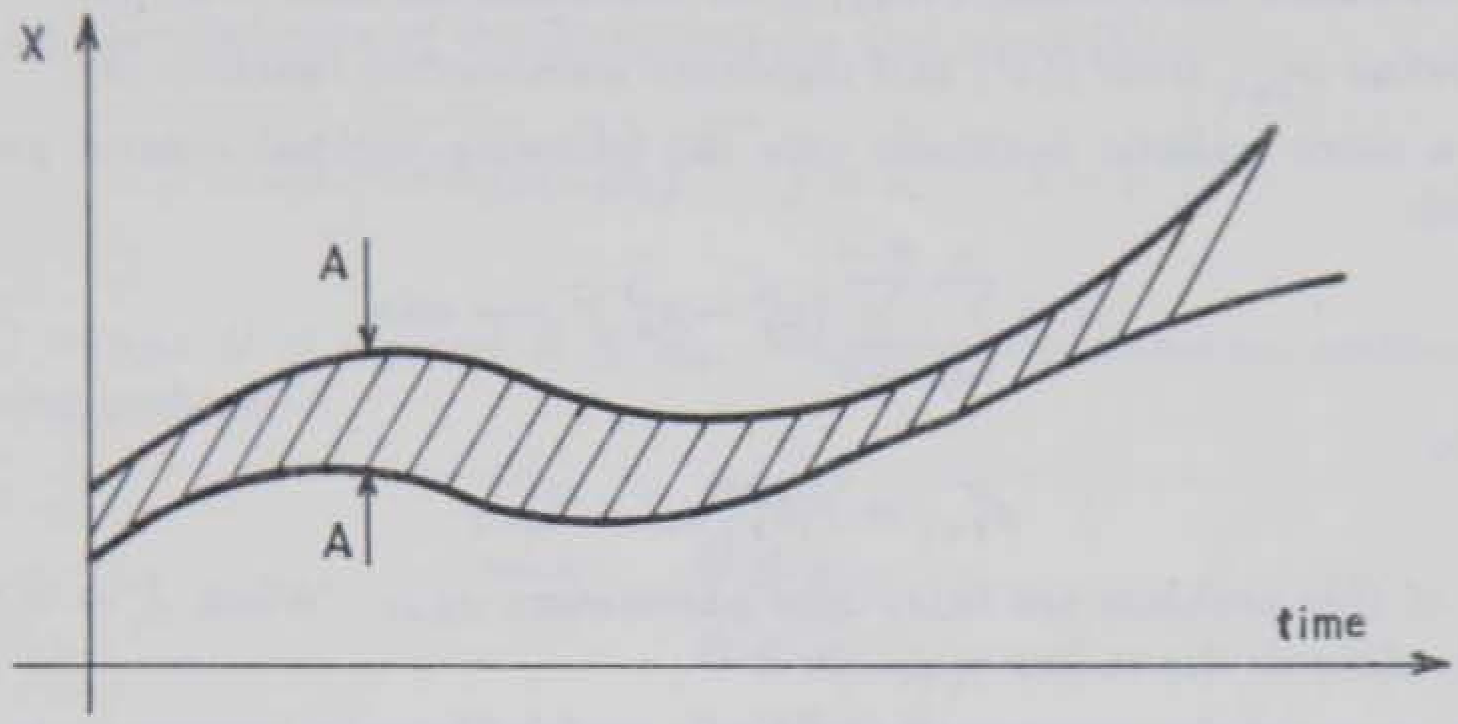

Fig. 8. Intersection of the duct

c) great violations of (1^b) are not acceptable $\mu_{k+1} = 0$.

We consider now the linear model

$$x_{k+1} = Ax_k + bu_k \qquad k = 0, \ldots, N-1 \tag{9^a}$$

The analogous fuzzy model of the type (4^b) is:

$$x^F_{k+1} = Ax^F_k + bu_k \qquad k = 0, \ldots, N-1 \tag{9^b}$$

where A, b are parameters, x — state vector according to fuzzy equation.

From (8) we have for the duct (where $\mu_{k+1} \in (0,1)$):

$$\mu_{k+1} = 1 - \frac{x^F_{k+1} - Ax^F_k - bu_k}{\alpha_{k+1}} \quad \text{or} \quad x^F_{k+1} = Ax^F_k + bu_k + \alpha_{k+1}(1 - \mu_{k+1}).$$

For $\delta_{k+1} = \alpha_{k+1}(1 - \mu_{k+1})$ we have $x^F_{k+1} = Ax^F_k + bu_k + \delta_{k+1}$ or in more detail:

$$x^F_1 = Ax_0 + bu_0 + \delta_1 = Ax_0 + bu_0 + \delta_1 = x_1 + \gamma_1 \qquad \gamma_1 = \delta_1$$

$$x^F_2 = Ax^F_1 + bu_1 + \delta_2 = Ax_1 + a\delta_1 + bu_1 + \delta_2 = x_2 + \gamma_2 \qquad \gamma_2 = A\delta_1 + \delta_2$$

....

$$x^F_k = x_k + \gamma_k \qquad k = 0, \ldots, N-1, \qquad \gamma_{k+1} = A\gamma_k + \delta_{k+1} \tag{10^a}$$

For the duct boundaries $\mu_{k+1} = 0$ and $\delta_{k+1} = \alpha_{k+1}$:

$$\gamma_{k+1} = A\gamma_k + \alpha_{k+1}, \quad \gamma_1 = \alpha_1 \quad k = 0, \dots, N-1 \tag{11^a}$$

$$\text{or} \quad \alpha_{k+1} = \gamma_{k+1} - A\gamma_k, \quad \gamma_1 = \alpha_1 \quad k = 0, \dots, N-1 \tag{11^b}$$

The algorithm for design of fuzzy sets which represents model with non-stochastic uncertainties has the following steps:

1) Determine duct radius (γ_{k+1}) from experimental data (Fig. 5).

2) Define α^j_{k+1} from (11^b) and construct membership function (8).

For a more realistic nonlinear case the following optimal control problem is considered:

$$J = \sum_{i=1}^{n} \sum_{k=0}^{N-1} (x_k^{ie} - x_k^{iF})^2 \longrightarrow \min \tag{12}$$

subject to

$$x_{k+1}^F = f(x_k^F, u_k) + \alpha_{k+1}. \tag{13}$$

Solution of this problem are fuzzy sets parameters α_{k+1}. When $J \to 0$ they are extremely close to the radius γ_{k+1}.

This approach for construction of fuzzy model allows to express more realistic ally and fully the dynamics of the systems: more appropriate is one curve but other curves are also acceptable for the description of the model (with a decreasing degree).

4. PROBLEM SOLUTION

Various FOC problems ((2), (5) and (6)) can be generalized as:

$$\begin{aligned} J &\longrightarrow \min \\ F_j &\simeq 0 \\ C_1 &= 0 \end{aligned} \tag{14}$$

when

$$\begin{aligned}
\text{for (2):} \quad & F_j = S_i(x_N) \qquad j = 1, \dots, q; \\
& C_l = x_{k+1} - f(x_k, u_k) \quad l = 1, \dots, N; \\
\text{for (5):} \quad & F_j = x_{k+1} - f(x_k, u_k) \qquad j = 1, \dots, N; \\
& C_l = S_i(x_N) \quad l = 1, \dots, q; \\
\text{for (6):} \quad & F_j = x_{k+1} - f(x_k, u_k) \qquad j = 1, \dots, N; \\
& F_j = S_i(x_N) \qquad j = N+1, \dots, N+q.
\end{aligned}$$

These optimal control problems can be transformed in to the mathematical programming (FMP) problems with $N \times (n+m)$ variables ($N \times n$ state variables

$(x_1, \ldots, x_N)$ and $N \times m$ control variables $(u_0, \ldots, u_{N-1})$). According to Bellmann-Zadeh's approach for decision making in a fuzzy environment, decision membership function μ_D is determined as an conjunction between membership function(s) of the fuzzy objective function(s) μ_G and membership function(s) of the fuzzy constraint(s) μ_C^j:

$$\mu_D = \mu_G \cap \mu_C^1 \cap \mu_C^2 \cap \ldots \cap \mu_C^v \qquad (15^a)$$

where $v = q$ for (2); $v = N$ for (5) and $v = N + q$ for (6). Optimal defuzzification is $\max_{x,u} \mu_D$. Conjunction of the fuzzy sets μ_G and μ_C^j is [1]:

$$\mu_D = \min\{\mu_G, \mu_C^j\} \qquad j = 1, \ldots, v - 1 \qquad (15^b)$$

For $D = \mu_D$: $D \leq \mu_G$ and $D \leq \mu_G^j$. We derive the following mathematical programming problem:

$$\begin{aligned} \text{Find} \quad & \max D \\ \text{s.t.c.} \quad & D \leq \mu_G \\ & D \leq \mu_C^j \\ & 0 \leq D \leq 1 \end{aligned} \qquad (16)$$

It is a crisp nonlinear mathematical programming problem which can be solved in a routine way.

5. NUMERICAL EXAMPLE

We consider the following two dimensional fuzzy optimal control problem: Find such a control sequence which satisfies fuzzy objective function (4) where

$$J = \frac{1}{5}\left[x^2 + y^2 + 0.2\sum_{k=0}^{4} u_k^2\right], \quad \alpha_0 = 1.8 \quad \text{and} \quad \beta_0 = 1, \quad \text{subject to:}$$

$$x_{k+1} = x_k + 0.2y_k \qquad k = 0, \ldots, 4$$

$$y_{k+1} = x_k + 0.2(-x_k + u_k + (1 - x_k^2)y_k^2) \qquad x^1 = x \quad x^2 = y$$

Transversality condition is $S(x_5) = x_5 + y_5 - 1.5$. Initial condition is $x_0 = (0; 1)$. We suppose that experimental data are such that there is a duct which includes all data and its radius γ_{k+1} determines α_{k+1}^j (according to (11^b)) is as constant: $\alpha_{k+1}^j = 0.01$ and $\alpha_{k+1}^2 = 0.1$ ($k = 0, \ldots, N-1$). On the basis of α_{k+1}^j we construct membership functions of type (8).

This fuzzy optimal control problem can be transformed in to the following crisp mathematical programming problem:

$$
\begin{array}{lll}
\text{Find} & \max D & \\
\text{s.t.c.} & D \le 1 - (J - 1.8)/1 & \\
 & D \le 1 - (x_{k+1} - x_k - 0.2y_k)/0.01 & k = 0, \dots, 4 \\
 & D \le 1 - (y_{k+1} - x_k - 0.2(-x_k + u_k + (1 - x_k^2)y_k^2))/0.1 & \\
 & 0 \le D \le 1 & \\
 & x_5 - y_5 = 1.5 & \\
 & x_0 = 0 & \\
 & y_0 = 1. &
\end{array}
$$

This problem has been solved by the gradient method GRG-2 and software program GINO [3]. The solution is:

Table 1. Optimal (fuzzy optimal) control is:

k	0	1	2	3	4
u_k	3.869964	0.000027	0.000025	0.000025	0.000025

Table 2. Optimal (fuzzy optimal) trajectory is:

k	0	1	2	3	4	5
x_k	0	−0.171302	0.401706	0.603379	0.708839	0.749898
y_k	1	2.852036	0.995361	0.514289	0.192291	−0.750102

The control strategy satisfies fuzzy conditions with degree $\mu_D = 0.739927$ which is as large as possible. This degree of satisfaction gives the value of crisp objective function $J = 2.060162$. The solution of the analogous crisp problem (of type (1)) are:

Table 3. Optimal (crisp optimal) control is:

k	0	1	2	3	4
u_k	−0.042960	0.193739	−0.598161	1.240191	−1.505662

Table 4. Optimal (crisp optimal) trajectory is:

k	0	1	2	3	4	5
x_k	0	1.174222	1.329960	1.516313	1.678482	1.936532
y_k	1	0.780591	0.931768	0.810844	1.290252	0.436583

The objective function has the value $J = 1.390608$.

It is easy to see that FOC approach allows one to find a more effective solution due to the fact that all possible dynamic behaviour features of the system are considered. In this example we improve the value of the objective function by about 14%. All possible trajectories of the system are studied using the fuzzy model and a better control strategy is found instead of the unique crisp model one (1^b).

6. CONCLUSION

The representation of the dynamics of the system by strict equations leads to neglecting the information about possible system behaviour. A successful approach to represent all possible dynamic behaviour of the system using fuzzy sets is developed in the paper. A more realistic fuzzy optimal control problem is formulated and solved. The concept proposed provides more effective control of the system due to the fact that all possible state dynamics are considered. A class of optimal control problems of biotechnological processes in the presence of uncertainties is now attracting our attention.

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